A TVD Scheme using Roe's Flux and the Ambient Boundary Condition

AUTHOR(S):

SAWADA, Keisuke; SHIMA, Eiji; MATSUDA, Takuya

CITATION:

SAWADA, Keisuke ...[et al]. A TVD Scheme using Roe's Flux and the Ambient Boundary Condition. Memoirs of the Faculty of Engineering, Kyoto University 1989, 51(2): 124-134

ISSUE DATE:

1989-04-30

URL:

http://hdl.handle.net/2433/281394

A TVD Scheme using Roe's Flux and the Ambient Boundary Condition

By

Keisuke SAWADA*, Eiji SHIMA* and Takuya MATSUDA**

(Received December 26, 1988)

Abstract

Details of a second-order accurate TVD scheme using the Roe's Riemann solver is described for the three-dimensional Euler equation. The differential equations are discretized using a finite volume formulation. The ambient boundary condition proposed by us is also explained.

1. Introduction

In problems of cosmic gas dynamics, strong shock waves and contact surfaces as well as a large gradient of density stratification are expected to occur, so that a robustness of a scheme is a crucial factor. The recent advance of a class of TVD schemes for hyperbolic equations (Harten 1983, Chakravarthy 1986) provides a higher order spatial accuracy and a robustness of the scheme simultaneously. In the three-dimensional as well as the two-dimensional calculations of inhomogeneous accretion flow on to a gravitating compact object (Sawada, Matsuda, Anzer, Börner & Livio, 1988), we adopt an upwind biased finite volume version of the TVD scheme, details of which are described in the following sections.

2. Basic equations

We neglect the effect of viscosity in the present calculation so that the Euler equations written in the inertial frame of reference are solved. They can be written in the vector form as

$$q_t+E_x+F_y+G_z+R=0, \tag{2.1}$$

* Aircraft Engineering Division, Kawasaki Heavy Industries
** Department of Aeronautical Engineering

$$q=\begin{vmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ e\end{vmatrix},\quad E=\begin{vmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ (e+p)u\end{vmatrix},\quad F=\begin{vmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ (e+p)v\end{vmatrix},$$

$$G=\begin{vmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ (e+p)w\end{vmatrix},\quad R=\begin{vmatrix}0\\ -\rho f_x\\ -\rho f_y\\ -\rho f_z\\ -\rho(f_xu+f_yv+f_zw)\end{vmatrix}, \tag{2.2}$$

where ρ represents the density, (u, v, w) the Cartesian velocities, p the pressure, e the total energy per unit volume, and f_x, f_y, f_z are the Cartesian components of an external force. The equation of state is given by

$$p=(\gamma-1)\left[e-\frac{1}{2}\rho(u^2+v^2+w^2)\right], \tag{2.3}$$

in which γ represents the ratio of the specific heats.

The equations are made dimensionless using appropriate charactersitic physical quantities.

3. Finite volume formulation

We adopt the upwind biased finite volume method (cell method) in the present method. We integrate the basic equations (2.1) directly in the physical space using the divergence theorem over each cell (Jameson, 1982) as

$$\iiint q_t dv+\iint(En_1+Fn_2+Gn_3)ds+\iiint Rdv=0, \tag{3.1}$$

where (n_1, n_2, n_3) denote the components of the unit vector outward normal to the cell surface.

If we define the flux function normal to each cell face as

$$H=En_1+Fn_2+Gn_3, \tag{3.2}$$

then we can consider locally the one-dimensional problem for all cell faces. The numerical flux functions on these faces are determined from the solution of the Riemann problem in a unified fashion.

We integrate the equation (3.1) in time by the two-step explicit method as

$$
\begin{aligned}
V_{i,j,k}q^{n+\frac{1}{2}}_{i,j,k} &= V_{i,j,k}q^{n}_{i,j,k} \\
&\quad -\frac{\triangle t}{2}(S_{i+\frac{1}{2},j,k}H^{n}_{i+\frac{1}{2},j,k} - S_{i-\frac{1}{2},j,k}H^{n}_{i-\frac{1}{2},j,k}) \\
&\quad -\frac{\triangle t}{2}(S_{i,j+\frac{1}{2},k}H^{n}_{i,j+\frac{1}{2},k} - S_{i,j-\frac{1}{2},k}H^{n}_{i,j-\frac{1}{2},k}) \\
&\quad -\frac{\triangle t}{2}(S_{i,j,k+\frac{1}{2}}H^{n}_{i,j,k+\frac{1}{2}} - S_{i,j,k-\frac{1}{2}}H^{n}_{i,j,k-\frac{1}{2}}) \\
&\quad -\frac{\triangle t}{2}R^{n}_{i,j,k}V_{i,j,k}\,, \qquad (3.3) \\
V_{i,j,k}q^{n+1}_{i,j,k} &= V_{i,j,k}q^{n}_{i,j,k} \\
&\quad -\triangle t(S_{i+\frac{1}{2},j,k}H^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - S_{i-\frac{1}{2},j,k}H^{n+\frac{1}{2}}_{i-\frac{1}{2},j,k}) \\
&\quad -\triangle t(S_{i,j+\frac{1}{2},k}H^{n+\frac{1}{2}}_{i,j+\frac{1}{2},k} - S_{i,j-\frac{1}{2},k}H^{n+\frac{1}{2}}_{i,j-\frac{1}{2},k}) \\
&\quad -\triangle t(S_{i,j,k+\frac{1}{2}}H^{n+\frac{1}{2}}_{i,j,k+\frac{1}{2}} - S_{i,j,k-\frac{1}{2}}H^{n+\frac{1}{2}}_{i,j,k-\frac{1}{2}}) \\
&\quad -\triangle t R^{n+\frac{1}{2}}_{i,j,k}V_{i,j,k}\,, \qquad (3.4)
\end{aligned}
$$

where $V_{i,j,k}$ denotes the volume of the cell, $S_{i+\frac{1}{2},j,k} \cdots S_{i,j,k-\frac{1}{2}}$ the area of each cell face and n denotes the time level. This procedure has a second-order of accuracy in time. However, in the actual calculations, we may use the local time stepping method to accelerate the convergence. In the local time stepping method, the strict time evolution is not followed.

4. Roe's approximate Riemann solver

In the cell method, the numerical fluxes at the cell faces are related to the change of the averaged values of the conservative variables in the cell. We adopt Roe's approximate Riemann solver (Roe, 1981) to find similar solutions at the cell faces from which we constitute the numerical fluxes there.

Roe constructed his scheme by linearizing the Riemann problem first. Using spatially averaged physical quantities, he found the exact solutions to this approximated Riemann problem. Let us consider two states $q^{-}_{i+\frac{1}{2}}$ and $q^{+}_{i+\frac{1}{2}}$ separated by the cell-face located at $i+\frac{1}{2}$. Using these two states, Roe defined the averaged state as

$$
\begin{aligned}
\rho_{i+\frac{1}{2}} &= a\rho^{+}_{i+\frac{1}{2}} + b\rho^{-}_{i+\frac{1}{2}}\,, \\
u_{i+\frac{1}{2}} &= a u^{+}_{i+\frac{1}{2}} + b u^{-}_{i+\frac{1}{2}}\,, \\
v_{i+\frac{1}{2}} &= a v^{+}_{i+\frac{1}{2}} + b v^{-}_{i+\frac{1}{2}}\,, \qquad (4.1) \\
w_{i+\frac{1}{2}} &= a w^{+}_{i+\frac{1}{2}} + b w^{-}_{i+\frac{1}{2}}\,,
\end{aligned}
$$

$$h_{i+\frac{1}{2}}=ah^{+}_{i+\frac{1}{2}}+bh^{-}_{i+\frac{1}{2}},$$

where a, b are the coefficients defined by

$$a=\frac{\sqrt{\rho^{+}_{i+\frac{1}{2}}}}{\sqrt{\rho^{+}_{i+\frac{1}{2}}}+\sqrt{\rho^{-}_{i+\frac{1}{2}}}}, \tag{4.2}$$

$$b=\frac{\sqrt{\rho^{-}_{i+\frac{1}{2}}}}{\sqrt{\rho^{+}_{i+\frac{1}{2}}}+\sqrt{\rho^{-}_{i+\frac{1}{2}}}}, \tag{4.3}$$

respectively and h denotes the total enthalpy defined by

$$h=\frac{\gamma p}{(\gamma-1)\rho}+\frac{1}{2}(u^2+v^2+w^2). \tag{4.4}$$

The speed of sound can be derived as

$$c^2=(\gamma-1)\left[h-\frac{1}{2}(u^2+v^2+w^2)\right]. \tag{4.5}$$

We introduce the local Cartesian coordinate in which the x-axis is taken to the normal direction of the cell face. The basic equations in the non-conservative form using this local Cartesian base can be written as

$$q_t+Aq_x+Bq_y+Cq_z=0. \tag{4.6}$$

Forming the Jacobian matrices $A\left(=\frac{\partial E}{\partial q}\right)$, $B\left(=\frac{\partial F}{\partial q}\right)$ and $C\left(=\frac{\partial G}{\partial q}\right)$ by the averaged values defined by (4.1), Roe found the exact solution of this linearized Riemann problem. Contributions from the y and z derivative terms can be neglected and thus we consider only the second term of the above equation.

The Jacobian matrix A has real eigenvalues $\lambda_k (k=1, 2, ...5)$ and a complete set of right eigenvectors r_k. The flux differences as well as the state differences between these two states, $q_{i+\frac{1}{2}}{}^{+}$ and $q_{i+\frac{1}{2}}{}^{-}$, can be expanded using these right eigenvectors as a set of base vectors. Denoting the eigenvalues as

$$\lambda_1=u-c,\ \lambda_2=\lambda_3=\lambda_4=u,\ \lambda_5=u+c, \tag{4.7}$$

then we have the right eigenvectors

$$r_1=\begin{vmatrix}1\\u-c\\v\\w\\h-cu\end{vmatrix},\ r_2=\begin{vmatrix}0\\0\\1\\0\\v\end{vmatrix},\ r_3=\begin{vmatrix}0\\0\\0\\1\\w\end{vmatrix}, \tag{4.8—1.10}$$

$$r_4=\begin{pmatrix}1\\u\\v\\w\\\frac{1}{2}(u^2+v^2+w^2)\end{pmatrix},\quad r_5=\begin{pmatrix}1\\u+c\\v\\w\\h+cu\end{pmatrix} \tag{4.11—4.12}$$

It is convenient to define a set of left eigenvectors $l_i\,(i=1,2...5)$ orthonormal to the right eigenvectors as

$$l_1=\left(\frac{cu+\frac{1}{2}(\gamma-1)\theta}{2c^2},\ -\frac{(\gamma-1)u+c}{2c^2},\ -\frac{(\gamma-1)v}{2c^2},\ -\frac{(\gamma-1)w}{2c^2},\ \frac{\gamma-1}{2c^2}\right), \tag{4.13}$$

$$l_2=(-v,\ 0,\ 1,\ 0,\ 0), \tag{4.14}$$

$$l_3=(-w,\ 0,\ 0,\ 1,\ 0), \tag{4.15}$$

$$l_4=\left(1-\frac{(\gamma-1)\theta}{2c^2},\ \frac{(\gamma-1)u}{c^2},\ \frac{(\gamma-1)v}{c^2},\ \frac{(\gamma-1)w}{c^2},\ -\frac{(\gamma-1)}{c^2}\right), \tag{4.16}$$

$$l_5=\left(\frac{-cu+\frac{1}{2}(\gamma-1)\theta}{2c^2},\ \frac{c-(\gamma-1)u}{2c^2},\ -\frac{(\gamma-1)v}{2c^2},\ -\frac{(\gamma-1)w}{2c^2},\ \frac{\gamma-1}{2c^2}\right). \tag{4.17}$$

where $\theta=u^2+v^2+w^2$. Using these left eigenvectors, the state changes, $\delta q_{i+\frac{1}{2}}=q_{i+\frac{1}{2}}{}^{+}-q_{i+\frac{1}{2}}{}^{-}$, can be expressed as

$$\delta q_{i+\frac{1}{2}}=\alpha_1 r_1+\alpha_2 r_2+\alpha_3 r_3+\alpha_4 r_4+\alpha_5 r_5 \tag{4.18}$$

where

$$\alpha_k=l_k\cdot\delta q_{i+\frac{1}{2}},\qquad (k=1,2\ldots5) \tag{4.19}$$

In the equation (4.18), $\alpha_k r_k$ $(k=1,2\ldots5)$ corresponds to the state changes across each simple wave. The intermediate states can be derived from these relations from which we can obtain the flux differences. However, Roe's scheme can give these flux differences in compact forms as

$$\delta f_k{}^{+}=\lambda_k{}^{+}\alpha_k r_k=\frac{1}{2}(\lambda_k+|\lambda_k|)\alpha_k r_k, \tag{4.20}$$

$$\delta f_k{}^{-}=\lambda_k{}^{-}\alpha_k r_k=\frac{1}{2}(\lambda_k-|\lambda_k|)\alpha_k r_k, \tag{4.21}$$

from which we can determine the numerical flux function at $i+\frac{1}{2}$ as

$$f(q_{i+\frac{1}{2}}{}^{+},q_{i+\frac{1}{2}}{}^{-})=\frac{1}{2}[f(q_{i+\frac{1}{2}}{}^{+})+f(q_{i+\frac{1}{2}}{}^{-})]-\frac{1}{2}\sum_k(\delta f_k{}^{+}-\delta f_k{}^{-}). \tag{4.22}$$

As pointed out by Roe (1981) and Chakravarthy (1986), Roe's scheme may catch an

expansion shock as well as the ordinary shocks. This is due to the fact that, at the sonic expansion, the expansion fan is replaced by a single transition across a simple wave, and no mechanism works to break up this discontinuity. To avoid this situation, numerical viscosity is added to each wave field to fix entropy, only at the sonic expansion. We follow the procedure developed by Chakravarthy (1986).

At sonic expansions:

$$(\lambda_k)_i<0<(\lambda_k)_{i+1}, \tag{4.23}$$

λ_k^+ and λ_k^- in equation (4.20) and (4.21) are replaced by

$$\lambda_k^+=\frac{1}{2}(\lambda_k+|\lambda_k|)+\frac{1}{4}[(\lambda_k)_{i+1}-(\lambda_k)_i], \tag{4.24}$$

$$\lambda_k^-=\frac{1}{2}(\lambda_k-|\lambda_k|)-\frac{1}{4}[(\lambda_k)_{i+1}-(\lambda_k)_i]. \tag{4.25}$$

5. Slope limiting procedure

We adopt the pre-processing approach to attain a higher order of spatial accuracy, which is the so-called MUSCL approach. In this approach, appropriate structures of dependent variables are introduced in each cell to estimate the face values with a higher-order of accuracy. Then the upwind scheme is applied to find the numerical fluxes there. The resultant scheme attains the second-order of accuracy if the piecewise linear distributions are introduced. This spatial accuracy is estimated in the computational space, so that a sufficient smoothness of the grid system in the physical space is implicitly assumed for keeping this accuracy.

Turkel (1985) has shown that a non-uniformity of grid spacings degrades the spatial accuracy, and sometimes results in an inconsistent discretization in the physical space. In the present study, we incorporate a procedure which accounts for the non-uniformity of grid spacings in the estimation of spatial gradients (Sawada & Takanashi, 1987).

The adopted procedure which accounts for the effects of non-uniform grid spacing is as follows (Sawada & Takanashi, 1987): we first assume a piecewise parabolic profile of conservative variables for there successive zones to find the averaged slope in the central zone. Then, we reassume a piecewise linear profile based on this slope to find the values of the dependent variables at the cell faces.

Let us consider the procedure realizing the above statement. We adopt a dimensionally split procedure so that only one space direction is considered here. Other directions can be treated in the same way.

We take the differences $\delta q_{i+\frac{1}{2}}$, $\delta q_{i-\frac{1}{2}}$ defined by

$$\delta q_{i+\frac{1}{2}}=q_{i+1}-q_i,$$
$$\delta q_{i-\frac{1}{2}}=q_i-q_{i-1}, \tag{5.1}$$

where q_i represents the conservative variables. Since the conservative variables satisfy

$$q_i=\frac{1}{\triangle\xi_i}\int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}}q(\xi)d\xi, \tag{5.2}$$

we can assume a piecewise parabolic profile:

$$q(\xi)=a\xi^2+b\xi+c \tag{5.3}$$

restricted by equation (5.2) and

$$q_{i-1}=\frac{1}{\triangle\xi_{i-1}}\int_{\xi_{i-\frac{3}{2}}}^{\xi_{i-\frac{1}{2}}}q(\xi)d\xi, \tag{5.4}$$
$$q_{i+1}=\frac{1}{\triangle\xi_{i+1}}\int_{\xi_{i+\frac{1}{2}}}^{\xi_{i+\frac{3}{2}}}q(\xi)d\xi, \tag{5.4}$$

where

$$\triangle\xi_i=\xi_{i+\frac{1}{2}}-\xi_{i-\frac{1}{2}}. \tag{5.5}$$

The size of the cell length $\triangle\xi_i$ can be determined by an appropriate manner.

Using equation (5.3), we can estimate the averaged slopes of the conservative variables in the central zone as

$$\triangle q_i=[(2\triangle\xi_{i-1}+\triangle\xi_i)\mu+(2\triangle\xi_{i+1}+\triangle\xi_i)\nu]\triangle\xi_i/\chi, \tag{5.6}$$

where

$$\chi=\triangle\xi_{i+1}+\triangle\xi_i+\triangle\xi_{i-1}, \tag{5.7}$$
$$\mu=\frac{\delta q_{i+\frac{1}{2}}}{\triangle\xi_{i+1}+\triangle\xi_i}, \tag{5.8}$$
$$\nu=\frac{\delta q_{i-\frac{1}{2}}}{\triangle\xi_i+\triangle\xi_{i-1}}. \tag{5.9}$$

The face values are then estimated as

$$q_{i+\frac{1}{2}}^{-}=q_i+\frac{1}{2}\triangle q_i, \tag{5.10}$$
$$q_{i-\frac{1}{2}}^{+}=q_i-\frac{1}{2}\triangle q_i. \tag{5.11}$$

In order to attain the non-oscillatory property of the scheme, we modify the above procedures as follows (Chakravarthy, 1986). Using the complete set of right eigenvec-

tors, r_k, defined by the cell averaged conservative variables, q_i, and the unit vector pointing toward the averaged normal direction of the cell faces, the differences of conservative variables, $\delta q_{i+\frac{1}{2}}$ and $\delta q_{i-\frac{1}{2}}$, can be written as

$$\delta q_{i+\frac{1}{2}}=\Sigma_k(\alpha_k)_{i+\frac{1}{2}}r_k, \tag{5.12}$$

$$\delta q_{i-\frac{1}{2}}=\Sigma_k(\alpha_k)_{i-\frac{1}{2}}r_k, \tag{5.13}$$

where α_k is the coefficient of the right eigenvector termed as the state change parameter (Chakravarthy, 1986).

Replacing $\delta q_{i+\frac{1}{2}}$ and $\delta q_{i-\frac{1}{2}}$ by $(\alpha_k)_{i+\frac{1}{2}}$ and $(\alpha_k)_{i-\frac{1}{2}}$ respectively, we can estimate the averaged value of α_k as

$$(\alpha_k)_i=\frac{1}{2}[(2\triangle\xi_{i-1}+\triangle\xi_i)\mu'+(2\triangle\xi_{i+1}+\triangle\xi_i)\nu']\triangle\xi_i/\chi, \tag{5.14}$$

where

$$\mu'=\frac{(\alpha_k)_{i+\frac{1}{2}}}{\triangle\xi_{i+1}+\triangle\xi_i}, \tag{5.15}$$

$$\nu'=\frac{(\alpha_k)_{i-\frac{1}{2}}}{\triangle\xi_i+\triangle\xi_{i-1}}. \tag{5.16}$$

In order to preserve the non-oscillatory property of the scheme, we limit the magnitude of the state change parameters α_k as

$$(\hat{\alpha}_k)_i=\begin{cases} sign((\alpha_k)_i)\sigma, & \text{if } (\alpha_k)_{i+\frac{1}{2}}(\alpha_k)_{i-\frac{1}{2}}>0; \\ 0, & \text{otherwise} \end{cases} \tag{5.17}$$

where

$$\sigma=min[|(\alpha_k)_i|,|(\alpha_k)_{i+\frac{1}{2}}|,\ |(\alpha_k)_{i-\frac{1}{2}}|]. \tag{5.18}$$

The limited values of the conservative variables at the cell faces are thus

$$q_{i+\frac{1}{2}}^{-}=q_i+\Sigma_k(\hat{\alpha}_k)_i r_k, \tag{5.19}$$

$$q_{i-\frac{1}{2}}^{+}=q_i-\Sigma_k(\hat{\alpha}_k)_i r_k. \tag{5.20}$$

Note that the above procedure accounts only for the irregular grid spacings, and the skewness of the grid line is still neglected. A multi-dimensional surface fitting, which is a direct extension of the present one dimensional case, may improve the spatial accuracy. However, the overall spatial accuracy still remains to be first-order if the shape of the grid is arbitrarily determined (Roe, 1987). Furthermore, a multi-dimensional surface fitting would require a large amount of computations, so that a practical way to attain fine solutions may be to improve the quality of the grid system.

6. Ambient boundary conditions

On the outer numerical boundary, we apply the ambient boundary condition (Sawada et al., 1986) in which we assume fictitious cells located just outside the outer boundary surface. The physical quantities corresponding to the incident gas are specified in these fictitious cells. The numerical flux at the outer boundary is determined by solving the Riemann problem. Since we solve the Riemann problem by the upwind scheme, it does not require any additional work.

The ambient boundary condition is a neat approach to realize the nonreflecting boundary condition. Inflows and outflows occur in quite natural ways according to the solution of the Riemann problem at the boundary surface. Influences of various numerical boundaries, including the present one, over the simulation of laboratory free jet flows can be found in the reference (Matsuda et al., 1987). A brief description of the ambient condition is given in the following (see also Sawada & Takanashi, 1987).

Let us first consider the system of a linear hyperbolic equations. It can be written in a characteristic form as

$$q_t + A q_x = 0, \tag{6.1}$$

where q represents the charactersitic variables of n components and the Jacobian matrix A can be written as

$$A = \begin{pmatrix} \Lambda^- & \cdots & 0 \\ \vdots & \Lambda^0 & \vdots \\ 0 & \cdots & \Lambda^+ \end{pmatrix}, \tag{6.2}$$

where Λ^-, Λ^0 and Λ^+ are the diagonal matrices which have negative components of $d_- \times d_-$, zero components of $d_0 \times d_0$ and positive components of $d_+ \times d_+$ dimensions, respectively.

The upwind difference of equation (6.1) becomes

$$q_i^{n+1} = q_i^n - \lambda [A^+ (q_i^n - q_{i-1}^n) + A^- (q_{i+1}^n - q_i^n)], \tag{6.3}$$

where A^+ and A^- are the projected Jacobian matrices as

$$A^+ = P^+ A = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & \Lambda^+ \end{pmatrix},$$

$$A^- = P^- A = \begin{pmatrix} \Lambda^- & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \tag{6.4}$$

in whibh P^+ and P^- are the natural projection operators and $\lambda=\Delta t/\Delta x$ (Osher & Chakravarthy, 1983)

If the numerical boundary locates at $i-\frac{1}{2}$, then q_i and q_{i+1} are the interior values, while q_{i-1} should be the exterior boundary value. A common treatment of the boundary condition is to first determine the values of the dependent variable q_{i-1}. Since we use characteristic variables, this reduces the relation to

$$q_{i-1}=P^+q_\infty+P^-q_i. \tag{6.5}$$

However, inserting (6.5) into (6.3), we have

$$q_i^{n+1}=q_i^n-\lambda[A^+(q_i^n-q_\infty)+A^-(q_{i+1}^n-q_i^n)]. \tag{6.6}$$

This equation implies that only the d_+ dependent variables are used explicitly and the rest are all discarded. Noting then above, we can set

$$q_{i-1}=q_\infty, \tag{6.7}$$

for the exterior value. Thus, we can solve the boundary cells with the interior scheme of equation (6.3) by assuming equation (6.7). This is the ambient condition for the linear case. It can be said that the exact boundary condition can be given if an upwind discretization together with the ambient conditions are employed for linear cases.

We next consider the nonlinear case. We closely mimic the linear case. A system of nonlinear hyperbolic equations can be written in the same from of equation (6.1), although the Jacobian matrix A is a function of q. The Jacobian matrix A has real eigenvalues and a complete set of right eigenvectors. Using these eigenvectors, we can find a similar transformation matrix T such as

$$A=T\Lambda T^{-1}, \tag{6.8}$$

where Λ is a diagonal matrix whose components are the eigenvalues of A. Then we have the characteristic form of equation (6.1) as

$$T^{-1}q_t+\Lambda T^{-1}q_x=0. \tag{6.9}$$

The interchange between T^{-1} and the difference operators may not be valid for the nonlinear case, but freezing the Jacobian matrix A as well as T^{-1} at the boundary value would lead to the same treatment as in the linear case. Defining the projection operators in equation (6.4), we can find the boundary values of dependent variables in characteristic form by using equation (6.5). This is the usually adopted boundary treatment of non-reflecting boundary conditions (Whitfield & Janus, 1984), although the detailed im-

plementations may differ from one case to another.

In the ambient condition combined with the upwind scheme, almost the same line is followed. However, one major point is different from the above procedure: the wave system at the boundary is not frozen to the inner one, but has dependence on both sides of the physical quantities because the Riemann problem is solved there. As a result of this procedure, the direction of the convection velocity, as well as the other flow variables deciding the numerical fluxes, is regulated by the circumferential conditions.

In the actual implementation, this ambient condition seems to be the simplest among various boundary conditions, since all we have to do is to put the reference state in the fictitious cell and keep the value unchanged during the integrations. There is no need to incorporate any switches to account for the directions of the characteristics, since the upwind scheme adopted in the numerical scheme actually does it.

References

Chakravarthy, S. R. 1986 The versatility and reliability of Euler solvers based on high-accuracy TVD formulations. *A.I.A.A. Paper* 86-0243.

Harten, A. 1983 High resolution schemes for hyperbolic conservation laws *J. Comp. Phys.* **49**, 357.

Jameson, A. 1982 Transonic aerofoil calculations using the Euler equations. *Numerical Methods in Aeronautical Fluid Dynamics* (ed. P. L. Roe). Academic Press.

Matsuda, T., Umeda, Y., Ishii, R, Yasuda, A. & Sawada, K. 1987 Numerical and experimental studies on choked underexpanded jets. *AIAA* paper, 87-1378.

Osher, S. & Chakravarthy, S. 1983 Upwind schemes and boundary conditions with applications to Euler equations in general geometries. *J. Comp. Phys.* **50**, 447.

Roe, P. L. 1981 Approximate Riemann solvers, parameter vectors, and difference schemes. *J. Comp. Phys.* **43**, 357.

Roe, P. L. 1987 Error estimates for cell-vertex solutions of the compressible Euler equations. *I.C.A.S.E. Rep.* 87-6.

Sawada, K., Matsuda, T., Anzer, U., Börner, G. & Livio, M. 1988 Inhomogeneous wind accretion: Comparison between 3D and 2D computations. to be published in *Astron. and Astrophys.*

Sawada, K., Shima, E., Matsuda, T. & Inaguchi, T. 1986 The Osher upwind scheme and its application to cosmic gas dynamics. *Mem. Fac. Eng. Kyoto Univ.* **48**, 240.

Sawada. K & Takanashi, S. 1987 A numerical investigation on wing / nacelle interfaces of USB configuration. *A.I.A.A. Paper* 87-0455.

Turkel, E. 1985 Accuracy of schemes with nonuniform meshes for compressible fluid flows. *I.C.A.S.E. Rep.* 85-43.

Whitefield, D. L. & Janus, J. M. 1984 Three-dimensional unsteady Euler equations solution using flux vector splitting. *A.I.A.A. Paper* 84-1552.