A Method of Determining Characteristic Functions for Cooperative Differential Games without Side Payment

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By

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Abstract

Many person games are treated as non-cooperative games or cooperative games. Cooperative games are divided into games with side payment and without side payment. It is known that cooperative games without side payment can be analyzed and solved in the form of characteristic functions. It is necessary to determine the characteristic functions for differential games which are not described in the form of the characteristic functions.

In this paper, a method of determining the characteristic functions is presented. Two kinds of characteristic functions are obtained according to the α-effectiveness and β-effectiveness, respectively. Determining the characteristic functions is reduced to solving the parametric minimax and maximin problems, two person differential games. The necessary conditions for the solutions of the problems are obtained.

1. Introduction

Differential games involving many players are studied by two approaches. One is to study non-cooperative games and the other is to study cooperative games.

In this paper, cooperative differential games are examined. Since a solution of cooperative games without side payments is given in the form of characteristic functions, a method of determining the characteristic functions is presented.

2. Notation and definitions

[1] Cooperative game

Let N be the set of players and n be the number of players. A subset of N is called a coalition. Let R_N be an n-dimensional Euclidean space whose coordinates are indexed by N. The points in R_N are called payoff vectors. The components of these vectors will be indexed with subscripts; e.g., $J_N \in R_N$, $J_N=(J_1, \cdots, J_n)$.

Let $J_N >_S J_N'$ denote $J_i > J_i'$ for all $i \in S$. Similarly, $J_N \geq_S J_N'$ and $J_N =_S J_N'$ are defined. When $S=N$, simply $J_N > J_N'$ or $J_N \geq J_N'$ is used.

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Definition

An n-person characteristic function is a set N with n members, together with a function v that carries each subset S of N into a subset $V(S)$ of R_N, where

(1) $V(S)$ is closed;

(2) $V(S)$ is comprehensive, i.e., $J_N \in v(S)$; $J_N \in R_N$, and $J_N' \leq_S J_N$ imply $J_N' \in v(S)$;

(3) $v(S)$ is convex;

(4) v is super-additive, i.e., if S and T are disjoint subsets of N, then $v(S \cup T) \supset v(S) \cap v(T)$;

(5) $v(\phi) = R_N$.

Intuitively, v is a function which maps a coalition S onto the set of payoff vectors for which S is "effective." Two notions of effectiveness that determine such a characteristic function are usually defined as follows:

(1) A coalition S is said to be α-effective for $J_N \in R_N$, if and only if there exists a (correlated mixed) strategy for S which assures each member i of S a payoff of at least J_i against any strategy of $N-S$.

(2) A coalition S is β-effective for $J_N \in R_N$, if and only if, for each strategy of $N-S$, there is a strategy for S which yields to each member i of S a payoff of at least J_i.

Definition

Suppose $J_N' \in v(S)$. If $J_N' >_S J_N$, it is said that J_N' dominates J_N via S, and write J_N' Dom_S J_N. Let J_N' Dom J_N denote J_N' Dom_S J_N for some S such that $J_N' \in v(S)$, and Dom K denote the set of J_N in R_N such that J_N' dominates J_N, i.e., J_N' Dom J_N, for some J_N in K.

It is easily shown that there is a number v_i (real, ∞, or $-\infty$) such that $v(\{i\}) = \{J_N \in R_N \mid J_i \leq v_i\}$. This leads to the following:

Definition

A point J_N is called individually rational if $J_i \geq v_i$ for all $i \in N$. The point J_N is called group rational if there is no $J_N' \in H$ such that $J_N' > J_N$, where H is a convex hull of possible outcomes of the game. Let A denote the set of points in H which are individually rational and group rational.

Definition

An n-person game is an n-person characteristic function (N,v) together with a subset $H \subset v(N)$ such that a payoff vecter J_N is in $v(N)$ if and only if there is a payoff vector J_N' in H such that $J_N' \geq J_N$. A solution to the game is a set $V(\subset A)$ such that

$$V = A - \text{Dom } V .$$

[2] Equation of a system

The state of the system is described by the differential equation

$$\frac{dx}{dt}=f(x, u_1, \cdots, u_n, t), \quad x(0)=x_0 ,$$

where variable x is a state variable, u_i's are control variables and t is a time variable. Time interval $[0, t_f]$ of the process is fixed. Player i decides u_i whose value lies in a closed subset of R^{m_i}. u_i should be included in a class of piece-wise continuous functions of time t. Let us define u_i as pure strategy and U_i as a class of pure strategies.

Function f and partial derivative $\partial f/\partial x$ are defined and assumed to be continuous in $X\times U_1\times U_2\times, \cdots, \times U_n$. In the following, U_i is assumed to be R^{m_i} space for simplicity.

[3] Payoff functions and payoff vectors

Player i has a payoff function

$$J_i = K_i(x(t_f))+\int_0^{t_f} L_i(x, u, t)\, dt$$

where K_i and L_i satisfy the same conditions as f. J_i is also described for a given pure strategy as

$$J_i = (J_1, \cdots, J_n) ,$$

$J_N=(J_1, \cdots, J_n)$, and $J_S=(J_i)_{i\in S}$ are the payoff vectors and Euclidean spaces which contain these payoff vectors which are described as R_N and R_S, respectively. If S and T are different coalitions, R_S and R_T are considered to be different spaces.

[4] Mixed strategy

Let us consider a mixed strategy by a pair of finite pure strategies and probability distributions

$$\begin{pmatrix} u_i^1, \cdots, u_i^k \\ p_1, \cdots, p_k \end{pmatrix} \quad \sum_{j=1}^{k} p_j = 1, \quad p_j \geq 0 .$$

When player i uses a mixed strategy and the other players use pure strategies, the payoff vector is defined by the expectation

$$\sum_{j=1}^{k} p_j J_N(u_1, \cdots, u_{i-1}, u_i^j, u_{i+1}, \cdots, u_n) .$$

The payoff vector is analogously defined when more than two players use mixed strategies.

3. Determination of the sets H and A.

Let G be a set of feasible payoff vectors for pure strategies. G is assumed to be a compacet set in R_N. By the theory of convex analysis it is easily shown that the set H of feasible payoff vectors for mixed strategies is a convex hull of G and also a compact set in R_N. Since H is a closed convex set in R_N, H is an intersection of all the closed half spaces which include H in R_N.

Theorem 3.1.

The set H of feasible payoff vectors is given as follows:

$$H = \bigcup_{\lambda_N \in \Lambda_N} \{J_N \mid (\lambda_N, J_N) \leq \max_{J_N \in G} (\lambda_N, J_N)\}$$

where

$$\Lambda_N = \{\lambda_N \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda_N = (\lambda_1, \cdots, \lambda_n)\}$$

and

$$(\lambda_N, J_N) = \sum_{i=1}^{n} \lambda_i J_i \, .$$

Proof.

Since (λ_N, J_N) is a continuous function of J_N, and G is a compact set in R_N, it achieves its maximum in G. It also achieves its maximum in H, meaning that these two maxima are equal to each other, as shown by the following:

$$\max_{J_N \in H} (\lambda_N, J_N) = \max_{J_N \in G} (\lambda_N, J_N) \, .$$

A closed half space in R_N which includes H and has the normal vector λ_N is expressed by an inequality,

$$(\lambda_N, J_N) \leq \max_{J_N \in H} (\lambda_N, J_N) = \max_{J_N \in G} (\lambda_N, J_N) \, .$$

Therefore, H is expressed as the intersection of the closed half spaces. This completes the proof.

Definition

For any subset S of N,

$$R_S^+ = \{J_S \mid J_i \geq 0, \; i \in S\} \, ,$$
$$R_S^- = \{J_S \mid J_i \leq 0, \; i \in S\} \, ,$$
$$R_S^+(J_S^*) = \{J_S \mid J_i - J_S^* \geq 0, \; i \in S\} \, ,$$
$$R_S^-(J_S^*) = \{J_S \mid J_i - J_S^* \leq 0, \; i \in S\} \, .$$

Theorem 3.2.

$$A = \bigcup_{\lambda_N \in \Lambda_N^+} \{J_N \mid (\lambda_N, J_N) = \max_{J_N \in G} (\lambda_N, J_N), J_N \in H\} \cap R_N^+(v_N)$$

where

$$\Lambda_N^+ = \{\lambda_N \mid \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0\ i \in N\}\ .$$

Proof.

It is shown that the set $A' = \bigcup_{\lambda_N \in \Lambda_N^+} \{J_N \mid (\lambda_N, J_N) = \max_{J_N \in G} (\lambda_N, J_N),\ J_N \in H\}$ is individually rational, because it is a subset of $R_N^+(V_N)$. It is sufficient to show that A' is a set of group rational points in H. Let J_N^* be an arbitrary element of A'. Then there exists λ_N in Λ_N^+such that

$$(\lambda_N, J_N^*) \geq (\lambda_N, J_N')$$

for $J_N' \in H$ which is different from J_N^*. Therefore, there exists an index i such that $\lambda_i > 0$ and $J_i^* \geq J_i'$. This means thta J_N^* is a group rational point.

Inversely, let J_N^* be an arbitrary group rational point in H. J_N^* is a boundary point of H, because no interior point of H can be a group rational point. Because of the group rationality of J_N^* in H, H includes no interior point of $R_N^+(J_N^*)$. Therefore, this means that the convex sets H and $R_N^+(J_N^*)$ are separated by the hyper plane which has non-negative coefficients and includes J_N^*, as shown by the following:

$$(\lambda_N, J_N) \leq (\lambda_N, J_N^*),\ \lambda_N \in \Lambda_N^+,\ J_N \in H\ .$$

Because J_N^* is a point of H,

$$(\lambda_N, J_N^*) = \max_{J_N \in G} (\lambda_N, J_N) = \max_{J_N \in H} (\lambda_N, J_N)$$

$$J_N \in \bigcup_{\lambda_N^+} \{J_N \mid (\lambda_N, J_N) = \max_{J_N \in G} (\lambda_N, J_N),\ J_N \in H\}\ .$$

This completes the proof.

By the Theorem 3.2. the problem of determining set A is reduced to the optimal control problem which has modified objective functions with parameter λ_N,

$$(\lambda_N, J_N) = \sum_{i=1}^{n} \lambda_i \{K_i(x(t_f)) + \int_0^{t_f} L_i(x, u_N, t) dt\}\ .$$

By the theory of optimal control, an optimal solution $u_N^*(t)$ satisfies the equation.

$$\max_{u_N(t)} H(\psi(t), x^*(t), u_N(t)) = H(\psi(t), x^*(t), u_N^*(t))$$

where

$$H(\psi, x, u_N) = (\psi, f(x, u_N, t)) + \sum_{i=1}^{n} \lambda_i L_i(x, u_N, t)\ ,$$

and adjoint variable $\psi(t)$ satisfies the differential equation,

$$\frac{d\psi(t)}{dt} = -\psi \frac{\partial f(x^*, u_N^*, t)}{\partial x}$$

with the terminal condition,

$$\vec{\psi}(\overleftarrow{t_f}) = \sum_{i=1}^{n} \lambda_i \frac{\partial K_i(x^*(t_f))}{\partial x(t_f)} .$$

4. Representation of $v_\alpha(S)$ and $v_\beta(S)$, I

Let $v_\alpha(S)$ be a characteristic function in the sense of α-effectiveness.

Theorem 4.1.

$$v_\alpha(S) = co\, w_\alpha(S) \times R_{N-S} ,$$

where

$$w_\alpha(S) = \bigcup_{U_N} \{ \bigcap_{U_{N-S}} (R_S^-(J_S(u_S, u_{N-S})))\}$$

and $w_\alpha(S)$ is assumed to be closed.

Proof.

Let J_S^* be an arbitrary element of co $w_\alpha(S)$. Then, by the convexity of co $w_\alpha(S)$,

$$J_S^* = \sum_{i=1}^{k} p_i J_S^i$$

where

$$J_S^i \in w_\alpha(S), \ \sum_{i=1}^{k} p_i = 1, \ p_i \geq 0, \ \ i = 1, \cdots, k ,$$

and there exists a control $u_S^i \in U_S$ such that

$$J_S^i \in \bigcap_{U_{N-S}} R_S^-(J_S(u_S^i, u_{N-S})) .$$

Therefore,

$$J_S^i \leq J_S(u_S^i, u_{N-S}) \qquad \text{for all } u_{N-S} \in U_{N-S} .$$

If a coalition S uses a mixed strategy,

$$\begin{pmatrix} u_S^1, u_S^2, \cdots, u_S^k \\ p_1, p_2, \cdots, p_k \end{pmatrix}$$

the coalition S assures J_S^* for any mixed strategy of the coalition $N-S$

$$\begin{pmatrix} u_{N-S}^1, u_{N-S}^2, \cdots, u_{N-S}^h \\ q_1 \ , \ q_2 \ , \cdots, \ q_h \end{pmatrix}$$

where

$$\sum_{i=1}^{k} q_i = 1, \ q_i \geq 0, \ i = 1, 2, \cdots, h.$$

Inversely, let us show that the coalition S cannot assure the payoff J_S', which is not contained in co $w_\alpha(S)$ when the coalition S decides a strategy first and the coalition $N-S$ decides a strategy next, knowing the strategy of S. It is impossible to express J_N' by any finite $J_S{}^i \in w_\alpha(S)$ as;

$$J_S' = \sum_{i=1}^{k} p_i J_S{}^i, \ \sum_{i=1}^{k} p_i = 1, \ p_i \geq 0.$$

Therefore, for a vertex $J_S(u_S{}^i, u_{N-S}{}^i)$ of $\bigcap_{U_{N-S}} R_S^-(J_S(u_S, u_{N-S}))$, $u_S \in U_S$, the inequality

$$J_S' \leq \sum_{i=1}^{k} p_i J_S(u_S{}^i, u_{N-S}{}^i)$$

cannot be satisfied. That is, there exists a subscript $j \in S$ such that

$$J_i' > \sum_{i=1}^{k} p_i J_j(u_S{}^i, U_{N-S}{}^i).$$

The coalition S cannot assure J_j' when the coalition $N-S$ uses a mixed strategy

$$\begin{pmatrix} u_S{}^1 & , \cdots, & u_S{}^k \\ u_{N-S}{}^1, & \cdots, & u_{N-S}{}^k \\ p_1 & , \cdots, & p_k \end{pmatrix}$$

in the superior game. Therefore, it is shown that the set considered above is α-effective for the coalition S and is the largest.

It is noted that the following three properties hold;

(1) $v_\alpha(S)$ is closed, because $w_\alpha(S)$ is assumed to be closed,

(3) $v_\alpha(S)$ is convex, because $v_\alpha(S)$ is the product of convex hull of $w_\alpha(S)$ and R_{N-S}.

(5) $v(\phi) = R_N$, when $S=\phi$ in R_{N-S}.

In the last case, let us prove the super-additivity of $v_\alpha(S)$.
If S and T are subsets of N and $S \cap T = \phi$, then

$$\begin{aligned} v_\alpha(S \cup T) &= \text{co } w_\alpha(S \cup T) \times R_{N-S\cup T}, \\ v_\alpha(S) \cap v_\alpha(T) &= \{\text{co } w_\alpha(S) \times R_{N-S}\} \cap \{\text{co } w_\alpha(T) \times R_{N-T}\} \\ &= [\{\text{co } w_\alpha(S) \times R_T\} \cap \{\text{co } w_\alpha(T) \times R_S\}] \\ &\quad \times R_{N-S\cup T}. \end{aligned}$$

For an arbitrary element $J_{S\cup T}{}^* = J_S{}^* \times J_T{}^*$ in $v_\alpha(S) \cap v_\alpha(T)$, there exist $u_S{}^*(u_{N-S}')$ and $u_T{}^*(u_{N-T}')$ such that

$$J_S(u_S{}^*(u_{N-S}'), u_{N-S}') \geq J_S{}^*,$$

$$J_T(u_T^*(u_{N-T}'), u_{N-T}')\geq J_T^*$$

for any u_{N-S}' and u_{N-T}'.
Therefore, if $u_{S\cup T}^*=u_S^*(u_{N-S})\times u_T^*(u_{N-T})$, then

$$J_{S\cup T}(u_{S\cup T}^*(u_{N-S\cup T}), u_{N-S\cup T})\geq T_{S\cup T}^* .$$

This shows the relation

$$v_\alpha(S)\cap v_\alpha(T)\subset v_\alpha(S\cup T) .$$

This completes the proof of the Theorem 4.1.

Next, let $v_\beta(S)$ be a characteristic function in the sense of β-effectiveness.

Theorem 4.2.

$$v_\beta(S) = \bigcap_{U_{N-S}} [\mathrm{co}\ \{\bigcup_{U_S} \{R_S^-(J_S(u_S, u_{N-S}))\}] \times R_{N-S}$$

where $\bigcup_{U_S} R_S^-(J_S(u_S, u_{N-S}))$ is assumed to be closed.

Proof.

Let us define

$$w_\beta(S) = \bigcap_{U_{N-S}} [\mathrm{co}\ \{\bigcup_{U_S} \{R_S^-(J_S(u_S, u_{N-S}))\}] .$$

Let us take $J_S^*\in w_\beta(S)$ arbitrarily. Then the relation

$$J_S\in \mathrm{co}\ \{\bigcup_{U_S} \{R_S^-(J_S^-(u_S, u_{N-S}))\}\}$$

is satisfied for any $u_{N-S}\in U_{N-S}$. By the definition of a convex hull, it holds that

$$J_S^* = \sum_{i=1}^{k} p_i J_S^i,\ J_S^i\in \bigcup_{U_S} R_S^-(J_S(u_S, u_{N-S})),\ \sum_{i=1}^{k} p_i = 1, p_i\geq 0$$

Moreover, there exist $u_S^{i\prime}(u_{N-S})$ for J_S^i and u_{N-S} such that

$$J_S(u_S^{i\prime}(u_{N-S}), u_{N-S})\geq J_S^i$$

by the definition of R_S^-. Therefore, the coalition S can assure the payoff J_S^* using a mixed strategy

$$\begin{pmatrix} u_S^1(u_{N-S}^1), & \cdots, & u_S^{k\prime}(u_{N-S}^k) \\ u_{N-S}' & , \cdots, & u_{N-S}^k \\ p_1 & , \cdots, & p_k \end{pmatrix}$$

in this superior game, because

$$\sum_{i=1}^{k} p_i J_S(u_S^i(u_{N-S}^i), u_{N-S}^i)\geq \sum_{i=1}^{k} p_i J_S^i = J_S^* .$$

Inversely, it should be shown that if J_S' is any element not contained in $w_\beta(S)$, then the coalition $N-S$ has a strategy which does not assure payoff J_S' for the coalition S. If $J_S' \in w_\beta(S)$, then there exists $u_{N-S}' \in U_{N-S}$ such that

$$J_S' \notin \omega\{\bigcup_{U_S} R_S^-(J_S(u_S, u_{N-S}'))\},$$

and it is impossible to satisfy the equation

$$J_S' = \sum_{i=1}^{k} p_i J_S^i,\ J_S^i \in \bigcup_{U_S} R_S^-(J_S(u_S, u_{N-S}')),\ \sum_{i=1}^{k} p_i = 1, p_i \geq 0.$$

It is possible to satisfy the inequality,

$$J_S(u_S^i(u_{N-S}'), u_{N-S}') \geq J_S^i,$$

taking $u_S^i \in U_S$ suitably for each J_S^i, but it is impossible to satisfy the inequality,

$$\sum_{i=1}^{k} p_i J_S(u_S^i(u_{N-S}'), u_{N-S}') \geq J_S'.$$

That is, there exists at least one subscript $j \in S$ such that

$$J_j' > \sum_{i=1}^{k} p_i J_j(u_S^i(u_{N-S}^i), u_{N-S}').$$

Therefore, coalition S cannot have a strategy which assures the payoff J_S' when the coalition $N-S$ uses u_{N-S}' as defined above. Properties (1) and (3) are trivial, (4) and (5) are shown by the same way as the proof of Theorem 4.1. This completes the proof.

5. Representation of $v(S)$ and $v(S)$, II

Theorem 5.1.

$$v_\alpha(S) = \bigcap_{\lambda_S \in \Lambda_S^+} \{J_N \mid (\lambda_S, J_S) \leq \sum \max_{u_S} \min_{\{u_{N-S}^i\}} \sum_{i \in S} \lambda_i J_i(u_S, u_{N-S}^i),$$
$$J_{N-S} \in R_{N-S}\}$$

Proof.

A supporting hyper plane of $v_\alpha(S)$ is given as follows,

$$(\lambda_S, J_S) = \max_{u_S} \min_{\{u_{N-S}^i\}} \sum_{i \in S} \lambda_i J_i(u_S, u_{N-S}^i),$$

because

$$\bigcap_{u_{N-S}} R_S^-(J_S(u_S, u_{N-S})) = R_S^-(J_S^*(u_S))$$

where

$$J_i^*(u_S) = \min_{u_{N-S}} J_i(u_S, u_{N-S}) = J_i(u_S, u(_{N-S}^{i\prime}(u_S))$$

and

$$w_\alpha(S) = \bigcup_{u_S} R_S^-(J_S^*(u_S)) .$$

Since a supporting hyper plane of $R_S^-(J_S^*(u_S))$ is

$$(\lambda_S, J_S) = (\lambda_S, J_S^*(u_S)) .$$

a supporting hyper plan of co $w_\alpha(S)$ is

$$(\lambda_S, J_S) = \max_{U_S} (\lambda_S, J_S^*(u_S))$$
$$= \max_{u_S} \min_{\{u_{N-S}^i\}} \sum \lambda_i J_i(u_S, u_{N-S}^i) .$$

This completes the proof.

An analogous Theorem for $v_\beta(S)$ is shown without a proof.

Theorem 5.2.

$$v_\beta(S) = \bigcap_{\lambda_S \in \Lambda_S^+} \{J_N \,|\, (\lambda_S, J_S) \leq \min_{u_{N-S}} \max_{u_S} (\lambda_S, J_S(u_S, u_{N-S})),\ J_{N-S} \in R_{N-S}\}$$

6. Optimal solution of two person differential games

In the previous sections, the problems of determining characteristic functions $v_\alpha(S)$ and $v_\beta(S)$ are reduced to maximin and minimax control problems with parameter λ_s, respectively.

In this section, the necessary conditions of optimal solutions are obtained by the theory of differential games.

I. Determination of $v_\alpha(S)$.

A system is described by the differential equations

$$\frac{dx^i}{dt} = f(x^i, u_S, u_{N-S}^i, t),\ x^i(0) = x_0,\ i \in S .$$

An objective function is

$$J_\alpha = \sum_{i \in S} \lambda_i J_i(u_S, u_{N-S}^i) = \sum_{i \in S} \lambda_i (K_i(x^i(t_f)) + \int_0^{t_f} L_i(x^i, u_S, u_{N-S}^i, t) dt .$$

The coalition S wants to maximize and $N—S$ wants to minimize the objective function.

I–1. When the objective function J_α has a saddle-point, that is, the equality

$$\max_{u_S} \min_{\{u_{N-S}^i\}} J_\alpha = \min_{\{u_{N-S}^i\}} \max_{u_S} J_\alpha$$

holds, and the optimal controls $u_S^*(t)$ and $u_{N-S}^{i*}(t)$ satisfy the following necessary conditions

$$\max_{u_S} \min_{\{u_{N-S}^i\}} \sum_{i \in S} H(\psi^i(t), x^{i*}(t), u_S(t), u_{N-S}^i(t)) = \sum_{i \in S} H(\psi^i(t), x^{i*}(t) ,$$

$u_S^*(t), u_{N-S}^{i*}(t))$

where

$$H(\psi^i, x^i, u_S, u_{N-S}^i) = (\psi^i, f(x^i, u_S, u_{N-S}^i)) + \lambda_i L_i(x^i, u_S, u_{N-S}^i, t) .$$

Adjoint variables $\psi^i(t)$ satisfy the differential equation

$$\frac{d\psi^i}{dt} = -\psi^i \frac{\partial f(x^{i*}, u_S^*, u_{N-S}^{i*}, t)}{\partial x^i} - \lambda_i \frac{\partial L_i(x^{i*}, u_S^*, u_{N-S}^{i*}, t)}{\partial x^i}$$

with the terminal conditions

$$\psi^i(t_f) = \lambda_i \frac{\partial K_i(x^{i*}(t_f))}{\partial x^i(t_f)} .$$

I–2. When the objective function J_α has no saddle-point, the optimal controls $u_S^*(x, t)$ and $u_{N-S}^i(x^i, t)$ satisfy the inequalities

$$\sum_{i\in S} H_u(\psi_u^i, x^{i*}, u_S, u_{N-S}^{i\prime}(x^i, u_S, t)) \leq \sum_{i\in S} H_u(\psi_u^i, x^{i*}, u_S^*(x^i, t), u_{N-S}^{i\prime}(x^i, t)) ,$$

and

$$\sum_{i\in S} H_v(\psi_v^i, x^{i*}, u_S^*(x^i, t), u_{N-S}^i) \geq \sum_{i\in S} H_v(\psi_v^i, x^{i*}, u_S^*(x^i, t), u_{N-S}^{i\prime}(x^i, t)) ,$$

where

$$H_u(\psi_u^i, x^i, u_S, u_{N-S}^i) = (\psi_u^i, f(x^i, u_S, u_{N-S}^i, t)) + \lambda_i L_i(x^i, u_S, u_{N-S}^i, t) ,$$

and

$$H_v(\psi_v^i, x^i, u_S, u_{N-S}^i) = (\psi_v^i, f(x^i, u_S, u_{N-S}^i, t)) + \lambda_i L_i(x^i, u_S, u_{N-S}^i, t) .$$

Adjoint variables ψ_u^i and ψ_v^i satisfy the differential equation

$$\frac{d\psi_u^i(t)}{dt} = -\psi_u^i \frac{\partial f(x^{i*}, u_S^*, u_{N-S}^{i\prime}, t)}{\partial x^i} - \lambda_i \frac{\partial L_i(x^{i*}, u_S^*, u_{N-S}^{i\prime}, t)}{\partial x^i}$$

and

$$\frac{d\psi_v^i(t)}{dt} = -\psi_v^i \frac{\partial f(x^{i*}, u_S^*, u_{N-S}^{i\prime}, t)}{\partial x^i} - \lambda_i \frac{\partial L_i(x^{i*}, u_S^*, u_{N-S}^{i\prime}, t)}{\partial x^i}$$

with the terminal conditions

$$\psi_u^i(t_f) = \psi_v^i(t_f) = \lambda_i \frac{\partial K_i(x^{i*}(t_f))}{\partial x^i(t_f)} .$$

II. Determination of $v(S)$.

A system equation is described by the differential equation

$$\frac{dx}{dt} = f(x, u_S, u_{N-S}, t), \; x(0) = x_0 .$$

An objective function is given as follows:

$$J_\beta = \sum_{i\in S} \lambda_i J_i(u_S, u_{N-S})$$

II–1. When the objective function J_β has a saddle-point, the optimal controls $u_S^*(t)$ and $u_{N-S}^*(t)$ satisfy the following necessary conditions:

$$\min_{\{u_{N-S}\}} \max_{u_S} H(\psi(t), x^*(t), u_S(t), u_{N-S}(t)) = H(\psi, x^*, u_S^*, u_{N-S}^*) ,$$

where

$$H(\psi, x, u_S, u_{N-S}) = (\psi, f(x, u_S, u_{N-S}, t)) + \sum_{i\in S} \lambda_i L_i(x, u_S, u_{N-S}, t) .$$

Adjoint variables satisfy the differential equation:

$$\frac{d\psi(t)}{dt} = -\psi \frac{\partial f(x^*, u_S^*, u_{N-S}^*, t)}{\partial x} - \sum_{i\in S} \lambda_i \frac{\partial L_i(x^*, u_S^*, u_{N-S}^*, t)}{\partial x}$$

with a terminal condition

$$\psi(t_f) = \sum_{i\in S} \lambda_i \frac{\partial K_i(x^*, (t_f))}{\partial x(t_f)} .$$

II–2. When the objective function has no saddle-point, optimal controls $u_S'(x, t)$ and $u_{N-S}^*(x, t)$ satisfy the following necessary conditions:

$$H_u(\psi_u, x, u_S, u_{N-S}^*(x, t)) \leq H_u(\psi_u, x, u_S'(x, t), u_{N-S}^*(x, t))$$

and

$$H_v(\psi_v, x, u_S'(x, u_{N-S}, t), u_{N-S}) \geq H_v(\psi_v, x, u_S'(x, t), u_{N-S}^*(x, t)) ,$$

where

$$H_u(\psi_u, x, u_S, u_{N-S}) = (\psi_u, f(x, u_S, u_{N-S}, t)) + \sum_{i\in S} \lambda_i L_i(x, u_S, u_{N-S}, t)$$

and

$$H_v(\psi_v, x, u_S, u_{N-S}) = (\psi_v, f(x, u_S, u_{N-S}, t)) + \sum_{i\in S} \lambda_i L_i(x, u_S, u_{N-S}, t) .$$

Adjoint variables $\psi_u(t)$ and $\psi_v(t)$ satisfy the differential equations:

$$\frac{d\psi_u(t)}{dt} = -\psi_u \frac{\partial f(x_i^*, u_S', u_{N-S}^*, t)}{\partial x} - \sum_{i\in S} \lambda_i \frac{\partial L_i(x^*, u_S^*, u_{N-S}^*, t)}{\partial x}$$

and

$$\frac{d\psi_v(t)}{dt} = -\psi_v \frac{\partial f(x^*, u_S', u_{N-S}^*, t)}{\partial x} - \sum_{i\in S} \lambda_i \frac{\partial L_i(x^*, u_S', u_{N-S}^*, t)}{\partial x}$$

with the terminal conditions

$$\psi_u(t_f) = \psi_v(t_f) = \sum_{i\in S} \lambda_i \frac{\alpha K_i(x^*(t_f))}{\partial x(t_f)} .$$

7. Conclusion

In this paper, a method of determining the characteristic functions is discussed for cooperative differential games. The problem is reduced to the two person differential games or optimal control problems, with the objective functions of the maximin type and the minimax type.

It is shown that there are two different types of objective functions with parameter λ_N, one being

$$J_\alpha = \sum_{i \in S} \lambda_i J_i(u_S, u_{N-S}{}^i)$$

for α-effectiveness, and the other being

$$J_\beta = \sum_{i \in S} \lambda_i J_i(u_S, u_{N-S})$$

for β-effectiveness. The necessary conditions for the optimal controls are obtained for both cases.

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