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# The Monte Carlo Approach to State Estimation for Linear Dynamical Systems with State-Dependent Measurement Noise

By

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### Abstract

This paper is concerned with the state estimation of linear dynamical systems with state-dependent measurement noise. The minimum variance estimate of the state is obtained as the weighted mean of the outputs of Kalman filters parameterized by the state-dependent measurement noise sequences. The usual calculation for this estimate, however, becomes impractical since a very large amount of outputs of Kalman filters is required. Therefore, we regard the set of all the state-dependent measurement noise sequences as a population. Then, we evaluate the minimum variance estimate on the basis of a relatively small number of outputs of Kalman filters, parameterized by the state-dependent measurement noise sequences sampled at random from the population. The convergence of the algorithm is established. Then, by an approximation of a sampling procedure with a fast convergence property, a feasible sampling procedure is determined and a practical algorithm is designed. This policy of design leads to an efficient algorithm. Digital simulation results show a good performance of the proposed algorithm.

## I Introduction

The exact solution of the dynamical state estimation problem has been obtained by Kalman and Bucy [1], [2] for linear systems with additive white Gaussian noise. Since then, many papers have been written on state estimation problems.

The present work considers the state estimation for linear systems with state-dependent measurement noise (s. d. m. n.). By "state-dependent noise", we mean not an additive noise but a multiplicative noise. Such state-dependent noises are often seen in aerospace and process control systems [3].

It is well known that if there exists a s. d. m. n., then the a posteriori

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probability density function of the state becomes non-Gaussian, and therefore, the construction of the optimal estimator is very difficult. Hence, a suboptimal technique becomes necessary.

McLane [3] has proposed a sub-optimal estimation algorithm by converting the state estimation problem into the problem of the Wiener-Hopf integral equation. His approach, however, is applicable only when the influence of the s. d. m. n. is relatively small.

This paper presents a practical algorithm which gives an accurate estimate of the state even if the influence of the s. d. m. n. is relatively large.

The optimal estimate of the state is first obtained as the weighted mean of the outputs of Kalman filters parameterized by the s. d. m. n. sequences. The usual computation for this estimate, however, is impractical since a large amount of outputs of Kalman filters is required as time progresses.

In order to overcome this difficulty, the set of all the s. d. m. n. sequences is regarded as a population, and the optimal estimate is evaluated on the basis of a relatively small number of outputs of Kalman filters, parameterized by the s. d. m. n. sequences sampled at random from the population. The convergence is established. Then, by an approximation of sampling procedure with a fast convergence property, a feasible sampling procedure is determined and a practical algorithm is designed. This policy of design leads to an efficient algorithm.

Finally, the digital simulation studies are carried out and performance of the proposed estimator is compared with the sub-optimal estimator by McLane.

## II  Problem Statement

### 2.1  Problem Formulation

The system model is

$$x_{t+1} = F_t x_t + w_t \qquad\qquad (1)$$

$$y_t = g'_t x_t + h'_t x_t v_t + u_t \qquad\qquad (2)$$

where $x_t \in R^n$ is the state and $y_t \in R^1$ is the measurement data. Here, we will consider the scalar case. Generalization to the vector case is not difficult, but it complicates the presentation and would unnecessarily detract from the basic idea. $u_t, v_t \in R^1$ and $w_t \in R^n$ are Gaussian white noises, $F_t$ is a known $n \times n$ matrix, and $g_t, h_t \in R^n$ are known vectors. The noise sequences $u_t, v_t, w_t$, and the initial state $x_0$ are assumed to be mutually independent random variables with the following statistics.

$$p(u_t) \sim N\{0, Q_t\}, \qquad p(v_t) \sim N\{0, R_t\}$$
$$p(w_t) \sim N\{0, S_t\}, \qquad p(x_0) \sim N\{\bar{x}_0, \Sigma_0\} \qquad (3)$$

The second term on the right-hand side of (2) represents the s. d. m. n. effect.

The problem is, on the basis of the data sequence $Y_t \triangleq \{y_0, \cdots\cdots, y_t\}$, to find

the estimate $\hat{x}_t$ of $x_t$ which minimizes the average risk

$$J[Y_t] \triangleq E\{[x_t - \hat{x}_t]' W[x_t - \hat{x}_t] | Y_t\} \qquad (4)$$

where $W$ is a symmetric and positive definite matrix.

## 2.2 Optimal Estimator

It is well known that the optimal estimate $\hat{x}_t$ of the state $x_t$ is given by the conditional mean

$$\hat{x}_t = E\{x_t | Y_t\} \triangleq \int x_t p(x_t | Y_t) \, dx_t \qquad (5)$$

Equation (5) can be rewritten as

$$\hat{x}_t = \int \hat{x}_t(\tilde{V}_t) p(V_t = \tilde{V}_t | Y_t) d\tilde{V}_t \qquad (6)$$

where

$$\hat{x}_t(\tilde{V}_t) \triangleq E\{x_t | V_t = \tilde{V}_t, Y_t\} \qquad V_t \triangleq \{v_0, \cdots\cdots, v_t\} \qquad (7)$$

The symbol $\tilde{V}_t \triangleq \{\tilde{v}_0, \cdots\cdots, \tilde{v}_t\}$ represents an arbitrary $(t+1)$−long sequence. Of course, $\tilde{V}_t$ is not a random variable. It is readily apparent that $\hat{x}_t(\tilde{V}_t)$ can be determined recursively by the following Kalman filter equations:

$$\bar{x}_0(\tilde{V}_{-1}) = \bar{x}_0, \qquad \bar{\Sigma}_0(\tilde{V}_{-1}) = \bar{\Sigma}_0 \qquad (8)$$

$$q_t(\tilde{v}_t) \triangleq g'_t + h'_t \tilde{v}_t \qquad (9)$$

$$\tilde{V}_t = (\tilde{V}_{t-1}, \tilde{v}_t) \qquad (10)$$

$$K_t(\tilde{V}_t) = \bar{\Sigma}_t(\tilde{V}_{t-1}) q'_t(\tilde{v}_t) [q_t(\tilde{v}_t) \bar{\Sigma}_t(\tilde{V}_{t-1}) q'_t(\tilde{v}_t) + Q_t]^{-1} \qquad (11)$$

$$\hat{x}_t(\tilde{V}_t) = \bar{x}_t(\tilde{V}_{t-1}) + K_t(\tilde{V}_t)[y_t - q_t(\tilde{v}_t)\bar{x}_t(\tilde{V}_{t-1})] \qquad (12)$$

$$\hat{\Sigma}_t(\tilde{V}_t) = \bar{\Sigma}_t(\tilde{V}_{t-1}) - K_t(\tilde{V}_t) q_t(\tilde{v}_t) \bar{\Sigma}_t(\tilde{V}_{t-1}) \qquad (13)$$

$$\bar{x}_{t+1}(\tilde{V}_t) = F_t \hat{x}(\tilde{V}_t) \qquad (14)$$

$$\bar{\Sigma}_{t-1}(\tilde{V}_t) = F_t \hat{\Sigma}_t(\tilde{V}_t) F'_t + S_t \qquad (15)$$

On the other hand, using Bayes' rule, we obtain the recursive relation

$$p(V_t = \tilde{V}_t | Y_t) = \frac{p(y_t | V_t = \tilde{V}_t, Y_{t-1}) p(v_t = \tilde{v}_t) p(V_{t-1} = \tilde{V}_{t-1} | Y_{t-1})}{\int [numerator] d\tilde{V}_t} \qquad (16)$$

where $p(y_t | V_t = \tilde{V}_t, Y_{t-1})$ is Gaussian with the statistic:

$$N\{q_t(\tilde{v}_t)\bar{x}_t(\tilde{V}_{t-1}), \; q_t(\tilde{v}_t)\bar{\Sigma}_t(\tilde{V}_{t-1})q'_t(\tilde{v}_t) + Q_t\} \qquad (17)$$

Introducing (16) into (6) recursively, we obtain

$$\hat{x}_t = \frac{\int \hat{x}_t(\tilde{V}_t) f_t(\tilde{V}_t) d\tilde{V}_t}{\int f_t(\tilde{V}_t) d\tilde{V}_t} \qquad (18)$$

where

$$f_t(\tilde{V}_t) \triangleq \prod_{\tau=0}^{t} p(y_\tau | V_\tau = \tilde{V}_\tau, Y_{\tau-1}) p(v_\tau = \tilde{v}_\tau)] \qquad (19)$$

Equaton (18) shows that the optimal estimate $\hat{x}_t$ is obtained as the weighted mean of the outputs of Kalman filers parameterized by the s. d. m. n. sequences $\tilde{V}_t$.

Therejore, $\hat{x}_t$ can be evaluated in principle by calculating $\hat{x}_t(\tilde{V}_t)$ and $f_t(\tilde{V}_t)$ at the lattice points in the space $\Omega_t$ of all possible s. d. m. n. sequence $\tilde{V}_t$, and using a numerical integration technique such as the Gaussian quadrature formula. It should, however, be noted that since the denominator and the numerator in (18) involve $(t+1)$-dimensional multiple integrals, the above algorithm requires a

growing amount of computation with the lapse of time, and hence, becomes impractical for a large value of $t$. To alleviate this difficulty, we propose an algorithm based on the Monte Carlo method in the next section. In the algorithm, the space $\Omega_t$ is regarded as a populaton and the optimal estimate $\hat{x}_t$ is evaluated by a relatively small number of sequences sampled at random from the population $\Omega_t$.

## III  Algorithm Based on the Monte Carlo Method

Let $\alpha_\tau^\nu$ and $\beta_\tau^\nu$, $\nu=1, \cdots\cdots, N$; $\tau=0, 1, \cdots\cdots$, be independent random numbers uniformly distributed in the interval $[0, 1]$. It is assumed that these random numbers are independent of $\{u_t\}$, $\{v_t\}$, $\{w_t\}$, and $x_0$. Let $\Theta_t$ denote the set of these uniform random numbers $\alpha_\tau^\nu$ and $\beta_\tau^\nu$, $\nu=1, \cdots\cdots, N$; $\tau=0, \cdots\cdots, t$. Note that the set $\Theta_{t^*}$ is the sub-set of $\Theta_t$ if $t^* \leqq t$.

In the algorithm proposed here, the optimal estimate $\hat{x}_t$ is evaluated according to the following procedures.

*Step 1*: Introduce the functions $\phi_{t, \tau}(\tilde{V}_\tau; Y_t)$, $\tau=1, \cdots\cdots, t$, which satisfy the following conditions.

(C1)  For each $\tau$, $\phi_{t, \tau}(\tilde{V}_\tau; Y_t)$ is a function of $\tilde{V}_\tau \in \Omega_\tau$, the functional form of which is determined by $Y_t$ or a subsequence of $Y_t$.

(C2)　　$\phi_{t, \tau}(\tilde{V}_\tau; Y_t) \geqq 0$, and

$$\int \phi_{t, \tau}(\tilde{V}_\tau; Y_t) d\tilde{v}_\tau = 1 \text{ for all } \tilde{V}_{\tau-1} \in \Omega_{\tau-1} \tag{20}$$

(C3)  The probability density $\Phi_t$ on $\Omega_t$ which is defined by

$$\Phi_t(\tilde{V}_t; Y_t) \underset{\tau=0}{\overset{t}{\triangleq \Pi}} \phi_{t, \tau}(\tilde{V}_\tau; Y_t) \tag{21}$$

differs from zero (see Fig. 1).

*Step 2 :* Let $-\infty = \varepsilon_{-1} < \varepsilon_0 < \varepsilon_1 < \cdots\cdots < \varepsilon_{l-1} < \varepsilon_{l+1} = +\infty$. Suppose that the $\tau$-long subsequence

$$\hat{x}_0 \leftarrow \Phi_0(\hat{\tilde{V}}_0; Y_0) = \phi_{0,0}(\hat{\tilde{V}}_0; Y_0)$$

$$\hat{x}_1 \leftarrow \Phi_1(\hat{\tilde{V}}_1; Y_1) = \phi_{1,0}(\hat{\tilde{V}}_0; Y_1)\phi_{1,1}(\hat{\tilde{V}}_1; Y_1)$$

$$\vdots \qquad\qquad \vdots$$

$$\hat{x}_t \leftarrow \Phi_t(\hat{\tilde{V}}_t; Y_t) = \phi_{t,0}(\hat{\tilde{V}}_0; Y_t)\phi_{t,1}(\hat{\tilde{V}}_1; Y_t)\times\cdots\times\phi_{t,t}(\hat{\tilde{V}}_t; Y_t)$$

$$\vdots \qquad\qquad \vdots$$

Fig. 1　The construction of $\phi_t(\hat{\tilde{V}}_t; Y_t)$

$$\tilde{V}^{\nu}_{t,\,\tau-1} \triangleq \{\tilde{v}^{\nu}_{t,\,0},\; \cdots\cdots,\; \tilde{v}^{\nu}_{t,\,\tau-1}\} \tag{22}$$

has been sampled. Then, find $\varepsilon_i$ that satisfies

$$F(i-1) \triangleq \int_{-\infty}^{\varepsilon_{i-1}} \phi_{t,\,\tau}(\tilde{V}^{\nu}_{t,\tau-1},\; \tilde{v}_{\tau}) ;\; Y_t)\; d\tilde{v}_{\tau} < \alpha^{\nu}_{\tau}$$

$$\leq \int_{-\infty}^{\varepsilon_i} \phi_{t,\,\tau}(\tilde{V}^{\nu}_{t,\,\tau-1},\; \tilde{v}_{\tau}) ;\; Y_t)\; d\tilde{v}_{\tau} \triangleq F(i) \tag{23}$$

Next, find $\rho \in [\varepsilon_{i-1},\; \varepsilon_i)$ that satisfies

$$D(\rho) \triangleq \frac{\int_{\varepsilon_{i-1}}^{\rho} \phi_{t,\,\tau}((\tilde{V}^{\nu}_{t,\,\tau-1},\; \tilde{v}_{\tau}) ;\; Y_t)\; d\tilde{v}_{\tau}}{\int_{\varepsilon_{i-1}}^{\varepsilon_i} \phi_{t,\,\tau}((\tilde{V}^{\nu}_{t,\,\tau-1},\; \tilde{v}_{\tau}) ;\; Y_t)\; d\tilde{v}_{\tau}} = \beta^{\nu}_{\tau} \tag{24}$$

and put

$$\tilde{v}^{\nu}_{t,\,\tau} = \rho \tag{25}$$

By sequentially performing these operations for $\tau=0,\; \cdots\cdots,\; t$, and $\nu=1,\; \cdots\cdots,\; N$, a number of $N$ of $(t+1)$-long sequences $\tilde{V}^{\nu}_{t,\,t}$, $\nu=1,\; \cdots\cdots,\; N$, can be sampled (see Fig. 2). In practical terms, (23) means the selection of an interval $[\varepsilon_{i-1},\; \varepsilon_i)$ with probability $F(i)-F(i-1)$, while (24) means the selection of a point $\rho$ from the interval $[\varepsilon_{i-1},\; \varepsilon_i)$ with a probability density $D(\rho)$.

*Step 3* : Use the following $\hat{x}^{*}_t(N)$ for $\hat{x}_t$ of (18).

$$\hat{x}^{*}_t(N) \triangleq \frac{N^{-1}\sum\limits_{\nu=1}^{N} [\hat{x}_t(\tilde{V}^{\nu}_{t,t})f_t(\tilde{V}^{\nu}_{t,t})/\Phi_t(\tilde{V}^{\nu}_{t,t};\; Y_t)]}{N^{-1}\sum\limits_{\nu=1}^{N} [f_t(\tilde{V}^{\nu}_{t,t})/\Phi_t(\tilde{V}^{\nu}_{t,t};\; Y_t)]} \tag{26}$$

Now let us denote the average risk (4) associated with the proposed algorithm by $J^{*}[Y_t,\; N]$, i. e.,

$$J^{*}[Y_t,\; N] \triangleq E\{[x_t-\hat{x}^{*}_t(N)]'W\,[x_t-\hat{x}^{*}_t(N)]\,|\,Y_t\} \tag{27}$$

Since the values of the elements of $\Theta_t$ are determined by the random numbers, $J^{*}[Y_t,\; N]$ can be calculated by regarding $\Theta_t$ as a random variable besides $\{u_t\}$,
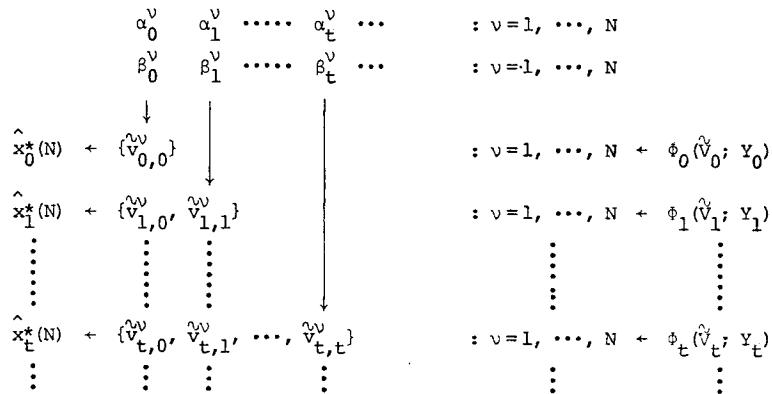


Fig. 2  Relation between uniform random numbers and sampled sequences.

$\{v_t\}$, $\{w_t\}$, and $x_0$.

We first obtain the following lemma.

*Lemma*

$$p(\tilde{V}_{t,\,t}^1,\,\cdots\cdots,\,\tilde{V}_{t,\,t}^N\,|\,x_t,\,Y_t) = \prod_{\nu=1}^{N}\,\Phi_t(\tilde{V}_{t,\,t}^{\nu}\,;\,Y_t) \tag{28}$$

$$p(\tilde{V}_{t,\,t}^1,\,\cdots\cdots,\,\tilde{V}_{t,\,t}^N\,|\,Y_t) = \prod_{\nu=1}^{N}\,\Phi_t(\tilde{V}_{t,\,t}^{\nu}\,;\,Y_t) \tag{29}$$

Proof: Given in Appendix 1.

It is easily seen from (29) that, for given $Y_t$, the numerator and the denominator of (26) are the unbiased estimators of the numerator and the denominator of (18), respectively. That is

$$E\left\{N^{-1}\sum_{\nu=1}^{N}\left[\frac{\hat{x}_t(\tilde{V}_{t,\,t}^{\nu})f_t(\tilde{V}_{t,\,t}^{\nu})}{\Phi_t(\tilde{V}_{t,\,t}^{\nu}\,;\,Y_t)}\right]\,\middle|\,Y_t\right\} = \int\hat{x}_t(\tilde{V}_t)f_t(\tilde{V}_t)d\tilde{V}_t \tag{30}$$

$$E\left\{N^{-1}\sum_{\nu=1}^{N}\left[\frac{f_t(\tilde{V}_{t,\,t}^{\nu})}{\Phi_t(\tilde{V}_{t,\,t}^{\nu}\,;\,Y_t)}\right]\,\middle|\,Y_t\right\} = \int f_t(\tilde{V}_t)d\tilde{V}_t \tag{31}$$

The following theorem provides the general expression for $J^*[Y_t,\,N]$.

*Theorem 1*

$$J^*[y_t,\,N] = J^{\circ}[Y_t] + \varDelta J^*[Y_t,\,N] \tag{32}$$

where

$$\begin{aligned}J^{\circ}[Y_t] &\triangleq E\{x_t-\hat{x}_t]'W[x_t-\hat{x}_t]\,|\,Y_t\} \\ \varDelta J^*[Y_t,\,N] &\triangleq E\{[\hat{x}_t^*(N)-\hat{x}_t]'W[\hat{x}_t^*(N)-\hat{x}_t]\,|\,Y_t\}\end{aligned} \tag{33}$$

Proof: Given in Appendix 2.

Theorem 1 shows that $J^*[Y_t,\,N]$ consists of two parts, one part being the optimal average risk $J^{\circ}[Y_t]$ and the other being the average risk $\varDelta J^*[Y_t,\,N]$, associated with the scattering due to the Monte Carlo method.

We observe that $\varDelta J^*[Y_t,\,N]$ is the mean square loss associated with the ratio estimator of (26). Therefore, the direct application of Cramér's theorem on characteristics of sampling distributions to the loss of the ratio estimator [4] gives the following theorem.

*Theorem 2*

$$J^*[Y_t,\,N] = J^{\circ}[Y_t] + 0(N^{-1}) \tag{34}$$

Theorem 2 shows that $J^*[Y_t,\,N]$ converges to $J^{\circ}[Y_t]$ as $N\to\infty$ regardless of the choice of $\phi_{t,\,\tau}(\tilde{V}_\tau\,;\,Y_t)$, $\tau=0,\,\cdots\cdots,\,t$.

Now, in the remainder of this section, we discuss the choice of $\phi_{t,\,\tau}(\tilde{V}_\tau\,;\,Y_t)$ to ensure fast convergence of $J^*[Y_t,\,N]$ to $J^{\circ}[Y_t]$. The following theorem is useful in the determination of $\phi_{t,\,\tau}(\tilde{V}_\tau\,;\,Y_t)$.

*Theorem 3* If we choose

$$\phi_{t,\tau}(\tilde{V}_\tau; \ Y_t) = p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, \ Y_t) \text{ for } \tau = 0, \ \cdots\cdots, \ t \qquad (35)$$

then

$$J^*[Y_t, \ N] \leqq [1 + (1/N)] J^\circ[Y_t] \qquad (36)$$

where $p(v_\tau = \tilde{v}_\tau | \tilde{V}_{\tau-1} = \tilde{V}_{\tau-1}, \ Y_t)$'s are the factors in the following expressions for $p(V_t = \tilde{V}_t | Y_t)$:

$$p(V_t = \tilde{V}_t | Y_t) = \prod_{\tau=0}^{t} p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, \ Y_t) \qquad (37)$$

Proof: Given in Appendix 3.

It follows from Theorem 3 that if we choose $N = 100$, the average risk $J^*[Y_t, N = 100]$ increases less than one percent compared with the optimal average risk $J^\circ[Y_t]$. Thus, it is possible to obtain a desirable estimate $\hat{x}_t^*(N)$ by picking a substantially small number of sequences from the space $\Omega_t$.

Unfortunately, the direct application of Theorem 3 will not be practical because of the following two disadvantages.

( i ) For instance, the calculation of $\phi_{t,0}(\tilde{V}_0; \ Y_t) = p(V_0 = \tilde{V}_0 | Y_t)$ involves $(t+1)$—and t-dimensional multiple integrals, i. e.,

$$p(V_0 = \tilde{V}_0 | Y_t) = \int p(V_t = \tilde{V}_t | Y_t) d\tilde{v}_1 \cdots\cdots d\tilde{v}_t = \frac{\int f_t(\tilde{V}_t) d\tilde{v}_1 \cdots\cdots d\tilde{v}_t}{\int f_t(\tilde{V}_t) d\tilde{V}_t} \qquad (38)$$

where the second equality is obtained by using (16) and (19).

(ii)  Since in general

$$\tilde{V}_{t, \ t-1}^\nu \neq \tilde{V}_{t-1, \ t-1}^\nu \qquad (39)$$

every time a new measurement $y_t$ is obtained, we must pick $N \times (t+1)$ components $\tilde{v}_{t, \ \tau}^\nu, \ \tau = 0, \ \cdots\cdots, \ t; \ \nu = 1, \ \cdots\cdots, \ N$.

Nevertheless, Theorem 3 is useful in suggesting that in order to obtain a fast convergence with respect to $N$, $\phi_{t,\tau}(\tilde{V}_\tau; \ Y_t)$ should be chosen to approximate $p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, \ Y_t)$ as close as possible. In this sense, Theorem 3 has the same role as the well known optimal "importance" in the area of the importance sampling [5] and gives valuable information on constructing an efficient algorithm.

Now, let us consider determining $\phi_{t,\tau}(\tilde{V}_\tau; \ Y_t)$ based on $Y_\tau$ so that $\phi_{t,\tau}(\tilde{V}_\tau; Y_t)$ approximates $p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, \ Y_t)$. The technique to be employed in this paper is to approximate in the minimum mean square sense $p(V_\tau = \tilde{V}_\tau | Y_\tau)$ and $p(V_{\tau-1} = \tilde{V}_{\tau-1} | Y_\tau)$ appearing in the expression

$$p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, \ Y_t) = \frac{p(V_\tau = \tilde{V}_\tau | Y_\tau)}{p(V_{\tau-1} = \tilde{V}_{\tau-1} | Y_\tau)} \qquad (40)$$

and to determine $\phi_{t,\tau}(\tilde{V}_\tau; Y_t)$ as a result of these approximations.

It is easily seen that $\Psi^n(\tilde{V}_\tau, \ Y_\tau)$ which minimizes the mean squared error

$$E\{\int [p(V_\tau = \tilde{V}_\tau | Y_\tau) - \Psi^n(\tilde{V}_\tau, \ Y_\tau)]^2 \ d\tilde{V}_\tau\}$$

$$= \int \{\int [p(V_\tau = \tilde{V}_\tau | Y_\tau) - \Psi^n(\tilde{V}_\tau, \ Y_\tau)]^2 \ d\tilde{V}_\tau\} p(Y_t) dY_t \qquad (41)$$

is given by the conditional mean

$$\Psi^n(\tilde{V}_\tau, Y_\tau) = \int p(V_\tau = \tilde{V}_\tau | Y_t) p(y_{\tau+1}, \cdots\cdots, y_t | Y_\tau) dy_{\tau+1}\cdots\cdots dy_t$$
$$= p(V_\tau = \tilde{V}_\tau | Y_\tau) \tag{42a}$$

Similarly, $\Psi^a(\tilde{V}_{\tau-1}, Y_\tau)$ which approximates $p(V_{\tau-1} = \tilde{V}_{\tau-1} | Y_t)$ is given by

$$\Psi^a(\tilde{V}_{\tau-1}, Y_\tau) = P(V_{\tau-1} = \tilde{V}_{\tau-1} | Y_\tau) \tag{42b}$$

As a result of these approximations, we obtain

$$\phi_{t, \tau}(\tilde{V}_\tau; Y_t) = \frac{p(V_\tau = \tilde{V}_\tau | Y_\tau)}{p(V_{\tau-1} = \tilde{V}_{\tau-1} | Y_\tau)} = p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, Y_\tau) \tag{43}$$

as an approximation to $p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, Y_t)$. Using Bayes' rule, we obtain

$$p(v_\tau = \tilde{V}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, Y_\tau) = \frac{p(y_\tau | V_\tau = \tilde{V}_\tau, Y_{\tau-1}) p(v_\tau = \tilde{v}_\tau)}{r_\tau(\tilde{V}_{\tau-1})} \tag{44}$$

where $r_\tau(\tilde{V}_{\tau-1})$ is defined by

$$r_\tau(\tilde{V}_{\tau-1}) \underline{\Delta} \int p(y_\tau | V_\tau = \tilde{V}_\tau, Y_{\tau-1}) p(v_\tau = \tilde{v}_\tau) d\tilde{v}_\tau \tag{45}$$

We see from (44) that $\phi_{t, \tau}(\tilde{V}_\tau; Y_t)$ given by (43) can be easily calculated from $p(v_\tau = \tilde{v}_\tau)$ and $p(y_\tau | V_\tau = \tilde{V}_\tau, Y_{\tau-1})$†. It is readily apparent that $\phi_{t, \tau}(\tilde{V}_\tau; Y_t)$ determined in this way satisfies conditions (C1), (C2), and (C3). Further, from (43), we have for each $\tau$

$$\phi_{t, \tau}(\tilde{V}_\tau; Y_t) = \phi_{t-1, \tau}(\tilde{V}_\tau; Y_{t-1}) = \cdots\cdots = \phi_{\tau, \tau}(\tilde{V}_\tau; Y_\tau) \tag{46}$$

Using (46) and examining (23), (24), and (25), we obtain without difficulty

$$\tilde{V}^\nu_{t, t-1} = \tilde{V}^\nu_{t-1, t-1} \qquad \upsilon = 1, \cdots\cdots, N \tag{47}$$

Therefore, the sequence $\tilde{V}^\nu_{t-1, t-1}$ which is used to calculate $\hat{x}^*_{t-1}(N)$ may again be used as the subsequence of $\tilde{V}^\nu_{t, t}$, so that $\tilde{V}^\nu_{t, t}$ can be obtained by sampling only the $(t+1)$th component $\tilde{v}^\nu_{t, t}$. It should be noted that since $\tilde{V}^\nu_{t-1, t-1}$ is the subsequence of $\tilde{V}^\nu_{t, t}$, eqs. (8) to (15) can be used recursively in the proposed algorithm.

From (19), (43), and (44), it turns out that

$$\Phi_t(\tilde{V}_t; Y_t) = f_t(\tilde{V}_t) / \prod_{\tau=0}^{t} r_\tau(\tilde{V}_{\tau-1}) \tag{48}$$

Substituting (48) into the general epuation (26) of the Monte Carlo method and noting (47), we have

$$\hat{x}^*_t(N) = \frac{N^{-1} \sum_{\nu=1}^{N} [\hat{x}_t(\tilde{V}^\nu_{t, t}) R_t(\tilde{V}^\nu_{t-1, t-1})]}{N^{-1} \sum_{\nu=1}^{N} R_t(\tilde{V}^\nu_{t-1, t-1})} \tag{49}$$

where $R_t(\tilde{V}^\nu_{t-1, t-1})$ is defined by

$$R_t(\tilde{V}^\nu_{t-1, t-1}) \underline{\Delta} \prod_{\tau=0}^{t} r_\tau(\tilde{V}^\nu_{t-1, \tau-1}) \tag{50}$$

The actual calculation may be carried out as follows.

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† As has been shown by (17), $p(y_\tau | V_\tau = \tilde{V}_\tau, Y_{\tau-1})$ is Gaussian with statistic $N\{q_\tau(\tilde{v}_\tau)\bar{x}_\tau(\tilde{V}_{\tau-1}), q_\tau(\tilde{v}_\tau)\Sigma_\tau(\tilde{V}_{\tau-1})q_\tau'(\tilde{v}_\tau) + Q_\tau\}$.

(1)  Set $R_{-1}(\tilde{V}^\nu_{-2,-2})=1$ for $\nu=1,\ \cdots\cdots,\ N$, and start from $t=0$.

(2)  Obtain $y_t$.

(3)  Find $\epsilon_i$ that satisfies

$$\int_{-\infty}^{\epsilon_{t-1}} p(v_t=\tilde{v}_t\,|\,V_{t-1}=\tilde{V}^\nu_{t-1,t-1},\ Y_t)d\tilde{v}_t < \alpha^\nu_t$$

$$\leqq \int_{-\infty}^{\epsilon_t} p(v_t=\tilde{v}_t\,|\,V_{t-1}=\tilde{V}^\nu_{t-1,t-1},\ Y_t)d\tilde{v}_t \tag{51}$$

find $\rho\epsilon[\epsilon_{t-1},\ \epsilon_t)$ that satisfies

$$\frac{\int_{\epsilon_{t-1}}^{\rho} p(v_t=\tilde{v}_t\,|\,V_{t-1}=\tilde{V}^\nu_{t-1,t-1},\ Y_t)d\tilde{v}_t}{\int_{\epsilon_{t-1}}^{\epsilon_t} p(v_t=\tilde{v}_t\,|\,V_{t-1}=\tilde{V}^\nu_{t-1,t-1},\ Y_t)d\tilde{v}_t} = \beta^\nu_t \tag{52}$$

and put

$$\tilde{v}^\nu_{t,\,t}=\rho \tag{53}$$

By performing these operations for $\nu=1,\ \cdots\cdots,\ N$, the sequence $\tilde{V}^\nu_{t,t}=\{\tilde{V}^\nu_{t-1,t-1},\ \tilde{v}^\nu_{t,t}\}$, $\nu=1,\ \cdots\cdots,\ N$, can be sampled. The calculation of $r_t(\tilde{V}^\nu_{t-1,t-1})$ of (45) and the selection of $\tilde{v}^\nu_{t,\,t}$ can be easily carried out by replacing $p(y_t\,|\,V_t=(\tilde{V}^\nu_{t-1,t-1},\ \tilde{v}_t),\ Y_{t-1})$ by the step function of $\tilde{v}_t$ which is constant over every interval $[\epsilon_{t-1},\ \epsilon_t)\ \underline{\underline{\Delta}}\ \{\tilde{v}_t|\epsilon_{t-1}\leqq\tilde{v}_t<\epsilon_t\}$, but has in each $\epsilon_t$ a step of the $p(y_t\,|\,V_t=(\tilde{V}^\nu_{t-1,t-1},\ \epsilon_t),\ Y_{t-1})$. Note here that, as a result of this discretization, $p(v_t=\tilde{v}_t\,|\,V_{t-1}=\tilde{V}^\nu_{t-1,t-1},\ Y_t)$ in (51) and (52) can also be replaced by an appropriate step function (see 44)).

(4)  For each $\tilde{V}^\nu_{t,\,t}$, compute $\hat{x}_t(\tilde{V}^\nu_{t,\,t})$ and calculate $R_t(\tilde{V}^\nu_{t-1,t-1})$ from

$$R_t(\tilde{V}^\nu_{t-1,t-1})=r_t(\tilde{V}^\nu_{t-1,t-1})R_{t-1}(\tilde{V}^\nu_{t-2,t-2}) \tag{54}$$

Use $\hat{x}^*_t(N)$ of (49) for $x_t$.

(5)  Replace $t$ with $t+1$ and return to (2).

## IV  NUMERICAL EXAMPLE

Let us consider the scalar system

$$x_{t+1}=Fx_t+w_t,\quad y_t=x_t+x_tv_t+u_t \tag{55}$$

$$p(x_0)\sim N\{15,\ 100\},\quad p(u_t)\sim N\{0,\ 400\}$$
$$p(v_t)\sim N\{0,\ 25\},\qquad p(w_t)\sim N\{0,\ 0.1\} \tag{56}$$

and compare the accuracy of the proposed estimator with that of the linear estimator by McLane [3]. The accuracy of the two estimators are compared on the basis of losses of 25 run averages

$$J^*_t[N]\ \underline{\underline{\Delta}}\ \left\{\frac{1}{25}\sum_{k=1}^{25}\,[x_t-\hat{x}^*_t(N)]^2_k\right\}^{1/2}$$

$$J^\#_t\ \underline{\underline{\Delta}}\ \left\{\frac{1}{25}\sum_{k=1}^{25}\,[x_t-\hat{x}^\#_t]^2_k\right\}^{1/2} \tag{57}$$

where $J_t^*[N]$ and $J_t^\#$ denote the average losses of the poposed algorithm and the algorithm due to McLane, respectively, and the subscript $k$ denotes the number of simulation runs. The number of samples used for the calculation of (49) is only 20 (i. e., $N=20$).

Figs. 3 and 4 show simulation results for the following cases:

Case (a)    $x_0=1$,   $F=0.98$

Case (b)    $x_0=30$, $F=0.98$

These cases correspond to the ones where the influence of the s. d. m. n. decreases with time. Examining Figs. 3 and 4, we observe that the convergence of the proposed algorithm is faster than McLane's algorithm.

Figs. 5 and 6 show the simulation results for the following cases:

Case (c)    $x_0=1$,   $F=1.01$

Case (d)    $x_0=30$, $F=1.01$

In these cases, the influence of the s. d. m. n. increses with time. We observe that the accuracy of linear estimator by McLane is not improved by the measurements at all, while the accuracy of the proposed estimator is well improved.
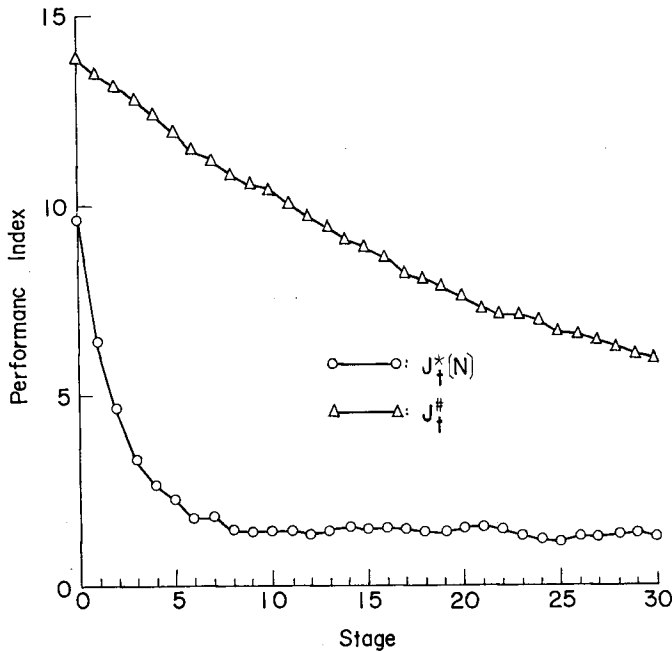


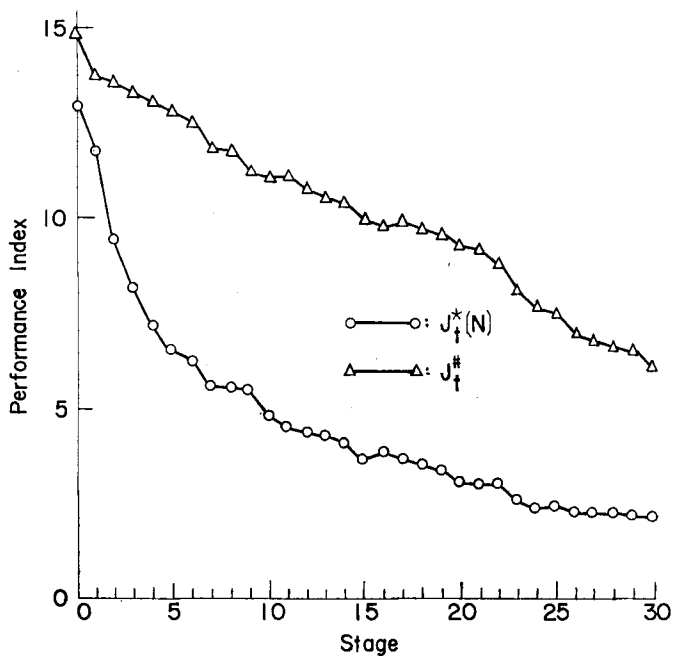Fig. 3  Comparison of the filter performances ($F=0.98$, $x_0=1$; 25 run average).

Fig. 4 Comparison of the filter persormances (F=0.98, $x_0$=30 ; 25 run average).
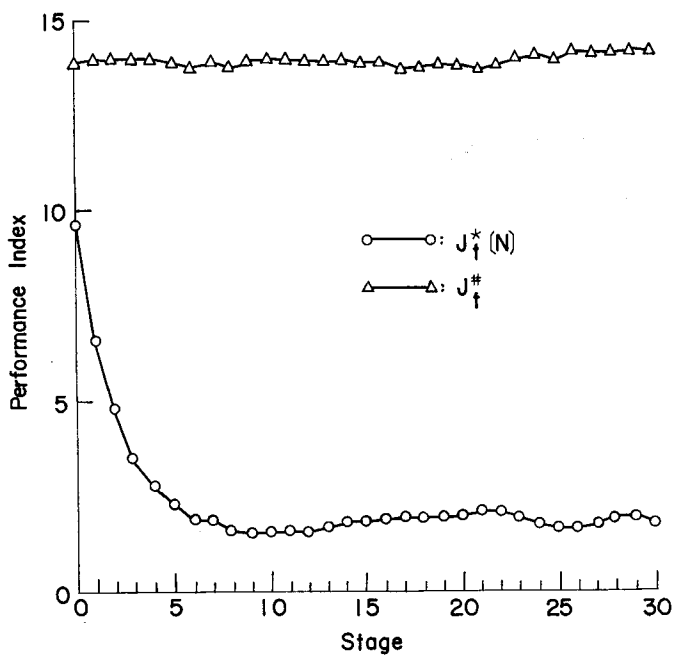


Fig. 5 Comparison of the filter performancesF=(1.01, $x_0$=1 ; 25 run average).
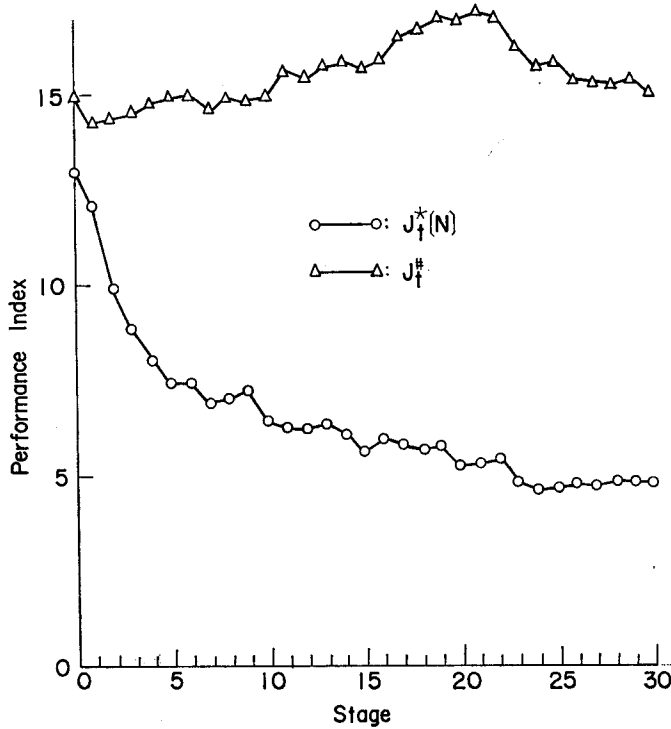
Fig. 6  Comparison of the filter performances (F=1.01, $x_0=30$; 25 run average).

## V  DISCUSSION

Due to increased computing speed of the newer electronic digital computers, the cost of designing and coding an algorithm becomes more and more expensive relative to the cost of machine running time. In the ordinary theoretical approach, the higher order moments of the a posteriori probability density of the state are required for improvement of the accuracy of the sub-optimal estimator. However, the introduction of higher order moments will destroy the computer program of the old algorithm, will increase the time spent in designing the new algorithm, and will complicate its coding.

From Theorem 2, the difference between $J^*[Y_t, N]$ and $J^o[Y_t]$ is of the order of $N^{-1}$, so that we may reach a sufficiently accurate estimate by simply increasing the number N of sampled sequences without changing the computer program of the algorithm. Thus the proposed algorithm, like many Monte Carlo methods, is suitable for the use of electronic digital computers.

Theorem 3 has the same role as the well known optimal "importance" in the

field of importance sampling [5], and gives valuable information on designing an efficient algorithm. As the result of the approximation of $p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, Y_t)$ with a fast convergence property, $\phi_{t,\tau}(\tilde{V}_\tau; Y_t)$ is determined and the practical algorithm is designed. Such a policy of designing the algorithm is commonly used in the area of Monte Carlo method [5]. The simulation results also show that this policy produces a fast convergence, and hence, reduces the machine running time. If necessary, we may perform the computation for each $\tilde{V}^\nu_{t,\tau}$ in parallel, further reducing the computing time.

## VI  CONCLUSION

The state estimation problem for linear dynamical systems with s. d. m. n. was discussed and an algorithm based on the Monte Carlo method was proposed. The average risk $J^*[Y_t, N]$ of the proopsed estimator is the sum of the optimal average risk $J^\circ[Y_t]$ and the risk $\Delta J^*[Y_t, N]$ associated with the scattering due to the Monte Carlo method (Theorem 1). The average risk $J^*[Y_t, N]$ converges to the optimal average risk $J^\circ[Y_t]$ as the number $N$ of randomly sampled s. d. m. n. sequences is increased (Theorem 2). As a result of the approximation of $p(v_\tau = \tilde{v}_\tau | V_{\tau-1} = \tilde{V}_{\tau-1}, Y_t)$ with a fast convergence property (Theorem 3), $\phi_{t,\tau}(\tilde{V}_\tau; Y_t)$ is determined and the feasible algorithm is designed. This policy of determination of $\phi_{t,\tau}(\tilde{V}_\tau; Y_t)$ enables us to design an efficient algorithm. The algorithm was simulated and it provided far better estimates than the algorithm by McLane [3].

## APPENDIX 1  PROOF OF LEMMA

Obviously, the pair $(x_t, Y_t)$ is uniquely determined by $\{u_t\}$, $\{v_t\}$, $\{w_t\}$, and $x_0$. Further, by assumption, both $\alpha^\nu_t$ and $\beta^\nu_t$ are independent of $\{u_t\}$, $\{v_t\}$, $\{w_t\}$, and $x_0$. Thus, under the condition that $(x_t, Y_t)$ is given, $\Theta_t$ is still a set of independent random variables each of which is uniformly distributed on the interval $[0, 1]$. Noting this and examining (23), (24), and (25), we have (28). We obtain (29) in a similar manner.

## APPENDIX 2  PROOF OF THEOREM 1

Let us rewrite (27) as

$$J^*[Y_t, N] = E\{[x_t - \hat{x}_t]'W[x_t - \hat{x}_t] | Y_t\} + E\{[\hat{x}^*_t(N) - \hat{x}_t]'W$$

$$\times [\hat{x}^*_t(N) - \hat{x}_t] | Y_t\} + 2E\{[x_t - \hat{x}_t]'W[x_t - \hat{x}^*_t(N)] | Y_t\} \qquad \text{(A. 1)}$$

From (28), we have

$$p(\tilde{V}^\nu_{t,\tau}, \cdots\cdots, \tilde{V}^N_{t,\tau}, x_t | Y_t) = p(x_t | Y_t) \prod_{\nu=1}^{N} \phi_{t,t}(\tilde{V}^\nu_{t,\tau}; Y_t) \qquad \text{(A. 2)}$$

Therefore, we lead to

$$E\{[x_t - \hat{x}_t]' W[\hat{x}_t - \hat{x}_t^*(N)] | Y_t\} = 0 \cdot W \cdot E\{[\hat{x}_t - \hat{x}_t^*(N)] | Y_t\} = 0 \quad \text{(A. 3)}$$

yielding (32) by (A. 1). Q. E. D.

## APPENDIX 3 PROOF OF THEOREM 3

From (21), (35), and (37), it follows that

$$\varPhi_t(\tilde{V}_{t,t}^\nu; Y_t) = p(V_t = \tilde{V}_t | Y_t) \Big|_{\text{evaluated with } \tilde{V}_t = \tilde{V}_{t,t}^\nu} \quad \text{(A. 4)}$$

On the other hand, from (16) and (19), after some manipulation we have

$$p(V_t = \tilde{V}_t | Y_t) \Big|_{\text{evaluated with } \tilde{V}_t = \tilde{V}_{t,t}^\nu} = \frac{f_t(\tilde{V}_{t,t}^\nu)}{\int f_t(\tilde{V}_t) d\tilde{V}_t} \quad \text{(A. 5)}$$

From (A. 5) and (26)

$$\hat{x}_t^*(N) = N^{-1} \sum_{\nu=1}^{N} \hat{x}_t(\tilde{V}_{t,t}^\nu) \quad \text{(A. 6)}$$

is obtained. Therefore, from (A. 4), (6), and (29), we have

$$J^*[Y_t, N] = \frac{1}{N} \int [\hat{x}_t(\hat{x}_t(\tilde{V}_t) - \hat{x}_t]' W[\hat{x}_t(\tilde{V}_t) - \hat{x}_t] p(V_t = \tilde{V}_t | Y_t) d\tilde{V}_t \quad \text{(A. 7)}$$

On the other hand, we have

$$J^\circ[Y_t] = \int \{\int [x_t - \hat{x}_t(\tilde{V}_t)]' W[x_t - \hat{x}_t(\tilde{V}_t)] p(x_t | V_t = \tilde{V}_t, Y_t) dx\} p(V_t = \tilde{V}_t | Y_t) d\tilde{V}_t$$

$$+ \int [\hat{x}_t(\tilde{V}_t) - \hat{x}_t]' W[\hat{x}_t(\tilde{V}_t) - \hat{x}_t] p(V_t = \tilde{V}_t | Y_t) d\tilde{V}_t \quad \text{(A. 8)}$$

From (A. 7), (A. 8) and the positive definite nature of the matrix $W$, it follows

$$J^*[Y_t, N] \leq \frac{1}{N} J^\circ[Y_t] \quad \text{(A. 9)}$$

Inequality (36) can easily be obtained from (A. 9) and (32). Q. E. D.

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