Analysis of Nonlinear Oscillations in Three-phase Circuits by Discrete Fourier Transform

By

Kohshi OKUMURA and Akira KISHIMA

(Received May 22, 1979)

Abstract

A method for analysing the nonlinear oscillations in three-phase circuits with nonlinearities of polynomials of a high degree is presented by use of the discrete Fourier transform (DFT). The stability of the oscillation is investigated by means of the DFT. Furthermore, this paper describes how to determine the sampling rate. Numerical examples by the conventional Fourier series method are compared.

1. Introduction

The analytical results of the sub-harmonic oscillations which occurred in three-phase circuits have been reported [1~5]. In these reports, the nonlinear characteristics are expressed as polynomials of the 3rd degree; and the conventional Fourier series is used to obtain the periodic solutions. However, it is tedious and time consuming to expand the nonlinear functions into the Fourier series, especially when they are given by a polynomial of a high degree.

In this report, we deal with such cases by applying the "discrete Fourier transform" (abbreviated as "DFT"), which is carried out by the fast Fourier transform algorithms (abbreviated as "FFT"), to the asymptotic method of Krylov, Bogoliubov and Mitropolsky (abbreviated as "KBM method"). The DFT is also utilized to obtain the characteristic equation of the variational equations when the stability of the periodic solutions is tested. Further, the sampling rate, which must be considered when the DFT is used, is discussed. The analytical results of the 1/3-harmonic oscillation by this method (the DFT method) and the conventional Fourier series method are compared.

2. Fundamental equation

The nonlinear oscillations in three-phase circuits are governed by the following equation:

* Department of Electrical Engineering II

$$\left.\begin{aligned}\frac{dx_1}{d\tau} &= x_2-x_3+\varepsilon X_1(x_1,\,x_2)\\ \frac{dx_2}{d\tau} &= -x_1-x_4+\varepsilon X_2(x_1,\,x_2)\\ \frac{dx_3}{d\tau} &= h_3x_1+x_4+\varepsilon X_3(x_1,\,x_2)\\ \frac{dx_4}{d\tau} &= h_1x_2-x_3+\varepsilon X_4(x_1,\,x_2)\end{aligned}\right\} \tag{1}$$

where

$$\begin{aligned}\varepsilon X_1(x_1,\,x_2) &= -\xi m_3x_1-\xi f_1(x_1,\,x_2)\\ \varepsilon X_2(x_1,\,x_2) &= -\xi m_1x_2-\xi f_2(x_1,\,x_2)\\ \varepsilon X_3(x_1,\,x_2) &= (\eta m_3-h_3)x_1+\eta f_1(x_1,\,x_2)\\ \varepsilon X_4(x_1,\,x_2) &= (\eta m_1-h_1)x_2+\eta f_2(x_1,\,x_2)\\ f_1(x_1,\,x_2) &= \sum_{\nu=0}^{n}\sum_{\gamma=2}^{\nu}\binom{\nu}{\gamma}\bar{c}_{2\nu+1}\rho_0^{2\nu-\gamma+1}(2x_1)^{\gamma}\\ &\quad+\sum_{\nu=0}^{n}\sum_{\gamma=1}^{\nu}\binom{\nu}{\gamma}\bar{c}_{2\nu+1}\rho_0^{2\nu-\gamma}(2x_1)^{\gamma}x_1\\ &\quad+\sum_{\nu=0}^{n}\sum_{\gamma=1}^{\nu}\binom{\nu}{\gamma}\bar{c}_{2\nu+1}\rho_0^{\nu-\gamma}(\rho_0+2x_1)^{\nu-\gamma}(x_1^2+x_2^2)^{\gamma}(\rho_0+x_1)\\ f_2(x_1,\,x_2) &= \sum_{\nu=0}^{n}\sum_{\gamma=1}^{\nu}\binom{\nu}{\gamma}\bar{c}_{2\nu+1}\rho_0^{2\nu-\gamma}(2x_1)^{\gamma}x_2\\ &\quad+\sum_{\nu=0}^{n}\sum_{\gamma=1}^{\nu}\binom{\nu}{\gamma}\bar{c}_{2\nu+1}\rho_0^{\nu-\gamma}(\rho_0+2x_1)^{\nu-\gamma}(x_1^2+x_2^2)^{\gamma}x_2\\ m_1 &= \sum_{\nu=0}^{n}\bar{c}_{2\nu+1}\rho_0^{2\nu}, \quad m_3 = \sum_{\nu=0}^{n}(1+2\nu)\bar{c}_{2\nu+1}\rho_0^{2\nu}\end{aligned}$$

Here, ε is a small parameter. The parameters ξ, η and ρ_0 correspond to the resistance, the capacitance and the amplitude of the fundamental frequency component of the flux-interlinkages, respectively. [See Appendix I, II, III]

3. Analysis by KBM method

In the first approximation, the value of the periodic solution of Eq. (1) at the sampling point $\tau=\tau_p$ $(p=1,\,2,\,\cdots,\,2N-1)$ is given by

$$\left.\begin{aligned}x_k^{(0)}(\psi_p) &= a\varphi_k^1e^{j\psi_p}+a\varphi_k^{1*}e^{-j\psi_p}+(x+jy)\varphi_k^2e^{j\Gamma\psi_p}+(x-jy)\varphi_k^{2*}e^{-j\Gamma\psi_p}\\ \psi_p &\triangleq \omega_1\tau_p\,,\quad \tau_p\triangleq \pi(p-1)/\omega_1N\,,\quad k=1,\,2,\,3,\,4\end{aligned}\right\} \tag{2}$$

where the asterisk indicates the complex conjugate value, the positive integer N represents half of the sampling rate and Γ is a positive integer. Applyng the

KBM method, we obtain

$$\left.\begin{aligned} \varepsilon A_1+ja\varepsilon D_1 &\simeq \frac{1}{2N}\sum_{k=1}^{4}\sum_{p=1}^{2N}\varphi_k^{1*}\varepsilon X_k(x_1^{(0)}(\psi_p),\ x_2^{(0)}(\psi_p))e^{-j\psi_p}/K_1 \\ \varepsilon_1 B-\Gamma y\varepsilon D_1+j(\varepsilon C_1+\Gamma x\varepsilon D_1) &\simeq \frac{1}{2N}\sum_{k=1}^{4}\sum_{p=1}^{2N}\varphi_k^{2*}\varepsilon X_k(x_1^{(0)}(\psi_p),\ x_2^{(0)}(\psi_p))e^{-j\Gamma\psi_p}/K_2 \end{aligned}\right\} \quad (3)$$

The reduction of Eq.(3) is given in Appendix IV. Therefore, the values of a, x and y are given by the singular points of the differential equation

$$\left.\begin{aligned} \frac{da}{d\tau} &= \varepsilon A_1(a, x, y) \\ \frac{dx}{d\tau} &= \varepsilon B_1(a, x, y) \\ \frac{dy}{d\tau} &= \varepsilon C_1(a, x, y) \end{aligned}\right\} \quad (4)$$

4. Numerical computation of singular point

The singular point of Eq.(4) is numerically obtained by the Newton method. The approximate value of the Jacobi matrix can be computed by differentiating both sides of Eq. (3), with respect to the variables a, x and y, respectively. For example, differentiating both sides of the first equation, with respect to the variable a, yields

$$\frac{\partial\varepsilon A_1}{\partial a}+j\left(\varepsilon D_1+a\frac{\partial\varepsilon D_1}{\partial a}\right)\simeq\frac{1}{2N}\sum_{k=1}^{4}\sum_{l=1}^{2}\sum_{p=1}^{2N}\varphi_k^{1*}\left(\frac{\partial\varepsilon X_k}{\partial x_l}\right)_{\psi=\psi_p}(\varphi_l^1+\varphi_l^{1*}e^{-j\Gamma\psi_p})/K_1 \quad (5)$$

Therefore, the elements J_{kl} of the Jacobi matrix are derived from the Fourier coefficients of the gradients $\partial\varepsilon X_k/\partial x_l$. If the first approximate solution is given by Eq. (2), the frequency components of the gradients needed to compute the Jacobi matrix are as follows:

J_{kl}	a	x	y
a	$0,\ \Gamma$	$1-\Gamma,\ 1+\Gamma$	$1-\Gamma,\ 1+\Gamma$
x	$0,\ \Gamma$ $\Gamma-1,\ \Gamma+1$	$1-\Gamma,\ 1+\Gamma$ $0,\ 2\Gamma$	$1-\Gamma,\ 1+\Gamma$ $0,\ 2\Gamma$
y	$0,\ \Gamma$ $\Gamma-1,\ \Gamma+1$	$1-\Gamma,\ 1+\Gamma$ $0,\ 2\Gamma$	$1-\Gamma,\ 1+\Gamma$ $0,\ 2\Gamma$

The integers in each block correspond to the frequency components. Thus, equating both sides of Eq. (5) gives the elements. The other elements are given in

Appendix V. The Newton iterative process is given by

$$\left.\begin{aligned} &\boldsymbol{a}_{(n+1)} = \boldsymbol{a}_{(n)} - \boldsymbol{J}_{(n)}^{-1}\boldsymbol{A}_{(n)} \qquad (n = 0,\ 1,\ 2,\ \cdots) \\ &\boldsymbol{a} \triangleq {}^t(a_n,\ x_n,\ y_n) \\ &\boldsymbol{A}_{(n)} \triangleq {}^t(\varepsilon A_{1(n)},\ \varepsilon B_{1(n)},\ \varepsilon C_{1(n)}) \end{aligned}\right\} \tag{6}$$

If the condition for a small ε_1

$$||\boldsymbol{a}_{(n+1)} - \boldsymbol{a}_{(n)}|| < \varepsilon_1 \tag{7}$$

is satisfied, then the approximate values of the singular points are numerically computed.

5. Determination of sampling rate

The sampling rate $2N$ (half period N) can be determined by applying the Sampling theorem. If the first approximate solution $x_k^{(0)}(\psi)$ is sampled by the Nyquist rate, then $x_k^{(0)}(\psi)$ can be recovered from the sampling sequence $\{x_k^{(0)}(\psi_p)\}$. The Nyquist rate of Eq. (2), denoted as R_A, is given by $R_A = 2\Gamma$. This rate is not necessarily sufficient. The Jacobi matrix must be computed as accurately as possible, since it is used to investigate the stability of the singular points. As demonstrated by Eq. (5), the elements of the Jacobi matrix must be recovered from the sample of the gradient sequence $\{\partial\varepsilon X_k(\psi_p)/\partial x_l\}$. Therefore, the gradient must be sampled at least by the Nyquist rate denoted as R_J. The rate is given by $R_J = 4\Gamma$. For allowable accuracy of the numerical solution, it is desirable to make the sampling rate $2N$ as small as possible from the standpoint of decreasing the computing time. From the above consideration, this minimum rate denoted as $R_{\min}$ can be determined from

$$R_{\min} = \max(R_A,\ R_J) = R_J \tag{8}$$

Therefore, the integer N must be at least 2Γ.

6. Stability investigation

In order to investigate the stability of the periodic solutions, the variational equations of Eq. (4) are considered. Putting the variations as δa's from the singular points a_0's, and neglecting any powers more than the $(\delta a)^2$'s, we have the linear equation

$$\frac{d\delta\boldsymbol{a}}{d\tau} = \boldsymbol{J}\delta\boldsymbol{a} \qquad \delta\boldsymbol{a} \triangleq {}^t(\delta a,\ \delta x,\ \delta y) \tag{9}$$

The characteristic equation of Eq.(9) is given by

$$\Delta(\lambda) \triangleq \det(\lambda \mathbf{1} - \boldsymbol{J}) = 0 \tag{10}$$

Eq. (10) can be written as

$$\Delta(\lambda) \triangleq \lambda^M + a_1 \lambda^{M-1} + \cdots + a_M = 0 \tag{11}$$

where $M=3$. Inserting $\exp\left(j\frac{2\pi}{M}p\right)$ into Eq. (11) leads to the equation

$$d_p \triangleq \Delta\left(\exp\left(j\frac{2\pi}{M}p\right)\right) - 1 = \sum_{k=1}^{M} a_k \exp\left(-j\frac{2\pi}{M}pk\right) \qquad p = 1, \cdots, M \tag{12}$$

Therefore, we can get

$$a_k = \frac{1}{M}\sum_{p=1}^{M} d_p \exp\left(j\frac{2\pi}{M}pk\right) \qquad k = 1, \cdots, M \tag{13}$$

Thus, the values a_k and d_p form the DFT pair (DFT and IDFT; the inverse of DFT is abbreviated as "IDFT"). For each p, the value of the determinant d_p is computed and the coefficient a_k is obtained by the IDFT[7]. Accordingly, by the roots of Eq. (11) the stability is investigated.

7. Algorithm

These results lead us to the following algorithms.

Step 1: Give the initial values of a, x and y.
Step 2: By the IDFT, get the sequence $\{x_k^{(0)}(\psi_p)\}$ for $k=1, \cdots, 4$.
Step 3: By the DFT, get the Fourier components of $\varepsilon X_k(x_1^{(0)}, x_2^{(0)})$ and $\partial\varepsilon X_k/\partial x_l$.
Step 4: Form Eq. (3) and (5), and get Eq. (4) and the Jacobi matrix $\boldsymbol{J}$.
Step 5: Carry out the Newton method, following Eq. (6).
Step 6: For the small value ε_1, if Eq. (7) is not satisfied, set a_n, x_n and y_n as the initial values and go to Step 2. Otherwise, go to the next step.
Step 7: Compute the sequence $\{d_p\}$ from Eq. (12).
Step 8: Get the coefficients a_k by the IDFT of $\{d_p\}$.
Step 9: Test whether the real parts of the roots of Eq. (11) are positive or not.
Step 10: Stop.

8. Numerical examples

This section illustrates the analytical results of the 1/3-harmonic oscillation by the DFT method, compared with those by the Fourier series method. We deal with a case in which the nonlinear characteristics of the inductors are given by $i=c_3\phi^3$.

Setting

$$\tau_1 = 0\,, \quad \tau_{2\nu+1} = 0 \qquad (\nu = 2, \cdots, n)\,,$$
$$\nu = 1\,, \quad n = 1\,, \quad \Gamma = 2$$
$$\xi \leftarrow \xi\tau_3\,, \quad \eta \leftarrow \eta\tau_3$$

we have

$$m_1 = \rho_0^2\,, \quad m_3 = 3\rho_0^2$$
$$f_1(x_1, x_2) = 2\rho_0 x_1^2 + (\rho_0 + x_1)(x_1^2 + x_2^2)$$
$$f_2(x_1, x_2) = 2\rho_0 x_1 x_2 + x_2(x_1^2 + x_2^2)$$

Table 1. Comparison of the DFT method and the Fourier Series method
$\xi=0.15$, $\eta=0.20$, $E=0.30$

	a	x	y
Fourier Series	±0.2491887	−0.1239301	0.0991561
DFT $N=16$	±0.2492006	−0.1239421	0.0992490
$N=12$	±0.2492006	−0.1239421	0.0992490
$N=8$	±0.2492006	−0.1239421	0.0992491
$N=4$	±0.2492005	−0.1239423	0.0992491
$N=2$	±0.2603325	−0.0860817	0.1008860

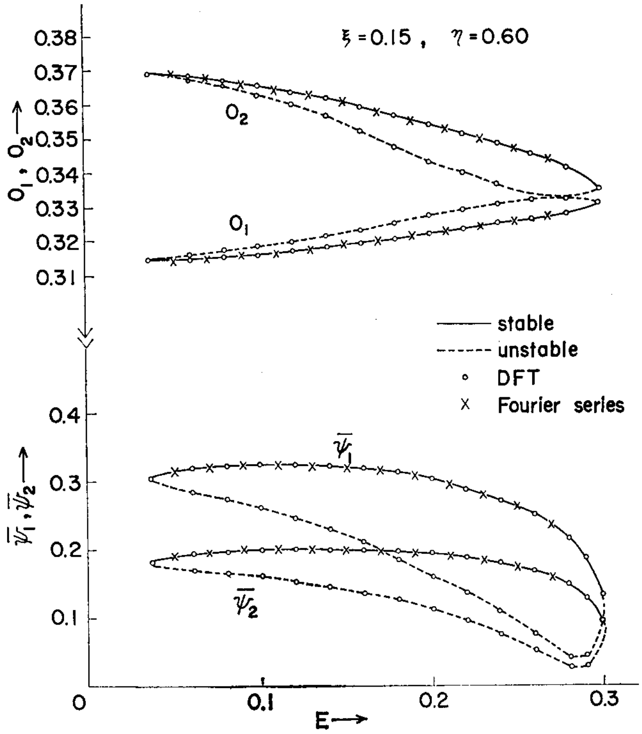

Fig. 1. The characteristics of the amplitudes and the frequencies of the 1/3-harmonic oscillation by the DFT method and the Fourier series method.

For the various values of N, the numerical solutions are shown in Table 1. For $N=16$, 12, 8 and 4, the values a, x and y are in good agreement with those by the Fourier series method.

Fig.1 shows the characteristics of the amplitudes and frequencies when the source voltage E is varied. The dashed curves represent the characteristics corresponding to the unstable solution (See Appendix VI). The points marked with '×' are obtained by the Fourier series method, and those with '○' by the DFT method. Both are in good agreement.

9. Conclusion

The fundamental equation for the analysis of nonlinear oscillations is given when the nonlinear characteristics are expressed by the polynomials. A method for computing the periodic solutions of the equation by use of the DFT is presented. This method is well confirmed by the Fourier series method, if the sampling rate is determined adequatly by the approach described.

The authors wish to thank Mr. Masafumi Mukai, a student of Kyoto University, for his cooperation in the numerical computation.

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Appendix I

The transformation from the three-phase variables to the zero-phase-sequence-, forward-, and backward-variables (abbreviated as o-, f-, and b-variables) is defined by

$$\left.\begin{aligned} w_0 &= \frac{1}{\sqrt{3}}(w_a+w_b+w_c) \\ w_f &= \frac{1}{\sqrt{3}}(w_a+aw_b+a^2w_c)e^{-j\theta} \\ w_b &= \frac{1}{\sqrt{3}}(w_a+a^2w_b+aw_c)e^{j\theta} \end{aligned}\right\} \tag{I.1}$$

or inversely

$$\left.\begin{aligned} w_a &= \frac{1}{\sqrt{3}}(w_0+e^{j\theta}w_f+e^{-j\theta}w_b) \\ w_b &= \frac{1}{\sqrt{3}}(w_0+a^2e^{j\theta}w_f+ae^{-j\theta}w_b) \\ w_c &= \frac{1}{\sqrt{3}}(w_0+ae^{j\theta}w_f+a^2e^{-j\theta}w_b) \end{aligned}\right\} \tag{I.2}$$

where in our case $\theta=\omega t+\varphi$ and $a=\exp(j2\pi/3)$. The forward- and backward-variables are always complex conjugates.

We deal with the nonlinear characteristics expressed as

$$\left.\begin{aligned} i_a &= c_{2\nu+1}\phi_a^{2\nu+1} \\ i_b &= c_{2\nu+1}\phi_b^{2\nu+1} \\ i_c &= c_{2\nu+1}\phi_c^{2\nu+1} \qquad \nu = 0,\ 1,\ 2,\ \cdots,\ n \end{aligned}\right\} \tag{I.3}$$

where i_a, i_b and i_c are the three-phase currents through nonlinear inductors, ϕ_a, ϕ_b and ϕ_c are the three-phase flux-interlinkages and $c_{2\nu+1}$ is the positive constant. The three-phase variables are expressed in terms of the *o*-, *f*- and *b*-variables as follows:

(1) $\nu\equiv 0 \pmod 3$

$$\begin{aligned} i_0 = \frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{ &\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m+2}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m+2}\phi_0^{3n}e^{j(2\nu-6m-3)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m}\phi_f^{2\nu-3m-3n}\phi_b^{3m}\phi_0^{3n+1}e^{j(2\nu-6m+1)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m+1}\phi_f^{2\nu-3m-3n-2}\phi_b^{3m+1}\phi_0^{3n+2}e^{j(2\nu-6m-1)(\omega t+\varphi)}\Bigg\} \end{aligned} \tag{I.4$_1$}$$

$$\begin{aligned} i_f = \frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{ &\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m}\phi_f^{2\nu-3m-3n+1}\phi_b^{3m}\phi_0^{3n}e^{j(2\nu-6m)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m+1}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m+1}\phi_0^{3n+1}e^{j(2\nu-6m-2)(\omega t+\varphi)} \end{aligned}$$

$$+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m+2}\phi_f^{2\nu-3m-3n-3}\phi_b^{3m+2}\phi_0^{3n+2}e^{j(2\nu-6m-4)(\omega t+\varphi)}\Bigg\} \tag{I.4$_2$}$$

$$\begin{aligned} i_b = \frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{&\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m+1}\phi_f^{2\nu-3m-3n}\phi_b^{3m+1}\phi_0^{3n}e^{j(2\nu-6m)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m+2}\phi_f^{2\nu-3m-3n-2}\phi_b^{3m+2}\phi_0^{3n+1}e^{j(2\nu-6m-2)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m}\phi_0^{3n+2}e^{j(2\nu-6m+2)(\omega t+\varphi)}\Bigg\} \end{aligned} \tag{I.4$_3$}$$

(2) $\nu\equiv 1(\text{mod } 3)$

$$\begin{aligned} i_0 = \frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{&\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m}\phi_f^{2\nu-3m-3n+1}\phi_b^{3m}\phi_0^{3n}e^{j(2\nu-6m+1)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m+1}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m+1}\phi_0^{3n+1}e^{j(2\nu-6m-1)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m+2}\phi_f^{2\nu-3m-3n-3}\phi_b^{3m+2}\phi_0^{3n+2}e^{j(2\nu-6m-3)(\omega t+\varphi)}\Bigg\} \end{aligned} \tag{I.5$_1$}$$

$$\begin{aligned} i_f = \frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{&\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m+1}\phi_f^{2\nu-3m-3n}\phi_b^{3m+1}\phi_0^{3n}e^{j(2\nu-6m-2)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m+2}\phi_f^{2\nu-3m-3n-2}\phi_b^{3m+2}\phi_0^{3n+1}e^{j(2\nu-6m-4)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m}\phi_0^{3n+2}e^{j(2\nu-6m)(\omega t+\varphi)}\Bigg\} \end{aligned} \tag{I.5$_2$}$$

$$\begin{aligned} i_b = \frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{&\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m+1}\phi_f^{2\nu-3m-3n}\phi_b^{3m+1}\phi_0^{3n}e^{j(2\nu-6m)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m}\phi_f^{2\nu-3m-3n}\phi_b^{3m}\phi_0^{3n+1}e^{j(2\nu-6m+2)(\omega t+\varphi)} \\ &+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m+2}\phi_f^{2\nu-3m-3n-3}\phi_b^{3m+2}\phi_0^{3n+2}e^{j(2\nu-6m-2)(\omega t+\varphi)}\Bigg\} \end{aligned} \tag{I.5$_3$}$$

(3) $\nu\equiv 2$ (mod 3)

$$i_0 = \frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m+1}\phi_f^{2\nu-3m-3n}\phi_b^{3m+1}\phi_0^{3n}e^{j(2\nu-6m-1)(\omega t+\varphi)}$$

$$
\begin{aligned}
&+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m+2}\phi_f^{2\nu-3m-3n-2}\phi_b^{3m+2}\phi_0^{3n+1}e^{j(2\nu-6m-3)(\omega t+\varphi)}\\
&+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m}\phi_0^{3n+2}e^{j(2\nu-6m+1)(\omega t+\varphi)}\Bigg\}
\end{aligned}
\qquad (\mathrm{I.6})_1
$$

$$
\begin{aligned}
i_f=\frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{&\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m+2}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m+2}\phi_0^{3n}e^{j(2\nu-6m-4)(\omega t+\varphi)}\\
&+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m}\phi_f^{2\nu-3m-3n}\phi_b^{3m}\phi_0^{3n+1}e^{j(2\nu-6m)(\omega t+\varphi)}\\
&+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m+1}\phi_f^{2\nu-3m-3n-2}\phi_b^{3m+1}\phi_0^{3n+2}e^{j(2\nu-6m-2)(\omega t+\varphi)}\Bigg\}
\end{aligned}
\qquad (\mathrm{I.6})_2
$$

$$
\begin{aligned}
i_b=\frac{1}{3^\nu}c_{2\nu+1}\sum_{n=0}^{2\nu+1}\Bigg\{&\sum_{m=0}^{2\nu-3n+1}\binom{2\nu+1}{3n}\binom{2\nu-3n+1}{3m}\phi_f^{2\nu-3m-3n+1}\phi_b^{3m}\phi_0^{3n}e^{j(2\nu-6m+2)(\omega t+\varphi)}\\
&+\sum_{m=0}^{2\nu-3n}\binom{2\nu+1}{3n+1}\binom{2\nu-3n}{3m+1}\phi_f^{2\nu-3m-3n-1}\phi_b^{3m+1}\phi_0^{3n+1}e^{j(2\nu-6m)(\omega t+\varphi)}\\
&+\sum_{m=0}^{2\nu-3n-1}\binom{2\nu+1}{3n+2}\binom{2\nu-3n-1}{3m+2}\phi_f^{2\nu-3m-3n-3}\phi_b^{3m+2}\phi_0^{3n+2}e^{j(2\nu-6m-2)(\omega t+\varphi)}\Bigg\}
\end{aligned}
\qquad (\mathrm{I.6})_3
$$

If the zero-phase sequence component ϕ_0 is negligibly small, the equations from Eq. $(\mathrm{I.4})_1$ to Eq. $(\mathrm{I.6})_3$ are summarized as follows:

$$
\begin{aligned}
i_0\simeq\frac{1}{3^\nu}c_{2\nu+1}\Bigg\{&\delta_{0k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m+2}\phi_f^{2\nu-3m-1}\phi_b^{3m+2}e^{j(2\nu-6m-3)(\omega t+\varphi)}\\
&+\delta_{1k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m}\phi_f^{2\nu-3m+1}\phi_b^{3m}e^{j(2\nu-6m+1)(\omega t+\varphi)}\\
&+\delta_{2k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m+1}\phi_f^{2\nu-3m}\phi_b^{3m+1}e^{j(2\nu-6m-1)(\omega t+\varphi)}\Bigg\}
\end{aligned}
\qquad (\mathrm{I.7})_1
$$

$$
\begin{aligned}
i_f\simeq\frac{1}{3^\nu}c_{2\nu+1}\Bigg\{&\delta_{k0}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m}\phi_f^{2\nu-3m+1}\phi_b^{3m}e^{j(2\nu-6m)(\omega t+\varphi)}\\
&+\delta_{1k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m+1}\phi_f^{2\nu-3m}\phi_b^{3m+1}e^{j(2\nu-6m-2)(\omega t+\varphi)}\\
&+\delta_{2k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m+2}\phi_f^{2\nu-3m-1}\phi_b^{3m+2}e^{j(2\nu-6m-4)(\omega t+\varphi)}\Bigg\}
\end{aligned}
\qquad (\mathrm{I.7})_2
$$

$$
i_b\simeq\frac{1}{3^\nu}c_{2\nu+1}\Bigg\{\delta_{0k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m+1}\phi_f^{2\nu-3m}\phi_b^{3m+1}e^{j(2\nu-6m)(\omega t+\varphi)}
$$

$$+\delta_{1k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m+2}\phi_f^{2\nu-3m-1}\phi_b^{3m+2}e^{j(2\nu-6m-2)(\omega t+\varphi)}$$
$$\left.+\delta_{2k}\sum_{m=0}^{2\nu+1}\binom{2\nu+1}{3m}\phi_f^{2\nu-3m+1}\phi_b^{3m}e^{j(2\nu-6m+2)(\omega t+\varphi)}\right\} \qquad (\mathrm{I.7})_3$$

where $k\equiv\nu$ (mod 3) and δ_{lk} (l, $k=0$, 1, 2) represents Kronecker's delta. If the nonlinear characteristics are given by

$$\left.\begin{aligned} i_a &= \sum_{\nu=0}^{n} c_{2\nu+1}\phi_a^{2\nu+1} \\ i_b &= \sum_{\nu=0}^{n} c_{2\nu+1}\phi_b^{2\nu+1} \\ i_c &= \sum_{\nu=0}^{n} c_{2\nu+1}\phi_c^{2\nu+1} \end{aligned}\right\} \qquad (\mathrm{I.8})$$

then the fundamental frequency components are expressed as

$$\left.\begin{aligned} i_f &= \sum_{\nu=0}^{n} \frac{1}{3^\nu}c_{2\nu+1}\binom{2\nu+1}{\nu}|\phi_f|^{2\nu}\phi_f \\ i_b &= \sum_{\nu=0}^{n} \frac{1}{3^\nu}c_{2\nu+1}\binom{2\nu+1}{\nu}|\phi_b|^{2\nu}\phi_b \end{aligned}\right\} \qquad (\mathrm{I.9})$$

Appendix II

Assuming that the three-phase circuit is symmetric, we can represent the circuit equations as either

$$\left.\begin{aligned} \frac{d\boldsymbol{\phi}}{dt} &= \boldsymbol{Au}-\boldsymbol{Ri}(\boldsymbol{\phi})+\boldsymbol{e}(t) \\ \boldsymbol{C}\frac{d\boldsymbol{u}}{dt} &= \boldsymbol{Ai}(\boldsymbol{\phi})+\boldsymbol{j}(t) \end{aligned}\right\} \qquad (\mathrm{II.1})$$

or

$$\left.\begin{aligned} \boldsymbol{A}\frac{d\boldsymbol{\phi}}{dt} &= \boldsymbol{u}-\boldsymbol{RAi}(\boldsymbol{\phi})+\boldsymbol{e}(t) \\ \boldsymbol{AC}\frac{d\boldsymbol{u}}{dt} &= \boldsymbol{i}(\boldsymbol{\phi})+\boldsymbol{j}(t) \end{aligned}\right\} \qquad (\mathrm{II.2})$$

where

$\boldsymbol{\phi}={}^t(\phi_a, \phi_b, \phi_c)$: flux interlinkage vector of three inductors

$\boldsymbol{u}={}^t(u_a, u_b, u_c)$: voltage vector of three capacitors

$\boldsymbol{i}(\boldsymbol{\phi})={}^t(i_a(\phi_a), i_b(\phi_b), i_c(\phi_c))$: vector-valued function to represent the magnetization characteristics of three inductors

$\boldsymbol{e}(t) = {}^t(e_a(t), e_b(t), e_c(t))$: balanced three-phase voltage source vector
$\boldsymbol{j}(t) = {}^t(j_a(t), j_b(t), j_c(t))$: balanced three-phase current source vector
$\boldsymbol{R} = diag(R, R, R)$: resistance matrix of three inductors
$\boldsymbol{C} = diag(C, C, C)$: capacitance matrix of three capacitors

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

The mark 't' denotes the transposed vector and diag( ) represents the diagonal matrix. If the voltage source is connected, we should set $\boldsymbol{j}(t)=\boldsymbol{0}$, and if the current source is connected, we should set $\boldsymbol{e}(t)=\boldsymbol{0}$ [4].

We apply the transformation of Eq. (I.1) to Eqs. (II.1) and (II.2). The following assumptions are made: (a) the zero-phase-sequence flux interlinkage is negligibly small, and (b) the inductors have no permanent magnetization. The variables v_f and v_b in the equations derived from Eqs. (II.1) and (II.2) are rewritten as

$$j\sqrt{3}\,u_f \rightarrow u_f\,, \quad -j\sqrt{3}\,u_b \rightarrow u_b$$

and

$$-j\frac{1}{\sqrt{3}}u_f \rightarrow u_f\,, \quad j\frac{1}{\sqrt{3}}u_b \rightarrow u_b$$

respectively. Furthermore, the derived equations are transformed by putting

$$\left.\begin{aligned} &\tau = \omega t+\varphi\,, \quad \alpha_\psi \boldsymbol{\phi} = \boldsymbol{\psi}\,, \quad \alpha_v \boldsymbol{u} = \boldsymbol{v}\,, \quad \alpha_i \boldsymbol{i} = \boldsymbol{I} \\ &\boldsymbol{\phi} \triangleq {}^t(\phi_f, \phi_b)\,, \quad \boldsymbol{\psi} \triangleq {}^t(\psi_f, \psi_b)\,, \quad \boldsymbol{u} \triangleq {}^t(u_f, u_b)\,, \quad \boldsymbol{v} \triangleq {}^t(u_f, u_b) \\ &\boldsymbol{i} \triangleq {}^t(i_f, i_b)\,, \quad \boldsymbol{I} \triangleq {}^t(I_f, I_b) \end{aligned}\right\} \tag{II.3}$$

where α_ψ, α_v and α_i are scale factors such that $\alpha_\psi = \omega\alpha_v$ and ω and φ are the angular frequency and initial phase angle of the power source, respectively. Thus, we have the following equation:

$$\left.\begin{aligned} \frac{d\psi_f}{d\tau} &= -j\psi_f - v_f - \xi I_f(\psi_f, \psi_b) + E \\ \frac{d\psi_b}{d\tau} &= j\psi_b - v_b - \xi I_b(\psi_f, \psi_b) + E \\ \frac{dv_f}{d\tau} &= -jv_f + \eta I_f(\psi_f, \psi_b) + J \\ \frac{dv_b}{d\tau} &= jv_b + \eta I_b(\psi_f, \psi_b) + J \end{aligned}\right\} \tag{II.4}$$

where the parameters ξ, η, E and J correspond to the resistance R, the elastance of the capacitor C, the amplitudes of the voltage and current sources, respectively. The variables ψ_b and v_b are the complex conjugate values of ψ_f and v_f. Therefore, the second and the fourth equations of Eq. (II.4) are superflous. Thus, the equation associated with the fundamental frequency components can be written as

$$\left.\begin{aligned} \frac{d\psi_f}{d\tau} &= -j\psi_f - v_f - \xi I_f(\psi_f) + E \\ \frac{dv_f}{d\tau} &= -jv_f + \eta I_f(\psi_f) + J \\ I_f(\psi_f) &= \sum_{\nu=0}^{n} \tilde{c}_{2\nu+1} \binom{2\nu+1}{\nu} |\psi_f|^{2\nu} \psi_f \\ \tilde{c}_{2\nu+1} &= \frac{1}{3^\nu} \frac{1}{\alpha_\psi^{2\nu+1}} \alpha_i c_{2\nu+1} \end{aligned}\right\} \qquad \text{(II.5)}$$

This is the fundamental equation of the three-phase circuits.

The three-phase circuit shown in Fig.A-1 has a no-loaded transformer with a wye-delta connection. Eq. (II.5) is also applied to the circuit when the resistance in the secondary winding is negligibly small.

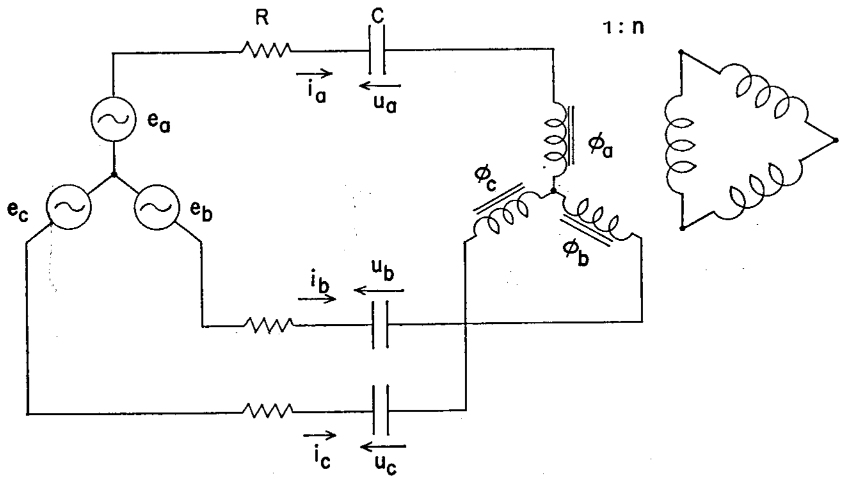

Fig. A-1. Three-phase circuit with no-loaded transformer (n: turn ratio)

Appendix III

We consider the state of the equilibrium (ψ_{f0}, v_{f0}) of Eq. (II.5). We deal with the case $J \equiv O$. Setting the right handside of Eq. (II.5) as zero, we have

$$\left.\begin{aligned} j\psi_{f0} + v_{f0} + \xi I_f(\psi_{f0}) - E &= 0 \\ jv_{f0} - \eta I_f(\psi_{f0}) &= 0 \end{aligned}\right\} \qquad \text{(III.1)}$$

Putting

$$\left.\begin{aligned} \psi_{f0} &= \rho_0 \exp(j\theta_0) \\ v_{f0} &= -j\eta\sigma_0 \exp(j\theta_0) \end{aligned}\right\} \tag{III.2}$$

we have

$$\left.\begin{aligned} &(\xi^2+\eta^2)\sigma_0^2-2\eta\rho_0\sigma_0+\rho_0^2-E^2=0 \\ &\sigma_0=\sum_{\nu=0}^{n} c_{2\nu+1}\rho_0^{2\nu+1} \\ &\tan\theta_0=\frac{1}{\xi}\left(-\eta+\frac{\rho_0}{\sigma_0}\right) \end{aligned}\right\} \tag{III.3}$$

Let us introduce the new variables $\Delta\psi_f$ and Δv_f defined by the following relation:

$$\left.\begin{aligned} \psi_f &= \psi_{f0}+\Delta\psi_f \exp(j\theta_0) \\ v_f &= v_{f0}+\Delta v_f \exp(j\theta_0) \end{aligned}\right\} \tag{III.4}$$

Then, Eq. (II.5) becomes

$$\left.\begin{aligned} \frac{d\Delta\psi_f}{d\tau} &= -j\Delta\psi_f-\Delta v_f-\xi\{I_f(\psi_f)-I_f(\psi_{f0})\} \\ \frac{d\Delta v_f}{d\tau} &= -j\Delta v_f+\eta\{I_f(\psi_f)-I_f(\psi_{f0})\} \end{aligned}\right\} \tag{III.5}$$

where

$$\begin{aligned} I_f(\psi_f)-I_f(\psi_{f0}) &= \frac{1}{2}(m_1+m_3)\Delta\psi_f+\frac{1}{2}(m_3-m_1)\Delta\psi_f^* \\ &+\sum_{\nu=0}^{n}\sum_{\gamma=2}^{\nu}\binom{\nu}{\gamma}c_{2\nu+1}\rho_0^{2\nu-\gamma+1}(\Delta\psi_f+\Delta\psi_f^*) \\ &+\sum_{\nu=0}^{n}\sum_{\gamma=1}^{\nu}\binom{\nu}{\gamma}c_{2\nu+1}\rho_0^{2\nu-\gamma}(\Delta\psi_f+\Delta\psi_f^*)^{\gamma}\Delta\psi_f \\ &+\sum_{\nu=0}^{n}\sum_{\gamma=1}^{\nu}\binom{\nu}{\gamma}c_{2\nu+1}\rho_0^{\nu-\gamma}(\rho_0+\Delta\psi_f+\Delta\psi_f^*)^{\nu-\gamma}(\Delta\psi_f\Delta\psi_f^*)^{\gamma}(\rho_0+\Delta\psi_f) \end{aligned}$$

$$m_1=\sum_{\nu=0}^{n} c_{2\nu+1}\rho_0^{2\nu}$$

$$m_3=\sum_{\nu=0}^{n}(1+2\nu)c_{2\nu+1}\rho_0^{2\nu}$$

Putting

$$\left.\begin{aligned} \Delta\psi_f &= x_1+jx_2 \\ \Delta v_f &= x_3+jx_4 \end{aligned}\right\} \tag{III.6}$$

we have Eq. (1) from Eq. (III.5).

Appendix IV

We shall get the periodic solution of Eq. (1) when ξ and $\tau_{2\nu+1}$ $(\nu=1, \cdots, n)$ are sufficiently small. The unpurturbed system of Eq. (1) is given by

$$\left.\begin{aligned}\frac{dx_1}{d\tau} &= x_2-x_3\\ \frac{dx_2}{d\tau} &= -x_1-x_4\\ \frac{dx_3}{d\tau} &= h_3x_1+x_4\\ \frac{dx_4}{d\tau} &= h_1x_2-x_3\end{aligned}\right\} \qquad \text{(IV.1)}$$

where h_1 and h_3 are the parameters introduced under the assumption that the unpurturbed system is in an internal resonance condition.

Let ω_1 and $\omega_2(\omega_1<\omega_2)$ be the eigen angular frequencies of Eq. (IV.1). Then the solution of Eq. (IV.1) becomes

$$x_k^{(0)} = a\varphi_k^1 e^{j\psi}+a\varphi_k^{1*}e^{-j\psi}+(x+jy)\varphi_k^2 e^{j\Gamma\psi}+(x-jy)\varphi_k^{2*}e^{-j\Gamma\psi} \qquad \text{(IV.2)}$$

where

$$\psi = \omega_1\tau\,, \quad \Gamma\psi = \omega_2\tau$$

Furthermore, we have

$$\varphi_1^l = \varphi^l\,, \quad \varphi_2^l = j\mu_l\varphi^l\,, \quad \varphi_3^l = j(\mu_l-\omega_l)\varphi^l\,, \quad \varphi_4^l = (\omega_l\mu_l-1)\varphi^l \qquad \text{(IV.3)}$$
$$l = 1, 2$$

where φ^l $(l=1, 2)$ are constants and

$$\mu_l = \frac{2\omega_l}{\omega_l^2+1-h_1} \qquad l=1, 2 \qquad \text{(IV.4)}$$

Following the KBM method, we assume that the periodic solution of Eq. (1) can be written as

$$x_k = x_k^{(0)}(a, x, y, \psi)+x_k^{(1)}(a, x, y, \psi)+\cdots \qquad \text{(IV.5)}$$

where $x_k^{(0)}$, $x_k^{(1)}$, $\cdots$ are periodic functions. As to the variables a, x and y themselves, we determine them from

$$\left.\begin{aligned}\frac{da}{d\tau} &= \varepsilon A_1(a, x, y)+\varepsilon^2 A_2(a, x, y)+\cdots\\ \frac{dx}{d\tau} &= \varepsilon B_1(a, x, y)+\varepsilon^2 B_2(a, x, y)+\cdots\\ \frac{dy}{d\tau} &= \varepsilon C_1(a, x, y)+\varepsilon^2 C_2(a, x, y)+\cdots\end{aligned}\right\} \qquad \text{(IV.6)}$$

and

$$\frac{d\psi}{d\tau} = \omega_1 + \varepsilon D_1(a, x, y) + \varepsilon^2 D_2(a, x, y) + \cdots \tag{IV.7}$$

Substituting Eqs. (IV.1), (IV.5) to (IV.7) into Eq. (1), we obtain

$$\varepsilon^0:\ \omega_1 \frac{\partial x_k^{(0)}}{\partial \psi} - \sum_{q=1}^{4} c_{kq} x_q^{(0)} = 0 \tag{IV.8$_0$}$$

$$\varepsilon^1:\ \omega_1 \frac{\partial x_k^{(1)}}{\partial \psi} - \sum_{q=1}^{4} c_{kq} x_q^{(1)} = X_k(x_1^{(0)}, x_2^{(0)}) - \frac{\partial x_k^{(0)}}{\partial a} A_1 - \frac{\partial x_k^{(0)}}{\partial x} B_1 - \frac{\partial x_k^{(0)}}{\partial y} C_1 - \frac{\partial x_k^{(0)}}{\partial \psi} D_1 \tag{IV.8}$$

where c_{kq} is an element of the coefficient matrix of the unperturbed system. In order for $x_k^{(1)}$ to be periodic, the following equations should hold:

$$\left.\begin{aligned} &\varepsilon A_1 + ja\varepsilon D_1 = \frac{1}{2\pi}\left\{\sum_{k=1}^{4} \bar{\varphi}_k^{1*} \int_0^{2\pi} \varepsilon X_k(x_1^{(0)}, x_2^{(0)}) e^{-j\psi} d\psi\right\} \Big/ K_1 \\ &\varepsilon B_1 - \Gamma y \varepsilon D_1 + j(\varepsilon C_1 + \Gamma x \varepsilon D_1) \\ &\qquad = \frac{1}{2\pi}\left\{\sum_{k=1}^{4} \bar{\varphi}_k^{2*} \int_0^{2\pi} \varepsilon X_k(x_1^{(0)}, x_2^{(0)}) e^{-j\Gamma\psi} d\psi\right\} \Big/ K_2 \end{aligned}\right\} \tag{IV.9}$$

where

$$K_l = \sum_{k=1}^{4} \bar{\varphi}_k^{l*} \varphi_k^l \qquad l = 1, 2$$

Here, $\bar{\varphi}_k^l$ is a characteristic function of the adjoint system of Eq. (IV.1). We derive Eq. (3) from the right-hand sides of Eq. (IV.9) in terms of the DFT.

Appendix V

The elements of the Jacobi matrix $\boldsymbol{J}$ are numerically obtained by the following computing processes. Differentiating partially both sides of Eq. (3), we obtain

$$\begin{aligned} \frac{\partial \varepsilon A_1}{\partial a} + j\left(\varepsilon D_1 + a\frac{\partial \varepsilon D_1}{\partial a}\right) &= \frac{1}{2\pi K_1} \sum_{k=1}^{4} \bar{\varphi}_k^{1*} \int_0^{2\pi} \sum_{l=1}^{2} \frac{\partial \varepsilon X_k}{\partial x_l} (\varphi_l^1 + \varphi_l^{1*} e^{-j\Gamma\psi}) d\psi \\ &\simeq \frac{1}{2NK_1} \sum_{k=1}^{4} \sum_{l=1}^{2} \sum_{p=1}^{2N} \bar{\varphi}_k^{1*} \left(\frac{\partial \varepsilon X_k}{\partial x_l}\right)_p (\varphi_l^1 + \varphi_l^{1*} e^{-j\Gamma\psi_p}) \end{aligned}$$

$$\begin{aligned} \frac{\partial \varepsilon A_1}{\partial x} + ja\frac{\partial \varepsilon D_1}{\partial x} &= \frac{1}{2\pi K_1} \sum_{k=1}^{4} \bar{\varphi}_k^{1*} \int_0^{2\pi} \sum_{l=1}^{2} \frac{\partial \varepsilon X_k}{\partial x_l} (\varphi_l^2 e^{-j(1-\Gamma)\psi} + \varphi_l^{2*} e^{-j(\Gamma+1)\psi}) d\psi \\ &\simeq \frac{1}{2NK_1} \sum_{k=1}^{4} \sum_{l=1}^{2} \sum_{p=1}^{2N} \bar{\varphi}_k^{1*} \left(\frac{\partial \varepsilon X_k}{\partial x_l}\right)_p (\varphi_l^2 e^{-j(1-\Gamma)\psi_p} + \varphi_l^{2*} e^{-j(\Gamma+1)\psi_p}) \end{aligned}$$

$$\frac{\partial \varepsilon A_1}{\partial y}+ja\frac{\partial \varepsilon D_1}{\partial y} = \frac{j}{2\pi K_1}\sum_{k=1}^{4}\bar{\varphi}_k^{1*}\int_0^{2\pi}\sum_{l=1}^{2}\frac{\partial \varepsilon X_k}{\partial x_l}(\varphi_l^2 e^{-j(1-\Gamma)\psi}-\varphi_l^{2*}e^{-j(\Gamma+1)\psi})d\psi$$

$$\simeq \frac{j}{2NK_1}\sum_{k=1}^{4}\sum_{l=1}^{2}\sum_{p=1}^{2N}\bar{\varphi}_k^{1*}\left(\frac{\partial \varepsilon X_k}{\partial x_l}\right)_p(\varphi_l^2 e^{-j(1-\Gamma)\psi_p}-\varphi_l^{2*}e^{-j(\Gamma+1)\psi_p})$$

$$\frac{\partial \varepsilon B_1}{\partial a}-\Gamma y\frac{\partial \varepsilon D_1}{\partial a}+j\left(\frac{\partial \varepsilon C_1}{\partial a}+\Gamma x\frac{\partial \varepsilon D_1}{\partial a}\right)$$

$$= \frac{1}{2\pi K_2}\sum_{k=1}^{4}\bar{\varphi}_k^{2*}\int_0^{2\pi}\sum_{l=1}^{2}\frac{\partial \varepsilon X_k}{\partial x_l}(\varphi_l^1 e^{-j(\Gamma-1)\psi}+\varphi_l^{1*}e^{-j(\Gamma+1)\psi})d\psi$$

$$\simeq \frac{1}{2NK_2}\sum_{k=1}^{4}\sum_{l=1}^{2}\sum_{p=1}^{2N}\bar{\varphi}_k^{2*}\left(\frac{\partial \varepsilon X_k}{\partial x_l}\right)_p(\varphi_l^1 e^{-j(\Gamma-1)\psi_p}+\varphi_l^{1*}e^{-j(\Gamma+1)\psi_p})$$

$$\frac{\partial \varepsilon B_1}{\partial x}-\Gamma y\frac{\partial \varepsilon D_1}{\partial x}+j\left\{\frac{\partial \varepsilon C_1}{\partial x}+\Gamma\left(\varepsilon D_1+x\frac{\partial \varepsilon D_1}{\partial x}\right)\right\}$$

$$= \frac{1}{2\pi K_2}\sum_{k=1}^{4}\bar{\varphi}_k^{2*}\int_0^{2\pi}\sum_{l=1}^{2}\frac{\partial \varepsilon X_k}{\partial x_l}(\varphi_l^2+\varphi_l^{2*}e^{-j2\Gamma\psi})d\psi$$

$$\simeq \frac{1}{2NK_2}\sum_{k=1}^{4}\sum_{l=1}^{2}\sum_{p=1}^{2N}\bar{\varphi}_k^{2*}\left(\frac{\partial \varepsilon X_k}{\partial x_l}\right)_p(\varphi_l^2+\varphi_l^{2*}e^{-j2\Gamma\psi_p})$$

$$\frac{\partial \varepsilon B_1}{\partial y}-\Gamma\left(\varepsilon D_1+y\frac{\partial \varepsilon D_1}{\partial y}\right)+j\left(\frac{\partial \varepsilon C_1}{\partial y}+\Gamma x\frac{\partial \varepsilon D_1}{\partial y}\right)$$

$$= \frac{j}{2\pi K_2}\sum_{k=1}^{4}\bar{\varphi}_k^{2*}\int_0^{2\pi}\sum_{l=1}^{2}\frac{\partial \varepsilon X_k}{\partial x_l}(\varphi_l^2-\varphi_l^{2*}e^{-j2\Gamma\psi})d\psi$$

$$= \frac{j}{2NK_2}\sum_{k=1}^{4}\sum_{l=1}^{2}\sum_{p=1}^{2N}\bar{\varphi}_k^{2*}\left(\frac{\partial \varepsilon X_k}{\partial x_l}\right)_p(\varphi_l^2-\varphi_l^{2*}e^{-j2\Gamma\psi_p})$$

where $K_l = \sum_{k=1}^{4}\bar{\varphi}_k^{l*}\varphi_k^l \qquad (l = 1,\ 2)$

Equating the real and imaginary parts of both sides of the equations gives the linear equations with respect to $\frac{\partial \varepsilon A_1}{\partial a}, \cdots, \frac{\partial \varepsilon C_1}{\partial y}$. The value of εD_1 is determined from Eq. (3). Therefore, the elements of the Jacobi matrix are numerically computed by the linear equations.

Appendix VI

The 1/3-harmonic components of the flux interlinkages in the original circuits are given by

$$\Delta\psi_a = \bar{\Psi}_1\cos(O_1\tau+\theta_0-\alpha_1)+\bar{\Psi}_2\cos\left(O_2\tau-\theta_0+\alpha_2+\psi_0+\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\Delta\psi_b = \bar{\Psi}_1\cos\left(O_1\tau+\theta_0-\alpha_1-\frac{2}{3}\pi\right)+\bar{\Psi}_2\cos\left(O_2\tau-\theta_0+\alpha_2+\psi_0+\tan^{-1}\left(\frac{y}{x}\right)+\frac{2}{3}\pi\right)$$

$$\Delta\psi_c = \overline{\Psi}_1 \cos\left(O_1\tau+\theta_0-\alpha_1+\frac{2}{3}\pi\right)+\overline{\Psi}_2 \cos\left(O_2\tau-\theta_0+\alpha_2+\psi_0+\tan^{-1}\left(\frac{y}{x}\right)-\frac{2}{3}\pi\right)$$

where $\overline{\Psi}_1 = 2a$, $\overline{\Psi}_2 = 2\sqrt{x^2+y^2}$,

$$O_1 = \frac{1}{3}-\varepsilon D_1(a, x, y), \quad O_2 = \frac{1}{3}+2\varepsilon D_1(a, x, y), \quad \alpha_l = \arg(\varphi^l) \quad l=1, 2$$

and ψ_0 is the initial value of ψ.