Realization of Pseudo-Rational Input/Output Maps and Its Spectral Properties

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By

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Abstract

Realization of a special class of input/output maps is considered. A linear input/output map which belongs to this class is called *pseudo-rational*, and it behaves in a way similar to the input/output maps of finite-dimensional systems in the following sense: To determine the canonical state space, only the *output data on a bounded time interval is needed*. Central examples of this class of input/output maps are those of delay-differential systems. A concrete representation for the canonical space is given; and then it is used to give an explicit differential equation description. Some spectral properties which are very similar to those of the delay-differential systems are also proved. Some examples are demonstrated to illustrate the realization procedure.

1. Introduction

Infinite-dimensional realization theory has been developed, guided mainly by the well-established finite-dimensional counterpart. One of the troubles in this respect is that there can be (and there are) many different candidates in generalizing the useful system-theoretic concepts to the infinite-dimensional context due to the very nature of infinite-dimensionality. For example, there are some different notions of canonicity, each leading (or not leading) to a different type of an existence and uniqueness theorem of canonical realizations. Faced with such a seemingly chaotic situation, one is no doubt led to the following basic suspicion: Is infinite-dimensional realization theory meaningful to study?

The basic theme of realization theory is of course to give an effective way of constructing a model out of a given external behavior. It seems to the author that most of the infinite-dimensional realization theories remain rather abstract and have

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not been very successful in this regard, especially when compared to the great success of the finite-dimensional theory.

Our objective here is a rather modest one. We isolate a small class of input/output maps which enables us to give a concrete description of the state space model in much the same way as in the finite-dimensional case.

In trying to generalize the finite-dimensional theory, one is of course interested in what property of finite-dimensional systems one should focus his attention on. To get some idea as to this, consider the following transfer function:

(1.1) $W(s)=1/(se^s-1)$.

Though the function W is not a rational function (hence does not admit a finite-dimensional realization), its numerator and denominator allow the following "coprime" condition:

(1.2) $1\cdot 1+(se^s-1)\cdot 0=1$.

In order words, if we allow the language of distribution theory, (1.2) may be rewritten as

(1.3) $\delta*\delta+(\delta'_{-1}-\delta*)0=\delta$

where δ_a denotes the Dirac distribution at a and $\delta=\delta_0$. Similar coprime notions have been used in the study of delay-differential systems. However, since coprime factorizations (though in the realm of the polynomial ring of one indeterminate) have played an essential role in the finite-dimensional theory, it is reasonable to expect that there can be a suitable generalization of this concept for the infinite-dimensional realization theory. Of course, one has to clarify the following questions:

1) How far can we go with such "coprime" factorizations?

2) What do we mean by "coprimeness" in this context?

In order to obtain some idea as to the first question, let us review what can be done to continuous-time finite-dimensional systems with coprime factorizations. It is meaningful to do so, because in the finite-dimensional theory, one usually does not deal with continuous-time systems, but bypasses the problem by resorting to the equivalent discrete-time problem, even with the presence of a coprime factorization.

Consider the following impulse response function:

$$A(t)=\sin t \quad (t\geq 0)$$

whose transfer function is of course

$$W(s)=1/(s^2+1).$$

Since the denominator and the numerator are coprime, it is a common understanding that the state space of the canonical realization must be completely determined by the denominator s^2+1. We shall now construct the canonical state space directly in the context of continuous-time systems.

Suppose that the present time is 0 without loss of generality, since the systems we consider are constant. The input space Ω is a function space consisting of functions having compact support in the past, i. e., $(-\infty, 0]$. The output space Γ is a function space consisting of functions which are locally L^2 on $[0, \infty)$. Then, the input/output map associated with $A(t)$

$$(1.4) \quad f(u)(t)=\int_{-\infty}^{0} \sin(t-\tau u(\tau)d\tau$$

gives a continuous linear map: $\Omega \to \Gamma$ which commutes with shifts. Now consider the factorization:

$$\begin{tikzcd} \Omega \arrow[rr, "f"] \arrow[dr, "f"'] & & \Gamma \\ & \operatorname{im} f \arrow[ur, "j"'] & \end{tikzcd}$$

The space $\operatorname{im} f$ can be taken to be the state space of a canonical model (KALMAN, FALB, and ARBIB [1969]; YAMAMOTO [1981b]). So let us compute $\operatorname{im} f$.

Differentiating both sides of (1.4), we obtain

$$\left(\frac{d^2}{dt^2}+1\right)f(u)(t)=0 \quad \text{for all} \quad t \geqq 0.$$

Conversely, one can easily check that if $(d^2/dt^2+1)y(t)=0$ for all $t \geq 0$ then $y=f(u)$ for some u in Ω. Hence we have

$$\begin{aligned} (1.5) \quad \operatorname{im} f &= \{y \in \Gamma: \ (d^2/dt^2+1)y(t)=0 \quad \text{for all} \quad t \geq 0\} \\ &= \{x_1 \sin t + x_2 \cos t: \ x_i \in \boldsymbol{R}\} \cong \boldsymbol{R}^2. \end{aligned}$$

Once such a representation is obtained, it is almost straightforward to obtain a matrix representation of the canonical realization. Indeed, take

$$\begin{aligned} (1.6) \quad & F:=d/dt, \\ & G:=A \ \text{(impulse response)}, \\ & H: \ y \mapsto y(0). \end{aligned}$$

Then, the triple (F, G, H) gives the canonical realization of $A(t)$. In the present case, (F, G, H) are expressed, in terms of the basis $\{\sin t, \ \cos t\}$, as

$$F=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad G=\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H=(0, \ 1).$$

Thus, the central problem is to obtain a concrete representation of $\overline{\operatorname{im} f}$ (which we employ as the canonical state space instead of $\operatorname{im} f$ in the infinite-dimensional case) like (1.5).

What we are to prove is the following result: Suppose that we are given an impulse response matrix $A(t)$ which admits a coprime factorization $A=Q^{-1}*P$ where Q and P are suitable matrices with distribution entries. Then $\overline{\operatorname{im} f}$ is represented as

$$\overline{\operatorname{im} f}=X^Q:=\{y \in \Gamma: \ Q*y(t)=0 \quad \text{for} \quad t \geq 0\}$$

(For a more precise statement, see Section 3, Theorem (3.13).) One will no doubt notice the similarty of this characterization to the Fuhrmann realization of finite-dimensional systems. We shall indeed give a realization of "Fuhrmann-type" in Section 3. In Section 4 we turn our attention to some spectral properties of such realizations. It will be shown that such realization have some remarkable spectral properties, including the spectral minimality. A complete characterization of the spectrum and the resolvent set will also be given. The final section is devoted to examples illustrating how the results in Section 3 can be used to explicitly compute the canonical realizations of delay-differential input/output maps.

2. Preliminaries

We start with some mathematical preliminaries which are needed in the subsequent developments. We then review some basic facts on the abstract realization theory for constant linear continuous-time systems.

In what follows, we deal only with systems over $\boldsymbol{R}$, but the generalization to the systems over $\boldsymbol{C}$ presents no esential difficulty. Every function and distribution is $\boldsymbol{R}$-valued.

Let us start by defining the input and output spaces. Assume the present time to be 0 without loss of generality, since we deal only with systems having constant structures. We assume that inputs and outputs are always locally L^2. Also, assume that each input has support in the past $(-\infty, 0]$ and each output has support in $[0, \infty)$. It is natural to assume that every input has compact support, since inputs are applicable only for a finite-time period. Assuming that there are m input and p output channels, and denoting the input space by Ω and output space by Γ, we have

$$
(2.1) \qquad \Omega = (\bigcup_{n>0} L^2[-n, 0])^m; \qquad \Gamma = (L_{loc}^2[0, \infty))^p.
$$

The topology of Ω is given as the strongest (finest) locally convex topology which makes all inclusions $L^2[-n, 0] \to \Omega$ continuous. This topology is called the strict inductive limit topology induced from the sequence $\{L^2[-n, 0]\}_{n>0}$. The topology of Γ is generated by the family of seminorms, each of which is nothing but the L^2-norm (though it is not a norm in the present context) on each bounded interval. These spaces are naturally equipped with the following shift operators which are strongly continuous semigroups.

$$
(2.2) \qquad (\sigma_t\omega)(s) := \begin{cases} \omega(s+t), & s \leq -t \\ 0, & s > -t, \end{cases} \quad s \leq 0, \ t \geq 0, \ \omega \in \Omega.
$$

$$
(\tilde{\sigma}_t)(s) := \gamma(s+t), \quad t, \ s \geq 0, \ \gamma \in \Gamma.
$$

In the subsequent developments, the use of distributions is crucial, and hence we prepare some basic notations regarding this theory.

As usual, $\mathcal{D}'(\boldsymbol{R})$ denotes the space of (scalar-valued) distributions on $(-\infty, \infty)$. $\mathcal{D}'_+(\boldsymbol{R})$ denotes the space of distributions with support bounded on the left. $\mathcal{E}'(\boldsymbol{R}^-)$ is the subspace of $\mathcal{D}'_+(\boldsymbol{R})$ (and of $\mathcal{D}'(\boldsymbol{R})$, of course) consisting of distributions having compact support contained in $(-\infty, 0]$. These spaces are each equipped with the standard topology based on duality. (For details, see SCHWARTZ [1966] or YAMAMOTO [1981a].) For any $\theta \in \mathcal{D}'_+$, $\ell(\theta)$ denotes the greatest lower bound of the support of θ, i. e.,

(2.3) $\ell(\theta) := \inf \{t : t \in \operatorname{supp} \theta\}$.

We need two notions of truncation mappings. Let $\mathcal{D}(\boldsymbol{R}^+)$ denote the space of C^∞-functions on $(-\infty, \infty)$ having compact support in $[0, \infty)$. Similarly, $\mathcal{D}[0, a]$ $(a>0)$ denotes the space of C^∞-functions having compact support in $[0, a]$. $\mathcal{D}(\boldsymbol{R})$ denotes, of course, the space of C^∞-function on $(-\infty, \infty)$, having compact support. Each space is endowed with the standard topology introduced by Schwartz. (SCHWARTZ [1966]). Let j and j_a denote the inclusion maps:

(2.4) $j: \mathcal{D}(\boldsymbol{R}^+) \to \mathcal{D}(\boldsymbol{R})$;

$j_a: \mathcal{D}[0, a] \to \mathcal{D}(\boldsymbol{R})$.

The desired truncations π and π_a are then defined as follows:

(2.5) $\pi: \mathcal{D}'(\boldsymbol{R}) \to \mathcal{D}'(\boldsymbol{R}^+)(=(\mathcal{D}(\boldsymbol{R}^+))') : \langle \pi \Lambda, \varphi \rangle := \langle \Lambda, j\varphi \rangle$;

$\pi_a: \mathcal{D}'(\boldsymbol{R}) \to \mathcal{D}'[0, a](=(\mathcal{D}[0, a])') : \langle \pi_a \Lambda, \varphi \rangle := \langle \Lambda, j_a \varphi \rangle$.

Note that the inclusions (2.4) are easily shown to be topological isomorphisms into $\mathcal{D}'(\boldsymbol{R})$. Hence the projections (truncations) (2.5) are surjective.

If π and π_a are applied to functions, then they clearly agree with the usual truncations:

$$\pi\varphi = \varphi|_{[0, \infty]}, \quad \pi_a \varphi = \varphi|_{[0, a]}.$$

We are now ready to give the definition of our input/output maps.

(2.6) **Definition.** Suppose that A is a $p \times m$ matrix whose entries are functions belonging to $L_{loc}^2[0, \infty)$. Then, the *constant linear input/output map associated with* A, denoted by f_A, is given by

$$f_A(u) := \pi(A * u), \quad u \in \Omega.$$

A is called the *impulse response* or *weighting pattern* of f_A.

For any such A, f_A gives a continuous linear map of Ω into Γ. Furthermore, f_A commutes with the shifts defined in (2.2), and hence the term "constant". (For details, see YAMAMOTO [1981b].)

(2.7) **Remark.** The input/output map defined above is a little more restricted than the one considered by YAMAMOTO [1981b]. It is also possible to consider an impulse response which is not necessarily a function but only a measure.

We employ the following simplified definition of systems.

(2.8) **Definition.** A *constant linear* (*continuous-time*) *system* is a 4-tuple $\Sigma = (x, \Phi, g, h)$ such that

1) the *state space* X is a complete locally convex space;
2) g: $\Omega \to X$, h: $X \to \Gamma$ are continuous linear maps;
3) $\{\Phi(t)\}_{t \geq 0}$ is a strongly continuous semigroup in X;
4) $g\sigma_t = \Phi(t)g$, $h\Phi(t) = \tilde{\sigma}_t h$ for all $t \geq 0$.

We understand that the *state-transition* is given by

(2.9) $\phi(t, x, u) := \Phi(t)x + g\sigma_t u)$

where $\phi(t, x, u)$ denotes the state at time $t \geq 0$ for a given initial state x at $t=0$ and input $u \in L^2[0, t]$, $(\sigma_t u)(s) := u(s+t)$. The linear map h gives the correspondence: initial states $\mapsto$ future outputs under the assumption that the input is identically 0 during the observation. The instantaneous readout map H may be defined by

(2.10) $Hx := h(x)(0)$

for those x such that $h(x)$ is continuous at 0. Note that this map H is often discontinuous for infinite-dimensional systems and cannot be defined on the whole of X.

Keeping in mind that X must always be equipped with a semigroup Φ, we often abbreviate (X, Φ, g, h) as (X, g, h). Note that for a finite-dimensional system (F, G, H), above g and h must be defined as

$$g(\omega) := \int_{-\infty}^{0} \exp(-Ft)G\omega(t)dt;$$

$$h(x)(t) := \mathrm{H}\exp\,(Ft)x.$$

(2.11) **Definition.** A constant linear system $\Sigma = (X, g, h)$ is *quasi-reachable* if g has dense image. It is *observable* if h is one-to-one. It is *topologically observable* if there exists a continuous inverse

h^{-1}: im $h \to X$.

It is *canonical* if it is both quassi-reachable and topologically observable. It is *topologically observable in bounded time* $T > 0$ if

$\pi_T \circ h$: $X \to \pi_T(\text{im } h) \subset L^2[0, T]$

admits a continuous inverse.

Given an input/output map f, we say that a system $\Sigma = (X, g, h)$ is a *realization* of f if the following diagram commutes:

$$\begin{tikzcd} \Omega \arrow[rr, "f"] \arrow[dr, "g"'] & & \Gamma \\ & X \arrow[ur, "h"'] & \end{tikzcd}$$

i.e., (X, g, h) is a "factorization" of the input/output map f.

Given an input/output map f, we can always find at least one factorization as follows:

$$\begin{tikzcd} \Omega \arrow[rr, "f"] \arrow[dr, "f"'] & & \Gamma \\ & \overline{\mathrm{im}\ f} \arrow[ur, "j"'] & \end{tikzcd}$$

The closure $\overline{\mathrm{im}\, f}$ is of course taken in Γ. The map j is the inclusion.

Since $\overline{\mathrm{im}\, f}$ is naturally equipped with a semigroup which is the restriction of $\{\sigma_t\}_{t\geq 0}$ in Γ, it is trivial to check that $(\overline{\mathrm{im}\, f}, f, j)$ is indeed a system. Furthermore, it is also canonical because f as above clearly has dense image in $\overline{\mathrm{im}\, f}$ and j is a topological isomorphism into Γ. It is also known that this is the *unique* canonical realization of f (YAMAMOTO [1981b]). We denote this canonical realization by Σ_f.

Actually, we can derive a differential equation description for Σ_f. Indeed, define

(2.12) $Fx:=dx/dt$, for $x\in\overline{\mathrm{im}\, f}\cap(H_{loc}{}^1[0, \infty))^p$,

$(H_{loc}{}^1[0, \infty)=\{\gamma\in L_{loc}{}^2[0, \infty): d\gamma/dt\in L_{loc}{}^2[0, \infty)\})$;

$G_i:=A_i(=i-$th column of $A)$;

$Hx:=x(0)$, $x\in\overline{\mathrm{im}\, f}\cap(C[0, \infty))^p$.

With these operators, system Σ_f is described by the following functional differential equation:

(2.13) $$(d/dt)x_t(\theta)=(d/d\theta)x_t(\theta)+\sum_{i=1}^{m}G_iu_i(t),$$

$$y(t)=Hx_t(\theta)=x_t(0).$$

where x_t is the state$\in\overline{\mathrm{im}\, f}$ (function of θ) at time t.

Our objective in the sequel is to give a concrete representation for (2.12) and (2.13).

3. Fuhrmann-type Realization for Input/Output Maps of Bounded Type

We have seen in the previous section that any input/output map f admits a unique canonical realization, which in turn can be represented by the differential equation description (2.13). Of course, this realization is highly abstract. What is then crucial to concrete realization is a concrete representation of the canonical state space $\overline{\mathrm{im}\, f}$ as the dimension of the canonical state space is crucial to the computation of canonical realizations in the finite-dimensional case.

Recall that $\overline{\mathrm{im}\, f}$ is merely a Fréchet space as a closed subspace of a Fréchet space Γ. It is in general hard to compute it. What kind of property can we impose on $\overline{\mathrm{im}\, f}$ (aside from the one $\overline{\mathrm{im}\, f}$ being finite-dimensional) so that it admits a nice representation?

An obvious mathematical choice is to impose $\overline{\mathrm{im}\, f}$ to be a Banach (or Hilbert) space. Another system theoretic one would be to require that $\overline{\mathrm{im}\, f}$ be completely determined by the partial data on a bounded interval $[0, T]$; then we need not consider the problem of determining the space which depends on the data on the infinite time interval $[0, \infty)$. Interestingly enough, these two directions entirely coincide. (YAMAMOTO [1982a, b]).

(3.1) **Definition.** A constant linear input/output map f (or its impulse response A) is of T-*bounded type* (or, simply, T-*bounded*) if its canonical realization Σ_f is topologically observable in bounded time T, i.e.,

$$\pi_T|_{\overline{\mathrm{im}\, f}}: \overline{\mathrm{im}\, f} \to \pi_T(\overline{\mathrm{im}\, f})$$

is continuously invertible.

A sufficient condition for an input/output map f to be T-bounded is given by YAMAMOTO [1982b, 1983]. We quote the following theorem:

(3.2) **Theorem.** *Let* A *be a* p×m *impulse response matirx. Suppose that there exists a distribution* β *such that*

1) *there exists a convolution inverse* $\beta^{-1} \in \mathcal{D}'_+(\boldsymbol{R})$;
2) ord $\beta^{-1} = -$ord β;
3) $\pi(\beta * A) = 0$.

Then A *is* T-*bounded for any* $T > -\ell(\beta)$. [*Here,* ord β *denotes the (global) order of distribution* β.]

Let us see the meaning of the above conditions. First we prepare the following two lemmas.

(3.3) **Lemma.** $\pi\beta = 0$ *if and only if* supp $\beta \subset (-\infty, 0]$.

Proof. Immediate from the definition of π. □

Now suppose $\beta \in \mathcal{E}'(\boldsymbol{R}^-)$ and $\zeta \in \mathcal{D}'_+(\boldsymbol{R})$. Since π is onto, there exists at least one $\tilde{\zeta} \in \mathcal{D}'(\boldsymbol{R})$, such that $\pi\tilde{\zeta} = \zeta$. Then define

(3.4) $\pi(\beta * \zeta) = \pi(\beta * \pi\tilde{\zeta}) := \pi(\beta * \tilde{\zeta})$.

This is well defined. Indeed, we have

(3.5) **Lemma.** *Let* $\beta \in \mathcal{E}'(\boldsymbol{R}^-)$ *and* $\zeta_1, \zeta_2 \in \mathcal{D}'(\boldsymbol{R})$. *Suppose* $\pi\zeta_1 = \pi\zeta_2$. *Then,* $\pi(\beta * \zeta_1) = \pi(\beta * \zeta_2)$ $(= \pi(\beta * \pi\zeta_i))$.

Proof. By Lemma (3.3), supp $(\zeta_1 - \zeta_2) \subset (-\infty, 0]$. Hence, supp $(\beta * (\zeta_1 - \zeta_2)) \subset (-\infty, 0]$. Again by Lemma (3.3), the conclusion follows. □

Now suppose that an impulse response A satisfies the conditions of Theorem (3.2). By condition 3), $\pi(\beta * A) = 0$. According to Lemma (3.3), this means that the support of each entry of $\beta * A$ is contained in $(-\infty, 0]$. Since the support of $\beta * A$ is obviously bounded on the left, it follows that $\beta * a_{ij}$ belongs to $\mathcal{E}'(\boldsymbol{R}^-)$. ($a_{ij}$ $= i-j$ entry of A). Writing $\beta * a_{ij} =: \alpha_{ij}$, we have

(3.6) $a_{ij} = \beta^{-1} * \alpha_{ij}, \quad \alpha_{ij} \in \mathcal{E}'(\boldsymbol{R}^-)$

Conversely, if each entry of A is of the form (3.6) for some $\alpha_{ij} \in \mathcal{E}'(\boldsymbol{R}^-)$ and β, that satisfies the conditions of Theorem (3. 2), A is T-bounded for any T greater than $-\ell(\beta)$.

In view of the above observations, we now give the following definition.

(3. 7) **Definition.** An impulse response matrix A (or its associated input/output map f_A) is *pseudo-rational* if it can be written as

(3. 8) $A=Q^{-1}*p$

for some $p\times p$ matrix $Q=(q_{ij})$ and $p\times m$ matrix $P=(p_{ij})$ such that

1) each q_{ij}, $p_{ij}\in\mathcal{E}'(\boldsymbol{R}^-)$;
2) Q is investible over $\mathcal{D}'_+(\boldsymbol{R})$ with respect to convolution, (i. e., there exists $(\det Q)^{-1}\in\mathcal{D}'_+(\boldsymbol{R})$); and
3) $\mathrm{ord}\,(\det Q)^{-1}=-\mathrm{ord}\,(\det Q)$.

(3. 9) **Remark.** Note that $\pi A=A$. Hence (3. 8) may be rewritten as $A=\pi(Q^{-1}*P)$.

If A is pseudo-rational, then

$$\begin{aligned}\pi((\det Q)*A)&=\pi((\det Q)*(Q^{-1}*P))\\&=\pi((\det Q)*((\det Q)^{-1}*(\mathrm{adj}\ Q)*P)\\&=\pi((\mathrm{adj}\ Q)*P)=0\end{aligned}$$

where adj Q denotes the adjoint (cofactor) matrix. Hence A is of bounded type in T greater than $-\ell(\det Q)$.

(3. 10) **Remark.** It is known (KAMEN [1975], YAMAMOTO [1982b]) that impulse responses of delay-differential systems are pseudo-rational. The input/output maps of finite-dimensional systems are, of course, pseudo rational. Another example is given by an impulse response function which is periodic. Though this class is certainly not too general, these examples suggest that it be of some theoretical interest for further study, especially since it enables us to study delay-differential systems in a unified general setting.

To state and prove our main result in this section, we need one more notion.

(3. 11) **Definition.** Let (Q, P) be a pair which satisfies the conditions of Definition (3. 7). The pair (Q, P) is called *left coprime* if there exist matrices R and S of suitable sizes with entries in $\mathcal{E}'(\boldsymbol{R}^-)$ such that

(3. 12) $Q*R+P*S=\delta I_p$

where I_p is the identity matrix of size p, and δ is the Dirac distribution at 0. In other literature on systems over rings, the condition (3. 12) is known as the Bezout identity.

It is of course well known that any finite-dimensional input/output maps admit left coprime representations. Some input/output maps of delay-differential systems also admit left coprime factorizations. Under what conditions these impulse responses admit left coprime factorizations does not seem to be totally known since our ring here is $\mathcal{E}'(\boldsymbol{R}^-)$-much larger than the polynomial rings in Dirac distributions, which are usually employed in the study of delay-differential systems. (For some related studies, see KHARGONEKAR [1982] and VIDYASAGAR et al. [1982].)

(3.13) **Theorem.** *Suppose that A is an impulse response matrix with the associated input/output map f. Suppose that A is pseudo-rational with the representation $A=Q^{-1}*P$. Then*

$$\overline{\mathrm{im}\, f}\subset X^Q:=\{\gamma\in\Gamma:\ \pi(Q*\gamma)=0\}.$$

If, further, (Q, P) is left coprime, $\overline{\mathrm{im}\, f}=X^Q$.

Proof. Take any ω in Ω. Then we have

$$\begin{aligned}\pi(Q*f(\omega))&=\pi(Q*\pi(Q^{-1}*P*\omega))=\pi(Q*Q^{-1}*P*\omega)\quad \text{(by Lemma (3.5))}\\ &=\pi(P*\omega)=0\end{aligned}$$

by Lema (3.3). Hence $\mathrm{im}\, f$ is contained in X^Q. Note that X^Q is a closed subspace of Γ due to the separate continuity of convolution and the continuity of π. Hence $\overline{\mathrm{im}\, f}$ is contained in X^Q.

Now suppose that (Q, P) satisfies (3.12). First, suppose that γ belongs to X^Q and each entry of γ is C^∞. Let $\hat{\gamma}$ be any C^∞ extension of γ to $(-\infty, \infty)$ such that supp $\hat{\gamma}$ is bounded on the left. Since each entry of γ belongs to $C^\infty[0, \infty)$, such an extension obviously exists. Then let $\omega:=S*Q*\hat{\gamma}$. Since $\hat{\gamma}$ is C^∞, so is ω. Also, since the supports of S, Q and $\hat{\gamma}$ are bounded on the left, the support of ω is also bounded on the left. Furthermore,

$$\pi\omega=\pi(S*Q*\hat{\gamma})=\pi(S*\pi(Q*\hat{\gamma})=0$$

by Lemma (3.5), and hence supp ω is contained in $(-\infty, 0]$ by Lemma (3.3). Therefore, ω belongs to Ω. Then we have

$$\begin{aligned}\pi(Q^{-1}*P*\omega)&=\pi(Q^{-1}*P*S*Q*\hat{\gamma})=\pi(Q^{-1}*(\delta I_p-Q*R)*Q*\hat{\gamma})\\ &=\pi(\hat{\gamma}-R*Q*\hat{\gamma})\\ &=\pi\hat{\gamma}-\pi(R*\pi(Q*\hat{\gamma}))=\gamma\end{aligned}$$

Hence, ω belongs to im f. If we prove $X^Q\cap(C[0, \infty))^p$ is dence in X^Q, it completes the proof.

Take a family of C^∞ functions (on $(-\infty, \infty)$ ρ_ε such that

1) supp ρ_ε is compact and contained in $(-\infty, 0]$;

2) $\rho_\varepsilon\to\delta$ as $\varepsilon\to 0$.

Take any γ in X^Q, and let $\gamma_\varepsilon:=\pi(\rho_\varepsilon*\gamma)$. It is well known (SCHWARTZ [1966, Chapter 6, Section 4]) that $\rho_\varepsilon\to\gamma$ in $L_{loc}^2(-\infty, \infty)$ and each $\rho_\varepsilon*\gamma\in C^\infty(\boldsymbol{R})$. Hence γ_ε belongs to $(C^\infty[0, \infty))^p$ and converges to γ in Γ. It remains only to prove that γ_ε belongs to X^Q. However, since supp ρ_ε is contained in $(-\infty, 0]$, we have

$$\pi(Q*\gamma_\epsilon)=\pi(Q*\pi(\rho_\epsilon*\gamma))=\pi(Q*\rho_\epsilon*\gamma)=\pi(\rho_\epsilon*Q*\gamma)$$
$$=\pi(\rho_\epsilon*\pi(Q*\gamma))=0$$

by Lemma (3.5). □

(3.14) **Remark.** The dual result of Theorem (3.13) in a different setting, using spaces of distributions as input and output spaces, was first given by KAMEN [1976]. Later it was extended to the multivariable case of delay-differential systems by DENHAM and YAMASHITA [1979]. However, the result in the latter does not seem to depend on the coprimeness of the fractional resresentation as (3.8), hence does not seem to be entirely right.

Having established the representation Theorem (3.13), we can apply (2.12) to obtain a differential equation description for a pseudo-rational input/output map.

(3.15) **Theorem.** *Let A be an impulse response matrix with the associated input/output map f. Suppose that A is pseudo-rational with the representation $A=Q^{-1}*P$. Then the fellowing system Σ^Q is a topologically observable realization of f.*

1) *State space* $=X^Q$;

2) *State transition*:

(3.16) $\frac{d}{dt}x_t(\theta)=\frac{\partial}{\partial\theta}x_t(\theta)+Au(t),\ \ x_t(\theta)\in X^Q\cap(H_{loc}{}^1[0,\ \infty))^p$;

3) *Output equation*:

(3.17) $y(t)=x_t(0)$.

If the pair $(Q,\ P)$ is left coprime, the above system is canonical.

Proof. The last statement follows from the first half and Theorem (3.13). So we need only to prove the first half.

We must give the semigroup, reachability map g, and the observability map h of Σ^Q. The semigroup genrerated by $(\partial/\partial\theta)$ is the shift operator $\tilde{\sigma}_\theta$ restricted to X^Q. This can be easily checked by computing the infinitesimal generator of $\tilde{\sigma}_\theta|X^Q$. Then the solution of (3.16) is given by

$$x_t(\cdot)=\tilde{\sigma}_t x_0(\cdot)+\int_0^t(\tilde{\sigma}_{t-\tau}A)\ \cdot))u(\tau)d\tau$$
$$=x_0(\cdot+t)+\int_0^t A(\cdot+t-\tau)u(\tau)d\tau$$

Hence the reachability map g is given by

$$(g(\omega)(\cdot)=\int_{-\infty}^0\tilde{\sigma}_{-\tau}(A(\cdot)\omega(\cdot)d\tau=\int_{-\infty}^0 A(\cdot-\tau)\omega(\tau)d\tau$$

The observability map induced by (3.17) is simply the inclusion map j: $X^Q\rightarrow\Gamma$. Therefore, the topological observability of Σ^Q is obvious. It remains only to prove that Σ^Q is a realization. However, we have

$$(jg(\omega))(t)=(g(\omega))(t)$$

$$=\int_{-\infty}^{0} A(t-\tau)\omega(\tau)d\tau$$
$$=f(\omega)(t)$$

This completes the proof. $\square$

In the final section we give some examples to illustrate how to apply this theorem to the realization of delay-differential systems.

4. Some Spectral Properties

In the previous section, we gave a topologically observable realization Σ^Q for a pseudo-rational impulse response $A=Q^{-1}*P$. It is of great theoretical importance to study the spectral properties of this system, especially those of the infinitesimal generator $F=(d/dt)$.

Let β be a distribution which is Laplace transformable. Denote by $\hat{\beta}$ the Laplace transform of β. We shall characterize $\sigma(F)$-the spectrum of F. Indeed, we shall prove

1) $\sigma(F)=\sigma_p(F)$, *i.e.*, every point in $\sigma(F)$ is an eigenvalue,
2) $\sigma(F)=\{\lambda\in \boldsymbol{C}: (\det Q)^\wedge(\lambda)=0\}$.

For simplicity, we assume, as opposed to other sections, that the systems and functions (distributions) are defined over $\boldsymbol{C}$.

Let $A=Q^{-1}*P$ be pseudo-rational, and let Σ^Q be the system given in Theorem (3.15). A complex number λ belongs to the resolvent set $\rho(F)$, if and only if for any $y(t)\in X^Q$ there exists a unique x in $X^Q\cap(H_{loc}{}^1[0, \infty))^p$ such that

(4.1) $\left(\lambda-\dfrac{d}{dt}\right)x(t)=y(t)$ for almost all $t\geqq 0$

Solving (4.1) in Γ, we have

(4.2) $x(t)=e^{\lambda t}x(0)-\int_0^t e^{\lambda(t-\tau)}y(\tau)d\tau$

If λ belongs to $\rho(F)$, $y=0$ must imply $x(0)=0$. Hence we have

(4.3) **Lemma.** *If a complex number λ belongs to $\rho(F)$, then the following statement holds*:

(4.4) $e^{\lambda t}v\in X^Q \Rightarrow v=0 \quad (v\in \boldsymbol{C}^p)$

Proof. Since $x(t)$ given by (4.2) always belongs to $(H_{loc}{}^1[0, \infty))^p$, λ belongs to the resolvent set only when no nonzero $e^{\lambda t}v$ belongs to X^Q. Hence the assertion follows. $\square$

We need the following lemma from distribution theory.

(4.5) **Lemma.** (Paley-Wiener-Schwartz Theorem for Laplace Transforms-A Special Case) *A distribution β has compact support contained in $[-n, 0]$ if and only if $\hat{\beta}$ is an entire function such that for some $C\geqq 0$ and a positive integer k*

(4.6) $|\hat{\beta}(s)| \leqq C(1+|s|)^k \exp(n \mathrm{Re}\, s)$ *if Re* $s \geqq 0$,

$\leqq C(1+|s|)^k$ *if Re* $s<0$.

Proof. Omitted. (See Kaneko [1976].) □

We can now prove the following proposition.

(4.7) **Proposition.** *A function* $e^{\lambda t}v$ *belongs to* X^Q *for some nonzero* $v \in C^p$ *if and only if* $(\det Q)^\wedge(\lambda)=0$.

Proof. Recall that $e^{\lambda t}v$ belongs to X^Q iff $\pi(Q*e^{\lambda t}v)=0$, i.e., the support of $Q*e^{\lambda t}v$ is contained in $[-n, 0]$ for some n. Then $L(Q*e^{\lambda t}v)$ is an entire function by the previous lemma. However, we have

(4.8) $$L(Q*e^{\lambda t}v)=\frac{\hat{Q}(s)v}{s-\lambda}$$

Since $\hat{Q}(s)$ is entire, expand $(\hat{Q}(s)$ in the powers of $(s-\lambda)$. Then the constant term is $\hat{Q}(\lambda)$. The right-hand side of (4.6) is entire iff $\hat{Q}(\lambda)v=0$. Since $v \neq 0$, $(\det Q)^\wedge(\lambda)=0$.

Conversely, suppose that $(\det Q)^\wedge(\lambda)=0$. Then by the same argument as above, $L(Q*e^{\lambda t}v)$ is an entire function for some $v \neq 0$. Since each entry of $\hat{Q}(s)$ satisfies the estimate (4.6) for some n, so does each entry of $\hat{Q}(s)v$. Since $1/|s-\lambda|$ is bounded for large enough s, $|\hat{Q}(s)v/(s-\lambda)|$ satisfies the same type of estimate as (4.6). Also, since $Q(s)v/(s-\lambda)$ is entire, it is bounded in a neighborhood of λ. Hence $\hat{Q}(s)v/(s-\lambda)$ satisfies the same type of estimate as (4.6). This implies $\pi(Q*e^{\lambda t}v)=0$, i.e., $e^{\lambda t}v$ belongs to X^Q. □

(4.9) **Remark.** In the above proof $e^{\lambda t}v$ must be considered to be $Y(t)e^{\lambda t}v$ when forming the convolution with Q. Here $Y(t)$ denotes the Heaviside unit step function.

(4.10) **Corollary.** *A complex number* λ *is an eigenvalue of* F *if and only if* $(\det Q)^\wedge(\lambda)=0$.

Proof. Suppose that $(\det Q)^\wedge(\lambda)=0$. Then by Proposition (4.7), there exists a nonzero $v \in C^p$ such that $e^{\lambda t}v$ belongs to X^Q. Since $e^{\lambda t}v$ clearly belongs to $(H_{loc}^1[0, \infty))^p$, it is an eigenfunction corresponding to λ.

Conversely, suppose that $x(t)$ is an eigenfunction corresponding to λ. By (4.2) $x(t)$ must be equal to $e^{\lambda t}v$ for some nonzero $v \in C^p$. Again by the previous Proposition (4.7), $(\det Q)^\wedge(\lambda)=0$. □

Finally, we prove that

$$\rho(F)=\{\lambda \in C: (\det Q)^\wedge(\lambda) \neq 0\}$$

This proves simultaneously that

1) $\sigma(F)=\{\lambda \in C: (\det Q)^\wedge(\lambda)=0\}$; and
2) $\sigma(F)=\sigma_p(F)$.

(4.11) **Theorem.** *The resolvent set* $\rho(F)$ *is given by*

$\rho(F)=\{\lambda\in C\colon (\det Q)^\wedge(\lambda)\neq 0\}$.

Proof. Lemma (4.3) and Proposition (4.7) have already shown that if λ belongs to $\rho(F)$ then $(\det Q)^\wedge(\lambda)$ must be nonzero. Hence we need only to show that $(\det Q)^\wedge(\lambda)\neq 0$ implies $\lambda\in\rho(F)$.

Suppose $(\det Q)^\wedge(\lambda)\neq 0$, and take any $y\in X^Q$. Then the solution of (4.1) with the initial value $x(0)$ is given by (4.2). The function $x(t)$ in (4.2) always belongs to $(H_{loc}{}^1[0, \infty))^p$. Hence it suffices to prove that (4.2) belongs to X^Q for one and only one choice of $x(0)$. Note that $x(t)$ is written as

$$x(t)=e^{\lambda t}\Big((x(0)-\int_0^t e^{-\lambda\tau}y(\tau)dt\Big).$$

This function belongs to X^Q iff

(4.12) $\pi(Q*(Y(t)e^{\lambda t}x(0)-(Y(t)e^{\lambda t})*(Y(t)y)))=0$.

Taking the Laplace transform of the left-hand side, we have the expression:

(4.13) $\hat{Q}(s)\dfrac{x(0)-\hat{y}(s)}{s-\lambda}=:\ \hat{Q}_1(s)$.

This function $\hat{Q}_1(s)$ is an entire function iff $(s-\lambda)$ divides $x(0)-\hat{y}(s)$, i.e., $x(0)$ must be equal to $\hat{y}(\lambda)$. [This is the only possible choice, because $(\det Q)^\wedge(\lambda)\neq 0$.] So let $x(0)=\hat{y}(\lambda)$. In order to show (4.12), we need only to prove that each entry of $\hat{Q}_1(s)$ satisfies the estimate of type (4.6). To see this, first note that y belongs to X^Q. Hence $\hat{Q}(s)\hat{y}(s)$ satisfies the estimate of type (4.6). Also, $Q(s)x(0)$ satisfies the estimate of type (4.6) since each entry of $\hat{Q}(s)$ already satisfies the same type of estimate. Since the set of all functions that satisfy the estimate of type (4.6) (for some k, n, C) constitutes a vector space, $\hat{Q}(s)(x(0)-\hat{y}(s))$ satisfies the same type of estimate. Since $1/|s-\lambda|$ is bounded for large enough s, $\hat{Q}_1(s)$ satisfies the same type of estimate. But since $\hat{Q}_1(s)$ is entire, it is bounded in a neighborhood of λ. Hence $\hat{Q}_1(s)$ satisfies the estimate of type (4.6). Therefore, the inverse Laplace transform has compact support contained in $(-\infty, 0]$. and this proves the theorem. □

(4.14) **Remark.** We took for granted that $y(t)$ is Laplace transformable. Proving this is not difficult, but rather messy to give here. The key idea is to note that X^Q is isomorphic to a closed subspace of $(L^2[0, T])^p$ for some $T>0$. Then the semigroup in X^Q, $\tilde{\sigma}_t$, has only exponettial growth. These two facts imply that y is Laplace transformable.

(4.15) **Corollary.** $\sigma(F)=\sigma_p(F)$

Proof. Obvious from Corollary (4.10) and Theorem (4.11). □

The above type of theorem has been obtained for delay-differential systems, (see, e.g., HALE [1977]). What is interesting here is, however, that we started with a "universal model" $\overline{\mathrm{im}\, f}$ and then derived these interesting spectral properties.

5. Examples

We begin by examining the transfer function given by (1.1).

(5.1) **Example.** Let $W(s)$ be the transfer function:

(5.2) $W(s)=1/(se^s-1)$

Then the impulse response $A(t)$ is given by

(5.3) $$A(t)=\begin{cases}0 & \text{for } 0\leq t<1,\\ \sum_{i=1}^{n-1}(t-i-1)^i/i! & \text{for } n\leq t<n+1\end{cases}$$

By (5.2), we have

(5.4) $A=(\delta'_{-1}-\delta)^{-1}*\delta=:\ \beta^{-1}*\alpha.$

It is easy to check that A is pseudo-rational (YAMAMOTO [1982b]), and the factorization (5.4) is of course (left) coprime. Hence Theorem (3.13) is applicable. Writing down the equation $\pi(\beta*\gamma)=0$ for a smooth γ, we have

(5.5) $\gamma'(t+1)-\gamma(t)=0$ for all $t\geq 0$.

In other words,

(5.6) $$\gamma(t)=\gamma(1)+\int_1^t\gamma(\tau-1)d\tau$$

for $1\leq t<2$. Iterating this formula successively, we see that the values of $\gamma(t)$ for $0\leq t<1$ and $\gamma(1)$ completely determine the values of $\gamma(t)$ for all $t\geq 0$, provided that γ is smooth and belongs to $\overline{\mathrm{im}\, f}$.

Taking the closure of all such γ's in $\Gamma\ (=L_{loc}^2[0,\ \infty))$, we can obtain $\overline{\mathrm{im}\, f}$. Note that $\gamma|_{[0,1]}$ and $\gamma(1)$ can be assigned arbitrarily without breaking rule (5.5) (in the sense of distributions). Therefore, we have

(5.7) $\overline{\mathrm{im}\, f}\cong L^2[0,\ 1]\times \boldsymbol{R}$.

Denote an element of $L^2[0,\ 1]\times\boldsymbol{R}$ by $(z(\theta),\ x)$ instead of $(\gamma(t),\ \gamma(1))$. Let us now apply Theorem (3.15) to obtain the differential equation description $(F,\ G,\ H)$. If we shift $(z(\theta),\ x)$ by ε to the left, then we have

(5.8) $$\tilde{\sigma}_\varepsilon(z,\ x)=\begin{cases}\left(z(\theta+\varepsilon),\ x+\int_0^\varepsilon z(\tau)d\tau\right) & \text{for } \theta\leq 1-\varepsilon\\ \left(x+\int_0^{\theta-1+\varepsilon}z(\tau)dt,\ x+\int_0^\varepsilon z(\tau)dt\right) & \text{for } 1-\varepsilon<\theta\leq 1\end{cases}$$

by using (5.6). Computing the limit of $(1/\varepsilon)(\hat{\sigma}_\varepsilon(z,\ x)-(z,\ x))$ as $\varepsilon\to 0$, we have

(5.9) $F(z,\ x)=((\partial/\partial\theta)z,\ z(0)),\ z\in H^1[0,\ 1]$ and $z(1)=x$.

Also, $G=(0,\ 1)$ since $A(t)=0$ for $t<1$ and $A(1)=1$. $H(z,\ x)=z(0)$ is readily obvious from definition.

Summarizing, we have obtained the following functional differential equation:

$$(5.10)\quad \frac{d}{dt}\begin{pmatrix} z_t \\ x_t \end{pmatrix} = \begin{pmatrix} (\partial/\partial\theta) z_t(\theta) \\ z_t(0) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$
$$y(t) = z_t(0),$$

which is nothing but the M_2-space model for such a retarded delay-differential system (See, e.g., DELFOUR and MITTER [1972], MANITIUS and TRIGGIANI [1978]).

(5.11) **Remark.** It should be noted that the derivation of the state space representation (5.7) is different from the standard procedure. Usually it is posed *a priori* as a starting point whereas it is a consequence of the canonical construction of Theorem (3.13) in the present context. Furthermore, the resulting system (5.10) is canonical without further verifications.

We now consider a multivariable case in the next example.

(5.12) **Example.** We consider the case $m=1$ and $p=2$. Let $W(s)$ be the following transfer function matrix:

$$(5.13)\quad W(s) = \begin{pmatrix} \dfrac{1}{e^s(e-1)} \\ \dfrac{1}{e^s(s-1)(se^s-1)} \end{pmatrix}$$

The corresponding impulse response matrix $A=(A_1, A_2)'$ for $0 \leqq t \leqq 2$ is given by

$$(5.14)\quad A_1(t) = \begin{cases} 0 & \text{for } 0 \leqq t < 1 \\ t & \text{for } 1 \leqq t \leqq 2; \text{ and} \end{cases}$$
$$A_2(t) = 0 \quad \text{for } 0 \leqq t \leqq 2.$$

It is easy to find a factorization of $W(s)$:

$$(5.15)\quad W(s) = \begin{pmatrix} s\lambda-\lambda & 0 \\ -1 & s\lambda-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: Q^{-1}P$$

where λ denotes e^s. Now take R and S as follows:

$$(5.16)\quad R := \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad S := [1 \ s\lambda-\lambda].$$

It is easy to check $QR+PS=I$. [To find R and S, consider the matrix $[Q, P]$ and transfer this matrix to $[I, 0]$ by fundamental column operations.] Hence $L^{-1}[Q]$ and $L^{-1}[P]$ are left coprime. For brevity of notation, also denote the inverse Laplace transforms of Q and P by Q and P, too.

It is readly seen that A is pseudo-rational. Hence we can apply Theorem (3.13). For a smooth $\gamma=(\gamma_1, \gamma_2)'$, the equation $\pi(Q^*\gamma)=0$ becomes

$$(5.17)\quad \gamma_1(t+1)-\gamma_1(t+1)=0 \quad \text{for all } t \geqq 0,$$

$$(5.18)\quad \gamma_2(t+1)-\gamma_2(t)-\gamma_1(t)=0 \quad \text{for all } t \geqq 0.$$

Solving (5.17), we have

$$(5.19)\quad \gamma_1(t) = e^t\gamma_1(1) \quad \text{for all } t \geqq 1.$$

Hence $\gamma_1(t)$ for $t \geqq 1$ is completely determined by specifying the value $\gamma_1(1)$. On the other hand $\gamma_1|_{[0,1]}$ can be arbitrarily chosen without violating rule (5.17). Hence $(\gamma_1|_{[0,1]}, \gamma_1(1))$ completely determines the values of $\gamma_1(t)$ for all $t \geqq 0$. Once $\gamma_1(t)$ is determined, it is easy to solve (5.18) as follows:

(5.20) $\gamma_2(t) = \gamma_2(1) + \int_1^t \gamma_2(t-1)dt + \int_1^t \gamma_1(\gamma-1)dt$ for $1 \leqq t \leqq 2$.

Iterating this formula, we see that $\gamma_2|_{[0,1]}$ and $\gamma_2(1)$ entirely determine the values of $\gamma_2(t)$ (note that $\gamma_1(t)$ is already known). Hence the pair $(\gamma_1|_{[0,1]}, \gamma_1(1))$ and $(\gamma_2|_{[0,1]}, \gamma_2(1))$ completely determine the values of $\gamma(t)$ for all $t \geqq 0$. Taking the closure of all such pairs in Γ, we have

(5.21) $\overline{\operatorname{im} f} \cong (L^2[0, 1] \times \boldsymbol{R})^2$.

Denote $(\gamma_1|_{[0,1]}, \gamma_2|_{[0,1]}, \gamma_1(1), \gamma_2(1))'$ by (z_1, z_2, x_1, x_2). Proceeding similarly as in Example (5.1), we obtain the following functional differential equation description:

$$(5.22) \quad \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\partial/\partial\theta)z_1(\theta) \\ (\partial/\partial\theta)z_2(\theta) \\ x_1 \\ z_1(0) + z_2(0) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u(t)$$

$$y(t) = (z_1(0), z_2(0))'.$$

[In reading the above equation, note that each z_i is a function of θ at each time t.] Thus, we again arrive at an M_2-space model.

In the following final example, we consider a system described by a neutral delay-differential equation.

(5.23) **Example.** Consider the following impulse response function.

(5.24) $A(t) = n-1$ for $n \leqq t < n+1$.

Its transfer function $W(s)$ is given by

$$W(s) = 1/s(e^s - 1).$$

In other words,

(5.25) $A = (\delta'_{-1} - \delta')^{-1} * \delta =: \beta^{-1} * \alpha$.

This is the impulse response of the system described by the following neutral delay-differential equation:

(5.26) $x'(t) = x'(t-1) + u(t)$

$$y(t) = x(t--1).$$

It is easily seen from (5.25) that A is pseudo-rational, and that the factorization (5.25) is (left) coprime. We apply Theorem (3.13) to the present case. For a smooth γ, the equation $\pi(\beta * \gamma) = 0$ becomes

(5.27) $\gamma'(t+1) - \gamma'(t) = 0$ for all $t \geqq 0$.

Solving (5.27), we have

(5.28) $\gamma(t) = \gamma(1) - \gamma(0) + \gamma(t-1)$ for $1 \leqq t < 2$.

Iterating this formula, we obtain all the values of $\gamma(t)$ for $t \geqq 0$ as long as $\gamma|_{[0,1]}$

and $(\gamma(1)-\gamma(0))$ are known. So denote the pair $(\gamma|_{[0,1]}, \gamma(1)-\gamma(0))$ by (z, x). Taking the closure of all such pairs in Γ, we have

(5.29) $\overline{\mathrm{im}\, f} \cong L^2[0, 1]\times \boldsymbol{R}$.

Note that

$$\gamma(1+\varepsilon)-\gamma(\varepsilon)=\gamma(1)-\gamma(0)$$

in view of (5.28). Hence the second coordinate of (z, x) does not change with time *without input*. Since $A|_{[0,1]}=0$ and $A(1)=1$, $G=(0, 1)$. Hence the following differential equation obtains:

(5.30) $$\frac{d}{dt}\begin{pmatrix} z_t(0) \\ x_t \end{pmatrix}=\begin{pmatrix} (\partial/\partial\theta)z_t(\theta) \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t)$$

$$y(t)=z_t(0).$$

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