Solvability of Linear Electrical Networks

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By

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Abstract

The problem of the solvability of a linear active network is discussed on the basis of the two-graph method. It is shown that the topological condition for the solvability is the existence of a common tree of the voltage and current graphs derived from the network. Several conditions for the existence of a common tree are given as well as an algorithm to check whether a common tree exists or not. The algorithm also gives a common tree, if one exists. Then a structure of two-graphs is defined and algorithms to determine the structure are given. The uniqueness and the stability of the structure are discussed. A decomposition of the coefficient matrix of the network equations is derived from the structure. Finally, a classification of the network solvability is given.

1. Introduction

Although the existence of a solution for a resistive network was proved by Kirchhoff,[1] and that for an RLC network with mutual inductance, by Seshu and Reed,[2] the equations for a network containing ideal transformers, ideal gyrators, controlled sources and/or other 2-port elements may or may not have a unique solution. For a network containing ideal transformers, Kuh, Layton and Tow,[3] and for a network containing ideal gyrators, Milic,[4] and Nitta and Kishima,[5] have derived a necessary and sufficient condition for solvability. The condition is of a topological nature and is stated in terms of the existence of a proper tree. No efficient algorithm to check the existence of such a tree is known. The solvability of a network containing controlled sources has been approached by many authors in various ways, depending on the restrictions imposed on the network elements and the network topology.[6)-11)]

Since the network equations for a linear time-invariant electrical network can be given as a set of simultaneous constant coefficient equations in the Laplace transformed domain, the condition for the solvability is often stated in terms of the singularity of the

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coefficient matrix. In general, it is very difficult, however, to check whether the determinant of a matrix is exactly zero or not. Then, a topological approach becomes useful as a first step investigation on solvability. As will be shown, the network singularity originating in the network topology has properties quite different from those caused by the special relations among the values of elements in the network.

We first show that the topological condition for the solvability of a network containing controlled sources is the existence of a common tree of the voltage and current graphs. Then, several conditions are given for the existence of a common tree. A partition of the voltage and current graphs together with its properties is derived. Based on the results derived, we give a classification of network solvability.

For the convenience of the discussions, we use the following notations. Let G be a graph. The set of edges, the rank and the nullity of G are denoted by E, ρ and μ, respectively. When we consider a partition of E into subsets $E_1, E_2, \cdots, E_k, \cdots, E_m$, we denote the graph by $G(E_1, E_2, \cdots, E_k, \cdots, E_m)$. If all the edges in a set, say E_k, are deleted or contracted, the derived graph is denoted by $G(E_1, E_2, \cdots, 0, \cdots, E_m)$ or $G(E_1, E_2, \cdots, 1, \cdots, E_m)$, respectively. We often pay attention to a particular set of edges only, and delete or contract the rest of the edges. In such cases the partition of the rest of the edges is insignificant. Thus, if we delete or contract all the branches except those in a set, say E_k, we denote the derived graph by $G(E_k; 0)$ or $G(E_k; 1)$, respectively. The rank and the nullity of $G(E_1, E_2, \cdots, E_k, \cdots, E_m)$ are denoted by $\rho(E_1, E_2, \cdots, E_k, \cdots, E_m)$ and $\mu(E_1, E_2, \cdots, E_k, \cdots, E_m)$, respectively. Similar notations are used for other graphs. For example, if we partition the edges of G into E_1, E_2 and E_3 and have $G(E_1, E_2, E_3)$, $\rho(E_1, 0, 1)$ is the rank of $G(E_1, 0, 1)$, the graph obtained from G by deleting all the edges in E_2 and contracting all the edges in E_3. The set of all the edges not belonging to a set, say E_k, is denoted by $\bar{E}_k$, and the number of edges in E_k, by $|E_k|$.

We write down the following useful formulae for $G(E_1, E_2)$:

$$\textit{max (number of tree-branches in } E_1) = \rho(E_1, 0) \tag{1}$$

$$\textit{min (number of tree-branches in } E_1) = \rho(E_1, 1) \tag{2}$$

$$\textit{max (number of chords in } E_1) = \mu(E_1, 1) \tag{3}$$

$$\textit{min (number of chords in } E_1) = \mu(E_1, 10). \tag{4}$$

The maximum or the minimum in the above equations is taken over all possible trees of G. Moreover,

$$\rho(E_1, 0) + \rho(1, E_2) = \rho(E_1, 1) + \rho(0, E_2) = \rho(E_1, E_2) \tag{5}$$

$$\mu(E_1, 0) + \mu(1, E_2) = \mu(E_1, 1) + \mu(0, E_2) = \mu(E_1, E_2). \tag{6}$$

2. Solvability and Common Trees

We denote the network to be considered by N^* and the corresponding graph by G^*. We assume that N^* satisfies the following restrictions.

(i) Restrictions of the network elements (The symbols in the parentheses indicate the number of elements.):

N^* contains only one-port passive elements, that is, inductors, capacitors and resistors (total number n_p), independent voltage sources (n_e), independent current sources (n_j), current-controlled voltage sources (n_β), and voltage-controlled current sources (n_δ). A current-controlled voltage source [voltage-controlled current source] is a two-port element consisting of a current sensor α [voltage sensor γ] and a controlled voltage source β [controlled current source δ]. A current sensor [voltage sensor] is a short-circuit[open-circuit], but it is represented by an edge in G^*. Thus, there are two edges for each two-port element in G^*. A controlled source is always controlled by a sensor.

(ii) Restriction on the network topology:

There is no loop consisting of independent voltage sources and/or current sensors, nor any cutset consisting of independent current sources and/or voltage sensors.

The restrictions (i) are imposed to simplify our discussion, but we suffer no loss of the generality of the network considered. The other elements which usually appear in an active network, such as transformers, gyrators and impedance convertors can be replaced by their equivalent controlled-source representations. A current-controlled current source can be replaced by a cascade connection of a current-controlled voltage source and a voltage-controlled current source. A voltage-controlled voltage source can be replaced similarly. Furthermore, if, for instance, many voltage sources are controlled by a current through an element, as many current sensors as there are sources are inserted in series with the element, so that a controlled source is always controlled by a sensor. The topological restriction (ii) is loose enough to allow the existence of a current [voltage] sensor which is not in series [parallel] with an element. Thus, we are considering wider varieties of network topology than was previously considered.[8)]

Now we derive the two-graphs, namely the voltage graph G_v and the current graph G_i. We first get the graph for the network without independent sources. It is denoted by G and is obtained from G^* by contracting all the independent voltage sources and deleting all the independent current sources. Next, the voltage graph G_v is obtained from G by contracting the current-sensor edges and deleting the controlled-current-source edges. The current graph G_i is obtained from G by deleting the voltage-sensor edges and contracting the controlled-voltage-source edges. An edge of G_v and an edge of G_i have a one-to-one correspondence to each other, and thus they are regarded as the same edges appearing in different graphs. The corresponding edges are given the same identification. The rank and the nullity of $G_v[G_i]$ are denoted by ρ_v and $\mu_v[\rho_i$ and $\mu_i]$ respectively.

We classify the unknown variables into two sets as follows.

Set 1. The voltages across the one-port passive elements, controlled voltage sources and voltage sensors, whose voltage vector is denoted by $\boldsymbol{v}_1$. Then, the currents through the one-port passive elements, current sensors and controlled current sources, whose current vector is denoted by $\boldsymbol{i}_1$.

Set 2. The voltages across the independent and controlled current sources. The currents through the independent and controlled voltage sources.

For the first set of variables we get the following network equations.

$$\begin{bmatrix} \boldsymbol{Y} & -\boldsymbol{Z} \\ \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{i}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{B}_e\boldsymbol{v}_e \\ -\boldsymbol{Q}_j\boldsymbol{i}_j \end{bmatrix} \tag{7}$$

where $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are diagonal matrices and $\boldsymbol{B}[\boldsymbol{Q}]$ is a fundamental loop[cutset] matrix obtained from $G_v[G_i]$. $-\boldsymbol{B}_e\boldsymbol{v}_e$ and $-\boldsymbol{Q}_j\boldsymbol{i}_j$ are the voltage and current vectors due to the independent voltage and current sources respectively.

Lemma 1

$$\mu_v = \mu - \mu(E_\delta;\, 1) \tag{8}$$

$$\rho_i = \rho - \rho(E_\beta;\, 0) \tag{9}$$

where $E_\delta[E_\beta]$ is the set of controlled-current-source[controlled-voltage-source] edges.

Proof: Consider the graph obtained from G by contracting all the current-sensor edges. Because of the topological restriction (ii) the nullity of this graph is still μ, but it is also equal to the sum of the nullity of G_v and $G(E_\delta$: 1) by (6). Thus we have (8). Equation (9) can be proved dually.

Now $\rho+\mu=n_1+n_\beta+n_\delta$, where $n_1=n_p+n_\beta+n_\delta$, $\rho_v+\mu_v=\rho_i+\mu_i=n_1$, $\rho(E_\delta;\,1)+\mu(E_\delta;\,1)=n_\delta$, and $\rho(E_\beta;\,0)+\mu(E_\beta;\,0)=n_\beta$. Then we get

$$\mu_v-\mu_i=\rho_i-\rho_v=\mu(E_\beta;\,0)+\rho(E_\delta;\,1) \tag{10}$$

$$\mu_v+\rho_i=n_1+\mu(E_\beta;\,0)+\rho(E_\delta;\,1). \tag{11}$$

An immediate consequence of (10) and (11) is the following.

Theorem 1

If and only if there is no loop consisting of controlled-voltage-source edges only nor a cutset consisting of controlled-current-source edges only in G, the ranks[nullities] of G_v and G_i are equal, and the number of equations in (7) is equal to the number of unknown variables in (7).

If the condition of Theorem 1 is satisfied, then the coefficient matrix of (7) is square and its determinant can be expanded as the sum of common-tree-immittance products.[12]

$$\begin{vmatrix} Y & -Z \\ B & 0 \\ 0 & Q \end{vmatrix} = \pm \sum_{all\ T} (sign\ T) \prod_{immit} \tag{12}$$

where T denotes a common tree of G_v and G_i, and *sign* T is the sign permutation of T. $\prod_{immit}$ signifies the common-tree-immittance product. The determinant becomes zero if there is no common tree of G_v and G_i, or if there are special relations among the values of network elements, so that all the common-tree-immittance products are canceled out. We will consider the former case only. If the condition in Theorem 1 is not satisfied, it is obvious that there is no common tree. Then it can be said that the topologieal condition for the solvability of (7) is the existence of a common tree of G_v and G_i.

In many cases the existence of a common tree is quaranteed by the following theorem.

Theorem 2

A sufficient condition for the existence of a common tree is that there is no loop consisting of controlled-voltage-source edges and/or current-sensor edges, nor a cutset consisting of controlled-current-source edges and/or voltage-sensor edges in G.

Thus, the restriction that a current [voltage] sensor be in series [parallel] with a passive element leads to the existence of a common tree, if the condition in Theorem 1 is satisfied.

If the condition in Theorem 1 is satisfied, but that in Theorem 2 is not, there may or may not be a common tree. We derive the voltage and current graphs and consider partitions of edges. The two-graphs are denoted by $G_v(A, B)$ and $G_i(A, B)$.

Theorem 3

A necessary and sufficient condition for the existence of a common tree is that there exists no partition of edges such that

$$\rho_v(A, 1) > \rho_i(A, 0) \tag{13}$$

or

$$\rho_i(A, 1) > \rho_v(A, 0). \tag{14}$$

Proof: As shown in (1) and (2), the number of edges in A and in a tree of G_v is no less than $\rho_v(A, 1)$, and the number of edges in A and in a tree of G_i is no more than $\rho_i(A, 0)$. Thus, if there is a partition satisfying (13) there can be no common tree. Similarly (14) is also necessary. The sufficiency of the condition follows from the discussion given below. We will show that if the two-graphs have no common tree, there is a partition satisfying (13) or (14).

Suppose G_v and G_i have no common tree. For any tree, say T_u, of G_v, there is, in G_t, at least one loop consisting only of the tree-branches of T_u, and/or one cutset, consisting only of the chords of T_u. The sets of edges belonging to such loops and cutsets are denoted by L_u and C_u respectively, and the numbers of such independent loops and cutsets are denoted by ξ_u and η_u respectively. Note that the same suffix is used to indicate the relation among the symbols T, L, C, ξ and η. The notations concerning other trees are given in the same way. We have

$$\xi_u+\rho_t=\eta_u+\rho_v. \tag{15}$$

Now a tree of G_t corresponding to T_u is defined to be a tree of G_t which is derived from T_u by removing proper ξ_u edges in L_u and adding proper η_u edges in C_u. It is denoted by T_h. T_u and T_h may be called a corresponding pair. There are, in G_v, loops consisting of the tree-branches of T_h and/or cutsets consisting of the chords of T_h. The sets of edges belonging to such loops and cutsets are denoted by L_h and C_h, respectively.

Assume a subset of edges A is given. Let ξ_{uA} and η_{uA} be the numbers of edges which are in A and also among the removed and added edges to obtain T_h from T_u, respectively, and let b_{uA} and b_{hA} be the numbers of tree-branches in T_u and T_h and also in A, respectively. Then we have

$$\xi_u\geqq\xi_{uA}=b_{uA}-b_{hA}+\eta_{uA}\geqq b_{uA}-b_{hA}\geqq\rho_v(A;1)-\rho_t(A;0). \tag{16}$$

The last relation in (16) can be derived by use of (1) and (2). Now if ξ_u is the minimum for all the trees of G_v, T_u is called a maximally-common tree (abbreviated as MCT) of G_v. It is easy to see that ξ_u is minimum if and only if η_u is minimum. A tree corresponding to an MCT of G_v is called an MCT of G_t. We denote the cotree of a tree, say T, by $\bar{T}$.

Now consider the sets of edges Ω and Γ which satisfy the following Condition 1 and Condition 2, respectively, with respect to a pair of corresponding MCT's, T_u of G_v, and T_h of G_t.

Condition 1. (i) Ω contains L_u in G_t. (ii) Ω does not contain C_u in G_t. (iii) $T_h\cap\Omega$ is a forest of $G_t(\Omega;0)$. (iv) $T_u\cap\Omega$ is a forest of $G_v(\Omega;1)$. (v) The set of edges Ω is a minimal one satisfying (i)–(iv).

Condition 2. (i) Γ contains C_u in G_t. (ii) Γ does not contain L_u in G_t. (iii) $T_h\cap\Gamma$ is a forest of $G_t(\Gamma;1)$. (iv) $T_u\cap\Gamma$ is a forest of $G_v(\Gamma;0)$. (v) The set of edges Γ is a minimal one satisfying (i)–(iv).

Let us regard Ω as set A in (16) and the corresponding MCT's as T_u and T_h in (16). Then $\xi_u=\xi_{u\Omega}$, $\eta_{u\Omega}=0$, $b_{u\Omega}=\rho_v(\Omega;1)$ and $b_{h\Omega}=\rho_t(\Omega;0)$ from (i), (ii), (iv) and (iii), respectively, in Condition 1, and we have

$$\xi_u=\rho_v(\Omega;1)-\rho_t(\Omega;0). \tag{17}$$

Similarly, from Condition 2 we get

$$\eta_u=\rho_i(\Omega;1)-\rho_v(\Omega;0) \tag{18}$$

for the MCT's.

From Conditions 1 and 2 we recognize that the subgraphs in G_v and G_i formed by the edges of $\Omega[\Gamma]$ have a strong similarity to $G_2[G_1]$ and $G_1[G_2]$, respectively, of the principal partition of a graph.[13] We also notice that an MCT resembles an extremal tree.[14] The existence of the set Ω or the set Γ when G_v and G_i have no common tree, and the fact that these sets are mutually disjoint can be seen from the following algorithm, which is an extension of that given in reference 14). In Algorithm 1, we first find the set of edges satisfying Condition 1, with respect to an arbitrary tree T_u of G_v. If T_u is not an MCT, a tree, T_w, of G_v with $\xi_w=\xi_u-1$, and thus having one more common edge with a tree of G_i is obtained. The new tree is used to find the set of ecges satisfying Condition 1 with respect to the tree. This operation is repeated until an MCT of G_v and the set of edges satisfying Condition 1 with respect to the MCT, that is, set Ω, are obtained.

Algorithm 1

Step 1. Set $G_{v1}=G_v$, $G_{i1}=G_i$, $T_{u1}=T_u$ and $m=1$. Go to step 2.

Step 2. Find L_{um} in G_{im}. If $L_{um}\neq\phi$, go to step 3. Otherwise go to step 7.

Step 3. Find, in G_{vm}, the fundamental cutsets defined by the edges of L_{um} with respect to T_{um}. Let F_{um} be the set of the edges in the cutsets. If F_{um} contains no edge in C_u, go to step 4. Otherwise go to step 5.

Step 4. Delete and contract the edges of F_{um} from G_{vm} and G_{im}, respectively, to obtain G_{vm+1} and G_{im+1}. Obtain also T_{um+1} from T_{um} by contracting the edges of L_{um}. Set $m=m+1$ and return to step 2.

Step 5. Choose an edge, say x, which is in $C_u\cap F_{um}$. In G_v, x is in the fundamental cutset defined by some edge in L_{um}. Let y be this edge. Obtain $T_w=T_u\cup y-x$ by an elementary tree transformation. If $m=1$, set $T_u=T_w$ and return to step 1. If $m>1$, go to step 6.

Step 6. $C_w\cap F_{um-1}\neq\phi$ in this case. Set $T_u=T_w$, $C_u=C_w$ and $m=m-1$. Return to step 5.

Step 7. Set $\Omega=F_{u1}\cup F_{u2}\cup\cdots\cup F_{um-1}$, and stop.

We can get Γ by the dual algorithm derived from Algorithm 1 by exchanging "G_v" and "G_i", and replacing "u" by "h" and "Ω" by "Γ". From step 3 we see that Ω and Γ are mutually disjoint. If there is no common tree, $L_{u1}\neq\phi$ or $L_{h1}\neq\phi$ and $\Omega\neq\phi$ or $\Gamma\neq\phi$. Then, $\xi_u\neq0$ or $\xi_h=\eta_u\neq0$, and we see the sufficiency of Theorem 3.

If there exists a common tree, an MCT must be a common tree. We can obtain a common tree by Algorithm 1, if one exists. If the MCT obtained by Algorithm 1 is not a common tree, there is no common tree

The set of edges which are neither in Ω nor in Γ is denoted by Λ Then the edges of the two-graphs are partitioned into three sets Ω, Γ and Λ. The two-graphs associated with this partition are denoted by $G_v(\Omega, \Gamma, \Lambda)$ and $G_i(\Omega, \Gamma, \Lambda)$. At least one common tree exists for $G_v(0, 1, \Lambda)$ and $G_i(1, 0, \Lambda)$.

3. Partition of Two-Graphs and Partially Ordered Set of Edges

Similar to the fact that a graph consisting of a pair of disjoint trees may have a finer structure,[15] two-graphs with a common tree may have a finer structure which follows a part of the conditions described in the previous section. The edges of two-graphs can be partitioned into several sets and a partial ordering can be given to these sets. The partition is based on the concept of the canonical decomposition of a bipartite graph.[16),17] For the simplicity of the notation, let G_v and G_i be two-graphs with a common tree. To find the structure of two-graphs, consider the sets of edges H and K satisfying the following Conditions 3 and 4 respectively, for an edge x with respect to a common tree T, which we assume has been found. Note that Condition 1[2] and Condition 3[4] are very much alike.

Condition 3. (i) $x \in H$. (ii) $T \cap H$ is a forest of $G_i(H; 0)$. (iii) $T \cap H$ is a forest of $G_v(H; 1)$. (iv) H is a minimal set satisfying (i)–(iii).

Condition 4. (i) $x \in K$. (ii) $T \cap K$ is a forest of $G_i(K; 1)$. (iii) $T \cap K$ is a forest of $G_v(K; 0)$. (iv) K is a minimal set satisfying (i)–(iii).

The sets H and K can be obtained by the following algorithms. If x is a chord, use Algorithms 2–1 and 2–2, and if a tree-branch, Algorithms 2–3 and 2–4.

Algorithm 2-1

Step 1. Let $D_1 = \{x\}$. Set $m=1$, and $H=D_1$. Go to step 2.

Step 2. Find the fundamental loops in G_i defined by the edges of D_m. If there are edges in the loops which are not in H yet, let D_{m+1} be the set of these edges, and set $H = H \cup D_{m+1}$. Set $m=m+1$ and go to step 3. If there is not such an edge, stop.

Step 3. Find the fundamental cutsets in G_v defined by the edges of D_m. If there are edges in the cutsets which are not in H yet, let D_{m+1} be the set of these edges, and set $H = H \cup D_{m+1}$. Set $m=m+1$ and go to step 2. If there is not such an edge, stop.

Algorithm 2-2

Exchange "G_v" and "G_i" and replace "H" by "K" in Algorithm 2–1.

Algorithm 2-3

Exchange "loops" and "cutsets", and replace "H" by "K" in Algorithm 2–1.

Algorithm 2-4

Exchange further "G_v" and "G_i", and replace "K" by "H" in Algorithm 2–3. From Conditions 3 and 4, the following equations hold for H and K respectively:

$$\rho_v(H;1)=\rho_i(H;0) \tag{19}$$

$$\rho_i(K;1)=\rho_v(K;0) \tag{20}$$

Now we denote the sets of edges H and K obtained by the above algorithms with x chosen at step 1 by H_x and K_x, respectively, to indicate the relation between the sets and the starting edge. Suppose we have such a set H_x, K_x, H_y, K_y and so on for edges x, y and so on, respectively.

Theorem 4

$K_y \supseteq K_x$ if and only if $H_y \supseteq H_x$.

proof: If $H_x \supseteq H_y$, then there is a string of edges $x=x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, x_m=y$ determined by Algorithm 2–1 or 2–4, so that x_{k+1} is in the fundamental loop or cutset defined by x_k. This string can be traced back by Algorithm 2–2 or 2–3.

Now let $M_x=H_x \cap K_x$, $M_y=H_y \cap K_y$ and so on.

Theorem 5

$H_x=H_y$ and $K_x=K_y$, if $y \in M_x$ or $x \in M_y$. The converse also holds.

Proof: If $y \in M_x$, then $H_x \supseteq H_y$, and $K_x \supseteq K_y$. From Theorem 4, then, $K_y \supseteq K_x$ and $H_y \supseteq H_x$, and we immediately have the result. The converse is obvious.

Theorem 6

If $y \notin M_x$ or $x \notin M_y$, then M_x and M_y are mutually disjoint.

Proof: If M_x and M_y have a common edge z, then $M_z=M_x$ and $M_z=M_y$ from Theorem 9. This obviously contradicts $y \notin M_x$ or $x \notin M_y$.

Corollary

$H_x \supset H_y$ if and only if $K_y \supset K_x$. $H_x=H_y$ if and only if $K_x=K_y$. $M_x=M_y$ if and only if $H_x=H_y[K_x=K_y]$.

From Theorem 6 we see that the set of edges of the two-graphs can be partitioned into disjoint subsets M_x, M_y, M_z and so on. A partial ordering can be given to these subsets according to the inclusion relation among K_x, K_y, K_z and so on. That is, $M_x < M_y$ if and only if $K_x \supset K_y$ etc. We define the structure of two-graphs as the partially ordered set of these subsets.

The structure of two-graphs having at least a common tree can be extended to two-graphs without a common tree. In general, Ω and Γ are also partitioned into subsets which are determined by the following algorithm. Consider $G_v(\Omega;1)$ and $G_i(\Omega;0)$. We get W for an edge, say a, in Ω.

Algorithm 3

Step 1. Let $D_1=\{a\}$. Set $m=1$ and $W=D_1$.
Step 2. Find all the nonseparable parts of $G_v(\Omega;1)$ which include any of the edge in D_m. If, in the nonseparable parts, there are edges which have not been in W yet, let D_{m+1} be the set of such edges. Set $W=W\cup D_{m+1}$, $m=m+1$, and go to step 3. If there is no such edge, stop.
Step 3. Find all the nonseparable parts of $G_i(\Omega;0)$ which include any of the edge in D_m. If, in the nonseparable parts, there are edges which have not been in W yet, let D_{m+1} be the set of such edges. Set $W=W\cup D_{m+1}$, $m=m+1$, and go to step 2. If there is no such edge, stop.

The subgraphs formed by the edges of W are called common nonseparable parts of $G_v(\Omega;1)$ and $G_i(\Omega;0)$. We denote the sets obtained by Algorithm 3 for edges a, b, c, etc., by W_a, W_b, W_c, etc., respectively. Ω is partitioned into disjoint subsets W_a, W_b, W_c and so on. The partition of Γ can be determined by the algorithm which is obtained from Algorithm 3, by replacing Ω by Γ and exchanging G_v and G_i. The structure of two-graphs is the partially ordered set of the subsets of Ω, Γ and Λ, as obtained above. In order to obtain the partial ordering, Algorithms 2–1∼2–4 must be modified. Instead of a common tree, a pair of corresponding MCT's of G_v and G_i are used, and if an edge of a common nonseparable part is included in $H[K]$, all the edges in the nonseparable part are added to $H[K]$.

4. Stability of the Structure

In order to remove the network singularity caused by the lack of a common tree, the network topology must be changed. We may add or remove pairs of edges to the two-graphs. Such changes may alter some part of the structure, but leave the other part unaltered. The concept of the stability of the structure given in references 16) and 17) is also applicable to the structure of two-graphs. Desirable changes in the network topology can be derived by use of the structure theory.

First, we show the uniqueness of the structure of two-graphs.

Theorem 7

Suppose G_v and G_i have more than one common tree. The sets H and K given by Conditions 3 and 4 respectively, are uniquely determined regardless of the common tree which is used to define the sets.

Proof. Let T and T' be common trees, and $H[H']$ be the set defined by Condition 3 with respect to $T[T']$. Also let b_H and $b_{\bar{H}}[b'_H$ and $b'_{\bar{H}}]$ be the numbers of the tree-branches of $T[T']$ in H and $\bar{H}$ respectively. We have $b_H+b_{\bar{H}}=b'_H+b'_{\bar{H}}$. In G_v, $b_H=\rho_v(H;1)$ and thus $b'_H\geqq b_H$ from (2). Assume $b'_H>b_H$. Then $b'_{\bar{H}}<b_{\bar{H}}$. From (2), we see that this is impossible since $b_{\bar{H}}=\rho_i(\bar{H};1)$ in G_i. Therefore, $b'_H=b_H=\rho_v(H;1)=\rho_i$

$(H; 0)$. This means $T' \cap H$ is a forest in $G_v(H; 1)$ and $G_i(H; 0)$. Then, from Condition 3, $H \subseteq H'$. Likewise, we can show $H' \subseteq H$ by exchanging the rolls of T and T'. Hence, $H = H'$ and we see H is unique. We can prove the uniqueness of K in the same way.

Corollary 1

Any common tree of G_v and G_i consists of the edges of a common tree of $G_v(H; 1)$ and $G_i(H; 0)$, and of a common tree of $G_v(\bar{H}; 0)$ and $G_i(\bar{H}; 1)$. No elementary common-tree transformation exchanging an edge of H for an edge of $\bar{H}$, is possible.

Corollary 2

The partitioned subsets M_x, M_y, M_z etc. of the set of edges in G_v and G_i are uniquely determined.

Suppose G_v and G_i have no common tree. The sets Ω, Γ and Λ are uniquely determined. The uniqueness of the partition of Ω and Γ can be easily seen from Algorithm 3. The partition of Λ and the partial ordering of the partitioned subsets are unique as shown above. Then:

Theorem 8

The structure of two-graphs is unique.

Let the partition of Ω, Γ and Λ be $\Omega = W_a \cup W_b \cup \cdots$, $\Gamma = V_l \cup V_m \cup \cdots$, and $\Lambda = M_x \cup M_y \cup \cdots$. Then the two-graphs can be written as $G_v(W_a, W_b, \cdots, M_x, M_y, \cdots, V_l, V_m, \cdots)$ and $G_i(W_a, W_b, \cdots, M_x, M_y, \cdots, V_l, V_m, \cdots)$. The ordering of $M_x, M_y, \cdots$ is in accordance with the partial ordering defined in the previous section, that is, if $M_x < M_y$, for example, M_x comes prior to M_y. If neither $M_x < M_y$ nor $M_x > M_y$, the relative positions of M_x and M_y are arbitrary. For the simplicity of the notations, let $G_v(0; M_x; 1)$ $[G_i(1; M_x; 0)] = G_v(0, 0, \cdots, M_x, 1, \cdots, 1, 1, \cdots)$ $[G_i(1, 1, \cdots, M_x, 0, \cdots, 0, 0, \ldots)]$, that is, the graph obtained from $G_v[G_i]$ by deleting [contracting] all the edges in the sets which are less than M_x and by contracting [deleting] all the edges in the sets which are greater than M_y. The edges in the set which is neither less nor greater than M_x are deleted.

From Corollary 1 of Theorem 7 we have:

Theorem 9

A pair of MCT's of G_v and G_i consists of the edges of the MCT's in $G_v(W_a; 1)$ and $G_i(W_a; 0)$, in $G_v(W_b; 1)$ and $G_i(W_b; 0)$, $\cdots$, in $G_v(0; M_x; 1)$ and $G_i(1; M_x; 0)$, in $G_v(0; M_y; 1)$ and $G_i(1; M_y; 0)$, $\cdots$, in $G_v(V_l; 0)$ and $G_i(V_l; 1)$, in $G_v(V_m; 0)$ and $G_i(V_m; 1)$, $\cdots$

Next, we consider the addition of an edge to each of G_v and G_i. Here, by addition we mean the addition of an edge to each of the graphs which does not result in the addition of a self-loop or a bridge.

Case 4–1. If the addition results only in the addition of an edge to each of $G_v(W_a;1)$ $[G_v(W_b;1)]\cdots[G_v(0;M_x;1)]$ $[G_v(0,M_y;1)]\cdots[G_v(V_l;0)]$ $[G_v(V_m;0)]\cdots$ and $G_i(W_a;0)$ $[G_i(W_b;0)]\cdots[G_i(1;M_x;0)]$ $[G_i(1;M_y:0)]\cdots[G_i(V_l;1)]$ $[G_i(V_m;1)]\cdots$, the structure of the new two-graphs is the same as that of the original one.

Case 4–2. If the addition results in the addition of an edge to $G_v(W_a;1)$ $[G_v(0;M_x;1)]$ and $G_i(1;M_x;0)$ $[G_i(V_l;1)]$, then W_a and $M_x[M_x$ and $V_l]$, together with the added edge and the subsets which are less [greater] than M_x, form a new subsets in $\Omega[\Gamma]$.

Case 4–3. Suppose $\rho_v(W_a;1)-\rho_i(W_a;0)=1$ and $\rho_i(V_l;1)-\rho_v(V_l;0)=1$. If the addition results in the addition of an edge to $G_v(W_a;1)$ and $G_i(V_l;1)$, then W_a, V_l and the added edge are partitioned into subsets in Λ.

Case 4–4. If $M_x<M_y$ and the addition results in the addition of an edge to $G_v(0;M_x;1)$ and $G_i(1;M_y;0)$, then M_x and M_y, together with the added edge and the subsets which are greater than M_x and less than M_y, form a new subset in Λ.

Case 4–5. If the addition results in the addition of an edge to $G_i(W_a;0)$ $[G_i(W_a;0)]$ $[G_i(1;M_x;0)]$ and $G_v(0;M_x;1)$ $[G_v(V_l;0)]$ $[G_v(V_l;0)]$, then the added edge forms a new subset in Λ which is greater than $W_a[W_a]$ $[M_x]$ and less than $M_x[V_l]$ $[V_l]$.

Case 4–6. Suppose $M_x<M_y$ or M_x and M_y are mutually independent. If the addition results in the addition of an edge to $G_i(1;M_x;0)$ and $G_v(0;M_y;1)$, then the added edge forms a new subset in Λ which is greater than M_x and less than M_y.

Case 4–7. If the addition results in the addition of an edge to $G_v(W_a;1)$ $[G_v(V_l;0)]$ and $G_i(W_b;0)$ $[G_i(V_m;1)]$, then W_a and $W_b[V_l$ and $V_m]$ together with the added edge form a subset in $\Omega[\Gamma]$.

The other part of the structure is not changed in Case 4–2～Case 4–7.

In Case 4–3, a part of the network singularity is removed, and $G_v(W_a;1)\cup G_v(V_l;0)\cup\{\tau\}$ and $G_i(W_a;0)\cup G_i(V_l;1)\cup\{\tau\}$, where τ is the added edge, have a common tree including τ as a tree-branch. An interesting fact is that W_a and V_l are generally decomposed into smaller subsets in Λ. If $\Omega=\phi$ and $\Gamma=\phi$, a common tree exists for the two-graphs, and the network equations are solvable, provided that there is no special relation among the element values. The partition of Λ is useful in solving the network equations, as will be shown in the next section.

5. Decomposition of the Coefficient Matrix

Let us consider again the network equations given in (7). Eliminating $\boldsymbol{v}_1$ or $\boldsymbol{i}_1$ from (7), we get, as the coefficient matrix for the remaining unknown variables, a matrix whose nonzero elements are located at the same positions as the nonzero elements of matrix $\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{Q} \end{bmatrix}$. We derive a canonical block-triangular form of this coefficient matrix as shown below.

$$\begin{array}{c|cccc|c} \Phi & & \Lambda & & \Gamma & \text{number of rows} \\ \hline \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{X} & \mu_v(\Omega;\,1)+\rho_i(\Omega;\,0)<|\Omega| \\ & \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{X} & \\ & & & \vdots & \vdots & \mu_v(0,\,1,\,\Lambda)+\rho_i(1,\,0,\,\Lambda)=|\Lambda| \\ & & & \boldsymbol{X} & \boldsymbol{X} & \\ \mathbf{0} & & & & \boldsymbol{X} & \mu_v(\Gamma;\,0)+\rho_i(\Gamma;\,1)>|\Gamma| \end{array} \tag{21}$$

All the nonzero elements are included in the submatrices indicated by $\boldsymbol{X}$. The edges of the two-graphs correspond to the columns as indicated above the matrix. If there is no special relation among the element values in the network, the upper block-triangular matrix has the following properties. The submatrices at the upper-left and lower-right corners are rectrangular and not square if $\Omega\neq\phi$ and $\Gamma\neq\phi$. They have at least one nonzero major determinant. There are excess equations for the unknown variables associated with the edges in Γ, but the number of equations to determine the unknown variables associated with the edges in Ω is not enough. The submatrices on the diagonal, except those mentioned above, are square, irreducible and nonsingular. The columns of these blocks on the diagonal correspond to the edges of the sets M_x, M_y, M_z etc. derived in section 3. The rows of the blocks correspond to the equations which are derived from the fundamental loops and cutsets defined by the edges in the sets. The corresponding MCT's of G_v and G_i are used to obtain **B** and $\boldsymbol{Q}$, arespectively. The The positions of the square blocks are determined in accordance with the partial ordering. Thus if $M_x<M_y$, the block corresponding to M_x is located to the upper-left of that corresponding to M_y.

If a common tree exists and if there is no special relation among the element values, the network equations are solvable. Then, the block-triangular form of the coefficient matrix is useful. The equations can be solved block by block, starting from the one at the lower-right corner. In general, this procedure becomes easier, if there are more blocks.

6. Classification of Network Solvability

On the basis of the results obtained in the previous sections, we give a classification of network solvability. We first consider the solvability of (7). For the brevity of the description we use the following notations.

CTE : the cases where at least a common tree of G_v and G_i exists.

CTN : the cases where no common tree of G_v and G_i exists.

SE : the cases where the unique solution of (7) exists.

SNP : the cases where no solution of (7) is possible for nonzero independent source voltages or currents.

SND : the cases where some of the unknown variables in (7) are not determinate.

RE : the cases where there exists special relations among the element values.

RN : the cases where there is no special relation among the element values.

Case 6–1. CTE. Usually we have Case 6–1–*a*

Case 6–1–*a*. RN and then it follows SE.

Case 6–1–*b*–1. RE and SNP.

Case 6–1–*b*–2. RE and SND. Example 6.1 in Fig. 1.

Case 6–1–*b*–3. RE and SE. The relation among the element values may have no effect on solvability, and therefore, we have such a case as this.

Case 6–2. The condition in Theorem 1 is satisfied but CTN. Usually we have Case

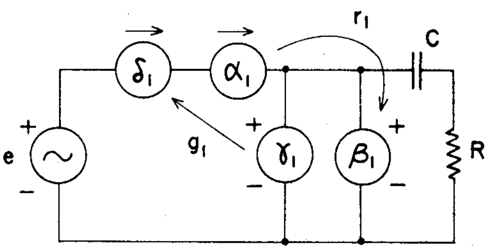

Fig. 1. Example 6.1. $r_1 g_1=1$.

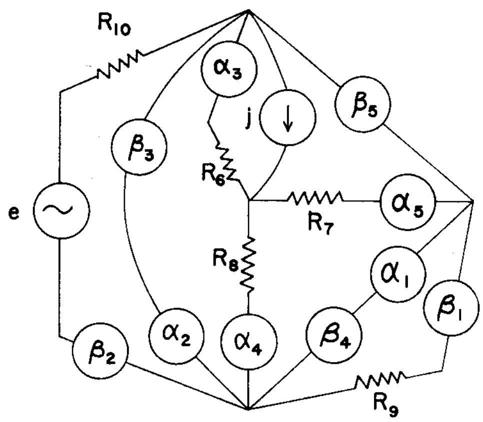

Fig. 2. Example 6.2.

6–2–*a*–1.

Case 6–2–*a*–1. RN, then it follows SNP and SND. An example is given in Fig. 2 (Example 6.2.).

Case 6–2–*a*–2. RN and SND. Example 6.3 in Fig. 3. Voltage $v_{\beta 1}$ is indeterminate.

Case 6–2–*b*–1. RE and SND. Example 6.4 in Fig. 4. Voltage $v_{\beta 3}$ is indeterminate.

Case 6–2–*b*–2. RE, SNP and SND. This case follows from Case 6–2–*a*–1, if the relations have no effect on the solvability.

Case 6–3. The condition in Theorem 1 is not satisfied. CTN. Usually we have Case 6–3–*a*–1.

Case 6–3–*a*–1. RN and SNP. Example 6.5 in Fig. 5.

Case 6–3–*a*–2. RN and SND. An example can be obtained by combining Example 6.3 in Fig. 3 and Example 6.5 in Fig. 5.

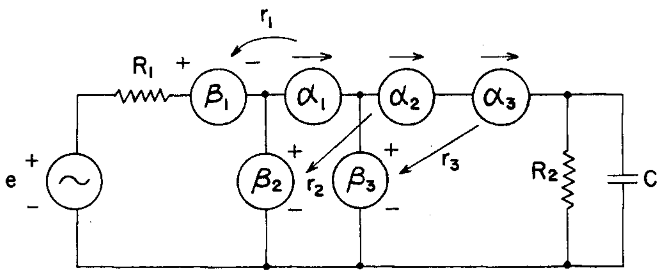

Fig. 3. Example 6.3.

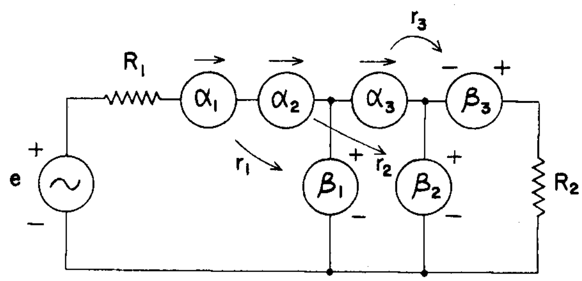

Fig. 4. Example 6.4. $r_1 = r_2$.

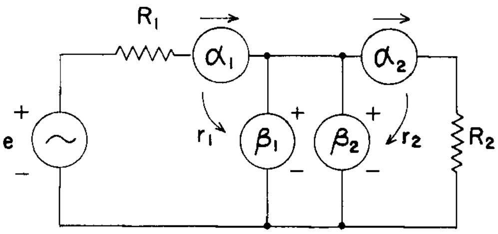

Fig. 5. Example 6.5.

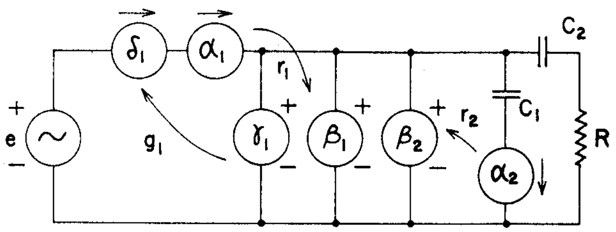

Fig. 6. Example 6.6.

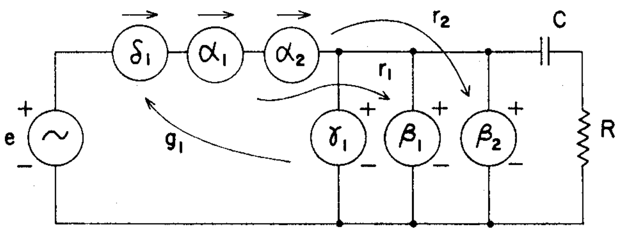

Fig. 7. Example 6.7. $r_1g_1=1$ and $r_1=r_2$.

Case 6–3–*a*–3. RN and SE. Example 6.6 in Fig. 6. Another example can be derived from Example 6.3 in Fig. 3 by contracting resistor R_1.

Case 6–3–*b*–1. RE and SNP. This case follows from Case 6–3–*a*–1, if the relations have no effect on the solvability.

Case 6–3–*b*–2. RE and SND. Example 6.7 in Fig. 7. If $r_1g_1=1$ and $r_1=r_2$, $v_{\beta 1}$ is indeterminate.

Case 6–3–*b*–3. RE and SE. Example 6.5 in Fig. 5 with $r_2=R_2$.

No other case than those given above, is possible.

Now, we consider the variables in Set 2 which is specified in section 2. If the condition in Theorem 1 is satisfied, there is a tree of G^* containing all the voltage sources, independent or controlled, but no current sources. Thus, if $\boldsymbol{v}_1$ and $\boldsymbol{i}_1$ are obtained, the unknown variables in Set 2 can be deternined from Kirchhoff's laws. Next, let us consider the cases where the condition in Theorem 1 is violated. Choosing a tree of G^* which contains a maximum of voltage sources and a minimum of current sources, we get $n_2-\mu(E_\beta;\, 0)-\rho(E_\delta;\, 1)$ equations for the variables in Set 2, where n_2 is the number of variables in Set 2. Therefore, some of the variables are indeterminate, even if $\boldsymbol{v}_1$ and $\boldsymbol{i}_1$ are obtained. In general $\boldsymbol{v}_1$ and $\boldsymbol{i}_1$ themselves can be obtained in very special cases only.

7. Concluding Remarks

Although the existence of a common tree is often guaranteed by checking the condition of Theorem 2, it is not true for an active network model using norators and nullators. The discussions in sections 2, 3, 4 and 5 are applicable to the two-graphs derived from such a network model, giving topological conditions and an algorithm to check the solvability.

Much similarity between the matching in a bipartite graph and the common tree in two-graphs can be recognized. The necessary and sufficient condition (13) or (14) for the existence of a common tree resembles Hall's famous condition for the existence of a complete matching. Equations (17) and (18) are mini=max type relations like König's Theorem. The structure of two-graphs has properties similar to the structure of a bipartite graph.

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