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By

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Abstract

The numerical solution obtained by Howarth for the Blasius equation for the laminar boundary layer of incompressible fluid along a flat plate is widely used. The numerical table is allegedly correct up to six significant figures. We show here that the accuracy of the last decimal is doubtful, and present a table which is correct with eight significant figures. This correction does not however, significantly affect the physical parameters which characterize the boundary layer.

1. Introduction

From the Navier-Stokes equation, Prandtl[1)] derived the equations which govern the laminar boundary layer flow. Later, Blasius[2)] solved the problem of uniform flow along a flat plate, and the solution was corrected by Bairstow[3)]. Toepfer[4)] and Goldstein[5)] attempted to obtain a numerical solution by expanding the unknown functions in series. Later, Howarth[6)] obtained a numerical solution with considerable accuracy, and his results are commonly adopted in standard references. (cf. Schlichting[7)], Takano[8)], etc.)

Howarth gives the numerical solution to the basic equation with six significant figures. While dealing with the basic Blasius equation in other contexts, we have come to wonder about the accuracy of the Howarth solution. Although the correction is always in the last figures, we thought it might be worthwhile to see if the numerical table adopted in the standard textbooks should be corrected. It is the object of this paper to present the standard numerical table, which we believe is more accurate than Howarth's.

2. Numerical Solution

The basic equations for the laminar boundary layer flow of incompressible

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fluid along a flat plate are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

with the boundary conditions

$$u=v=0;\quad \text{at}\quad y=0$$

$$u=U_\infty;\quad \text{as}\quad y\to\infty$$

with the standard notation (Batchelor 9). We define a new dimensionless parameter η and a stream-function $\psi(x,y)$ as follows:

$$\eta=y(U_\infty/\nu x)^{1/2},\quad \psi=(\nu x U_\infty)^{1/2}f(\eta)$$

so that,

$$u=\frac{\partial\psi}{\partial y}=U_\infty f'(\eta),\quad v=-\frac{\partial\psi}{\partial x}=\frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{1/2}[f'(\eta)\,\eta-f(\eta)]\,.$$

The velocity components u and v satisfy the continuity equation. Then, the problem is reduced to that of finding the solution $f(\eta)$ to the Blasius equation (cf. Schlichting[7], p. 136).

$$2f'''(\eta)+f''(\eta)\cdot f(\eta)=0 \tag{2.1}$$

with the following boundary conditions

$$f=0,\ \frac{df}{d\eta}=0\quad \text{for}\quad \eta=0, \tag{2.2}$$

$$\frac{df}{d\eta}\to 1\quad \text{as}\quad \eta\to\infty. \tag{2.3}$$

We note that the η used by Howarth is equivalent to the 2η in our notation. Therefore, $f'(\eta)$ and $f''(\eta)$ used by Howarth correspond to $2f'(\eta)$ and $4f''(\eta)$ in our expression, respectively. It is not explicitly stated how Howarth obtained his numerical solution, but one possibility is to start integrating the Equation (2.1) with the initial condition $f''(0)=A$ instead of the boundary condition (2.3), where A is a parameter, and to find a value of A such that the condition (2.3) is satisfied. It is then clear that to find an accurate solution to Eqn. (2.1) is equivalent to that of specifying an accurate value of A such that the condition (2.3) is satisfied.

Table 1. Accurate numerical solution of the Blasius equation

$\eta=y(U_\infty/(\nu x))^{1/2}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.0	0.0	0.0	0.33205734
0.1	$0.16602821\cdot10^{-2}$	$0.33205504\cdot10^{-1}$	0.33204815
0.2	$0.66409998\cdot10^{-2}$	$0.66407793\cdot10^{-1}$	0.33198384
0.3	$0.14941464\cdot10^{-1}$	$0.99598600\cdot10^{-1}$	0.33180935
0.4	$0.26559884\cdot10^{-1}$	0.13276416	0.33146985
0.5	$0.41492820\cdot10^{-1}$	0.16588526	0.33091096
0.6	$0.59734638\cdot10^{-1}$	0.19893726	0.33007913
0.7	$0.81276976\cdot10^{-1}$	0.23189024	0.32892207
0.8	0.10610822	0.26470914	0.32738927
0.9	0.13421301	0.29735396	0.32543263
1.0	0.16557173	0.32978004	0.32300712
1.1	0.20016009	0.36193838	0.32007152
1.2	0.23794872	0.39377611	0.31658919
1.3	0.27890276	0.42523694	0.31252887
1.4	0.32298158	0.45626177	0.30786540
1.5	0.37013854	0.48678930	0.30258051
1.6	0.42032077	0.51675679	0.29666346
1.7	0.47346911	0.54610082	0.29011165
1.8	0.52951804	0.57475815	0.28293102
1.9	0.58839578	0.60266655	0.27513638
2.0	0.65002438	0.62976574	0.26675155
2.1	0.71432003	0.65599827	0.25780926
2.2	0.78119335	0.68131038	0.24835092
2.3	0.85054978	0.70565289	0.23842607
2.4	0.92229014	0.72898194	0.22809176
2.5	0.99631112	0.75125971	0.21741159
2.6	$0.10725060\cdot10$	0.77245503	0.20645463
2.7	$0.11507652\cdot10$	0.79254385	0.19529422
2.8	$0.12309773\cdot10$	0.81150963	0.18400660
2.9	$0.13130294\cdot10$	0.82934352	0.17266941
3.0	$0.13968083\cdot10$	0.84604445	0.16136032
3.1	$0.14822008\cdot10$	0.86161906	0.15015546
3.2	$0.15690950\cdot10$	0.87608146	0.13912806
3.3	$0.16573807\cdot10$	0.88945289	0.12834712
3.4	$0.17469501\cdot10$	0.90176123	0.11787625
3.5	$0.18376986\cdot10$	0.91304039	0.10777264
3.6	$0.19295252\cdot10$	0.92332967	$0.98086279\cdot10^{-1}$
3.7	$0.20223330\cdot10$	0.93267297	$0.88859363\cdot10^{-1}$
3.8	$0.21160298\cdot10$	0.94111801	$0.80125918\cdot10^{-1}$
3.9	$0.22105284\cdot10$	0.94871547	$0.71911671\cdot10^{-1}$
4.0	$0.23057464\cdot10$	0.95551824	$0.64234121\cdot10^{-1}$

Table 1. (Continued)

$\eta=y(U_\infty/(\nu x))^{1/2}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
4.1	0.24016073·10	0.96158051	$0.57102804\cdot10^{-1}$
4.2	0.24980397·10	0.96695708	$0.50519747\cdot10^{-1}$
4.3	0.25949777·10	0.97170258	$0.44480053\cdot10^{-1}$
4.4	0.26923610·10	0.97587084	$0.38972610\cdot10^{-1}$
4.5	0.27901344·10	0.97951430	$0.33980888\cdot10^{-1}$
4.6	0.28882480·10	0.98268351	$0.29483772\cdot10^{-1}$
4.7	0.29866569·10	0.98542672	$0.25456434\cdot10^{-1}$
4.8	0.30853207·10	0.98778953	$0.21871186\cdot10^{-1}$
4.9	0.31842035·10	0.98981470	$0.18698304\cdot10^{-1}$
5.0	0.32832737·10	0.99154191	$0.15906798\cdot10^{-1}$
5.1	0.33825032·10	0.99300772	$0.13465122\cdot10^{-1}$
5.2	0.34818676·10	0.99424554	$0.11341789\cdot10^{-1}$
5.3	0.35813457·10	0.99528566	$0.95059139\cdot10^{-2}$
5.4	0.36809191·10	0.99615531	$0.79276596\cdot10^{-2}$
5.5	0.37805719·10	0.99687883	$0.65785930\cdot10^{-2}$
5.6	0.38802907·10	0.99747778	$0.54319575\cdot10^{-2}$
5.7	0.39800640·10	0.99797114	$0.44628649\cdot10^{-2}$
5.8	0.40798820·10	0.99837550	$0.36484135\cdot10^{-2}$
5.9	0.41797366·10	0.99870528	$0.29677441\cdot10^{-2}$
6.0	0.42796210·10	0.99897288	$0.24020398\cdot10^{-2}$
6.1	0.43795294·10	0.99918895	$0.19344820\cdot10^{-2}$
6.2	0.44794573·10	0.99936255	$0.15501706\cdot10^{-2}$
6.3	0.45794008·10	0.99950132	$0.12360164\cdot10^{-2}$
6.4	0.46793567·10	0.99961171	$0.98061507\cdot10^{-3}$
6.5	0.47793224·10	0.99969908	$0.77410929\cdot10^{-3}$
6.6	0.48792959·10	0.99976788	$0.60804424\cdot10^{-3}$
6.7	0.49792755·10	0.99982179	$0.47522262\cdot10^{-3}$
6.8	0.50792598·10	0.99986383	$0.36956255\cdot10^{-3}$
6.9	0.51792479·10	0.99989644	$0.28596154\cdot10^{-3}$
7.0	0.52792389·10	0.99992161	$0.22016894\cdot10^{-3}$
7.1	0.53792320·10	0.99994095	$0.16866819\cdot10^{-3}$
7.2	0.54792269·10	0.99995573	$0.12856980\cdot10^{-3}$
7.3	0.55792231·10	0.99996696	$0.97515437\cdot10^{-4}$
7.4	0.56792202·10	0.99997547	$0.73592978\cdot10^{-4}$
7.5	0.57792181·10	0.99998187	$0.55262174\cdot10^{-4}$
7.6	0.58792165·10	0.99998666	$0.41290308\cdot10^{-4}$
7.7	0.59792154·10	0.99999024	$0.30697064\cdot10^{-4}$
7.8	0.60792145·10	0.99999289	$0.22707750\cdot10^{-4}$
7.9	0.61792139·10	0.99999485	$0.16713981\cdot10^{-4}$
8.0	0.62792135·10	0.99999628	$0.12240925\cdot10^{-4}$

Table 1. (Continued)

$\eta=y(U_\infty/(\nu x))^{1/2}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
8.1	$0.63792132\cdot 10$	0.99999733	$0.89202520\cdot 10^{-5}$
8.2	$0.64792129\cdot 10$	0.99999810	$0.64679780\cdot 10^{-5}$
8.3	$0.65792128\cdot 10$	0.99999865	$0.46664700\cdot 10^{-5}$
8.4	$0.66792127\cdot 10$	0.99999904	$0.33499394\cdot 10^{-5}$
8.5	$0.67792126\cdot 10$	0.99999933	$0.23928417\cdot 10^{-5}$
8.6	$0.68792125\cdot 10$	0.99999953	$0.17006678\cdot 10^{-5}$
8.7	$0.69792125\cdot 10$	0.99999968	$0.12026896\cdot 10^{-5}$
8.8	$0.70792125\cdot 10$	0.99999978	$0.84628403\cdot 10^{-6}$
8.9	$0.71792124\cdot 10$	0.99999985	$0.59252580\cdot 10^{-6}$
9.0	$0.72792124\cdot 10$	0.99999990	$0.41278786\cdot 10^{-6}$
9.1	$0.73792124\cdot 10$	0.99999993	$0.28613770\cdot 10^{-6}$
9.2	$0.74792124\cdot 10$	0.99999996	$0.19735666\cdot 10^{-6}$
9.3	$0.75792124\cdot 10$	0.99999997	$0.13544314\cdot 10^{-6}$
9.4	$0.76792124\cdot 10$	0.99999998	$0.92489147\cdot 10^{-7}$
9.5	$0.77792124\cdot 10$	0.99999999	$0.62842443\cdot 10^{-7}$
9.6	$0.78792124\cdot 10$	1.0000000	$0.42485807\cdot 10^{-7}$
9.7	$0.79792124\cdot 10$	1.0000000	$0.28580065\cdot 10^{-7}$
9.8	$0.80792124\cdot 10$	1.0000000	$0.19129829\cdot 10^{-7}$
9.9	$0.81792124\cdot 10$	1.0000000	$0.12740529\cdot 10^{-7}$
10.0	$0.82792124\cdot 10$	1.0000000	$0.84429146\cdot 10^{-8}$
10.1	$0.83792124\cdot 10$	1.0000000	$0.55670593\cdot 10^{-8}$
10.2	$0.84792124\cdot 10$	1.0000000	$0.36524798\cdot 10^{-8}$
10.3	$0.85792124\cdot 10$	1.0000000	$0.23843957\cdot 10^{-8}$
10.4	$0.86792124\cdot 10$	1.0000000	$0.15488072\cdot 10^{-8}$
10.5	$0.87792124\cdot 10$	1.0000000	$0.10010250\cdot 10^{-8}$
10.6	$0.88792124\cdot 10$	1.0000000	$0.64375559\cdot 10^{-9}$
10.7	$0.89792124\cdot 10$	1.0000000	$0.41193210\cdot 10^{-9}$
10.8	$0.90792124\cdot 10$	1.0000000	$0.26227614\cdot 10^{-9}$
10.9	$0.91792124\cdot 10$	1.0000000	$0.16615769\cdot 10^{-9}$
11.0	$0.92792124\cdot 10$	1.0000000	$0.10473954\cdot 10^{-9}$
11.1	$0.93792124\cdot 10$	1.0000000	$0.65694554\cdot 10^{-10}$
11.2	$0.94792124\cdot 10$	1.0000000	$0.40999317\cdot 10^{-10}$
11.3	$0.95792124\cdot 10$	1.0000000	$0.25459649\cdot 10^{-10}$
11.4	$0.96792124\cdot 10$	1.0000000	$0.15731014\cdot 10^{-10}$
11.5	$0.97792124\cdot 10$	1.0000000	$0.96714046\cdot 10^{-11}$
11.6	$0.98792124\cdot 10$	1.0000000	$0.59163099\cdot 10^{-11}$
11.7	$0.99792124\cdot 10$	1.0000000	$0.36011467\cdot 10^{-11}$
11.8	$0.10079212\cdot 10^{2}$	1.0000000	$0.21810179\cdot 10^{-11}$
11.9	$0.10179212\cdot 10^{2}$	1.0000000	$0.13143353\cdot 10^{-11}$
12.0	$0.10279212\cdot 10^{2}$	1.0000000	$0.78810058\cdot 10^{-12}$

Table 1. (Continued)

$\eta=y(U_\infty/(\nu x))^{1/2}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
12.1	$0.10379212\cdot10^{2}$	1.0000000	$0.47020328\cdot10^{-12}$
12.2	$0.10479212\cdot10^{2}$	1.0000000	$0.27913750\cdot10^{-12}$
12.3	$0.10579212\cdot10^{2}$	1.0000000	$0.16488427\cdot10^{-12}$
12.4	$0.10679212\cdot10^{2}$	1.0000000	$0.96910051\cdot10^{-13}$
12.5	$0.10779212\cdot10^{2}$	1.0000000	$0.56674400\cdot10^{-13}$
15.0	$0.13279212\cdot10^{2}$	1.0000000	$0.16715186\cdot10^{-19}$
20.0	$0.18279213\cdot10^{2}$	1.0000000	$0.12332438\cdot10^{-36}$

By adopting the series expansion near $\eta=0$ and the asymptotic expansion as $\eta\to\infty$, Toepher found that $f''(0)=A=0.33206$ (in fact, $4f''(0)=1.32824$), and Howarth determined the numerical solutions $f(\eta), f'(\eta)$ and $f''(\eta)$ using this value. We integrated the Blasius equation by the fourth-order Runge-Kutta method with double precision, varying the integration step width ($h=10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5}), and obtained the following value;

$$A=f''(0)=0.33205734$$

On the other hand, Howarth's value is 0.33206 as mentioned earlier. Although the difference is apparently minute, it has been found that with $A=0.33206$, the limiting value of $f'(\eta)$ as $\eta\to\infty$ is *not* 1.00000, but 1.00000535. Hence, although Howarth's table may be valid for almost all practical purposes, the numerical values presented therein are not as accurate as they are claimed to be.

With this value of A, the numerical solution to the basic Equation (2.1) has been computed and is given in Table 1. It can be seen that the computed $f'(\eta)$ approaches unity with 8 significant figures at $\eta=9.6$, while the Howarth table shows that a similar value is attained at $\eta=7.8$ (Schlichting). We also noted that with $A=0.33206$, $f'(10.0)=1.000054$. Apart from these, Howarth's values are correct up to 5 figures, but the 6th figure is not always correct.

3. Discussions

With the revised table of the Blasius equation, it is possible to discuss the implications which the revision may have on various physical parameters in the theory of the laminar boundary layer.

First, we consider the thickness of the boundary layer δ. It is customary to define δ such that $u/U_\infty=f'(\eta)=0.99$ (Schlichting)[7] or 0.995 (Takano)[8]. With the former definition, Schlichting gives $\eta\sim5$, while Howarth's table has $\eta=4.9$. Takano gives $\eta=5.2$, while our table gives $\eta=5.28$. Thus, the thickness of the boundary layer is sensitive to the definition and numerical solution to the Blasius equation.

We list below some of the numerical coefficients which are relevant to the theory of the laminar boundary layer.

$$v_\infty = c_1\left(\frac{\nu U_\infty}{x}\right)^{1/2},$$

$$\tau_w = c_2\,\mu\,U_\infty\left(\frac{U_\infty}{\nu x}\right)^{1/2}, \quad c_\tau = \tau_w/\left(\frac{1}{2}\rho\,U_\infty^2\right) = c_3(Re_x)^{-1/2},$$

$$C_f = \int_0^x \tau_w dx = c_4(Re_x)^{-1/2}\frac{\rho U_\infty^2}{2}x,$$

$$\delta^* = c_5\left(\frac{\nu x}{U_\infty}\right)^{1/2}, \quad \theta = c_6\left(\frac{\nu x}{U_\infty}\right)^{1/2}.$$

Here, v_∞ is the vertical component of the velocity at the outer edge of the laminar boundary layer, τ_w, the local skin friction, c_τ, the coefficient of local skin friction, C_f the skin friction of one side, δ^*, the displacement thickness, and θ, the momentum thickness.

	Schlichting	Takano	Revised Value
c_1	0.8604		0.8603938
c_2	0.332	0.33206	0.3320573
c_3	0.664	0.66412	0.6641147
c_4	1.328	1.32824	1.3282294
c_5	1.7208		1.7207876
c_6	0.664		0.6641147

We may summarize the result of the newly obtained table of the Blasius equation as follows. The numerical table given by Howarth is not as accurate as is claimed. If the Blasius equation is integrated from $\eta=0$, with $f''(0)=0.33206$, then the solution asymptotically approaches $f'(\infty)=1.000005$, and not 1.000000. But if $f''(0)=0.33205734$, it is possible to satisfy the condition at infinity ($\eta\to\infty$) with eight significant figures. The revised values do not alter the physical parameters to any significant degree. However, some of the numerical values which are adopted in standard textbooks ought to be revised.

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