# Reliability Theory of Structures with Strength Degradation in Load History



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By

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**Synopsis**

The theory of structural reliability is developed for repeated loads with due consideration for strength degradation dependent on the load intensity. The probability distribution of the residual strength of the structure is treated as such that modified by successive application of loads in the sense of both the non-failure effect and the strength-degradation effect.

The numerical results of this study show some essential and interesting aspects as to the change in the structural strength and the reliability function through applications of repeated loads.

## 1. Introduction

In a rational design procedure of structures, it is of intrinsic importance to establish a method to make an adequate estimation of external loads and resisting strength of structural members and to combine them to realize functionally feasible structures with well-balanced mechanics and economy.

One of the difficulties involved in this subject is that critical external loads have in many cases a random nature, and so does the strength of structural materials. The theory of structural reliability has been developed to provide an appropriate method of analysis of the structural safety in practical design procedures by dealing with the uncertainties indicated above in a unified theory.

The foundation of the reliability theory was laid by A.M. Freudenthal[1] in 1947, which thereafter was developed also by many other researchers. A.M. Freudenthal, J.M. Garrelts and M. Shinozuka[2] presented in 1966 a synthesis of the field and have established the classical reliability theory. A.H.-S. Ang and M. Amin[3] discussed in 1968 the structural reliability under a repetition of loads and showed a monotonic property of the failure rate (termed also hazard function, intensity function or risk function) of structures. Recent efforts in this field have been made to apply these

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theories to more complex structures with high redundancy and to refine them by introducing, for example, unknown variables by which to evaluate artificial errors.

In spite of the remarkable accomplishments described above, some essential problems are left unsolved as to the structural reliability for repeated loads. We can expect a properly designed structure to withstand a strong load, a strong earthquake for example. It may, however, suffer internal damage in spite of its survival, so that the resisting strength may undergo a considerable degradation. In establishing the reliability theory for such a structural behaviour in strong loads, it is required to evaluate two different effects. One is that the survival of the structure in its strong load guarantees that its strength was higher than the load applied, and consequently that the original probability distribution of the strength adopted in design can be modified on the condition of its survival, which raises its reliability. The other is that the structure, however, may have undergone a strength degradation in the strong load, which would lower its reliability for future loads. In this study, these two effects are called, respectively, the non-failure effect and the strength-degradation effect, or simply the degradation effect. In the present paper, the theory of structural reliability is developed for a repetition of loads with due consideration for the non-failure effect and the strength-degradation effect dependent on the load intensity.

Not only the theoretical conclusions but also the numerical results show that the method of analysis in this study is more pertinent to the true structural behaviours in repeated strong loads than other studies so far worked out.

The theory in this study is expected to provide a helpful means in the structural design in determining, for example, the safety factor so that the structure may remain safe against every strong load during its life period. Also, it should be helpful in the prediction of the future reliability of exsisting structures which have withstood some past loads with known intensities.

## 2. Reliability of Load-Degraded Structures

### 2.1. Basic Concepts of the Reliability Theory[2)]

Some basic concepts of the structural reliability theory relevant to the present study are introduced below.

When the structural reliability under a single load is in question, then the probability of structural safety or the reliability function $L$ is represented by

$$L = \int_0^\infty F_S(y) f_R(y) dy \qquad \cdots\cdots\cdots\cdots\cdots (1)$$

where $F_S(y)$ is the probability distribution of the load intensity and $f_R(y)$ is the probability density of the strength of the structure.

Eq. (1) is applicable to a case where failure can occur either in symmetrical positive or negative ranges of load and to a case where failure occurs exclusively under a positive or negative load. In other cases, a simple modification is to be made.[3]

When the structural reliability for a series of loads is to be discussed, it is assumed that the loads applied in a sequence are independent although they may have different distribution functions and that the failure of a composite structure occurs when any one of its components fails.

If the loads are applied either in equal intervals or at prescribed instants, the life of the structure can be measured in terms of the number $N$, of load applications. In this case, the reliability function is defined as

$$L_N(n) = \boldsymbol{P}(N>n) \qquad \cdots\cdots(2)$$

And the probability of failure in the interval $[1, n]$, or the probability distribution of the number of load applications up to failure is represented by

$$F_N(n) = 1-L_N(n) = \boldsymbol{P}(N\leqslant n) \qquad \cdots\cdots(3)$$

Furthermore, let $f_N(n)$ and $h_N(n)$ be introduced so that $f_N(n)$ is the probability that a structure will fail exactly at the $n$ th application of load. Thus

$$f_N(n) = \boldsymbol{P}(N=n) = F_N(n)-F_N(n-1) \qquad \cdots\cdots(4)$$

whereas $h_N(n)$ is the probability that the structure which has survived $n-1$ applications of load will fail at the $n$ th load application.

$$\begin{aligned} h_N(n) &= \boldsymbol{P}(N=n\,|\,N>n-1) \\ &= \frac{\boldsymbol{P}\{(N=n)\cap(N>n-1)\}}{\boldsymbol{P}(N>n-1)} \\ &= \frac{f_N(n)}{L_N(n-1)} = \frac{f_N(n)}{1-F_N(n-1)} \end{aligned} \qquad \cdots\cdots(5)$$

Here $f_N(n)$ is termed the mortality function and $h_N(n)$, the failure rate; they play an important role in the reliability theory.

### 2.2 Classical Reliability Theory for Repeated Loads

In this section, a brief survey is made on the reliability theory so far developed in relation to repeated loads. For further detail, readers are referred to the reference.[2),3)]

### 2.2.1. Structure without Strength Degradation

First, we discuss the case where the probability distribution $F_S(x)$ of the load and that $F_R(y)$ of the strength of the structure are both time invariant, implying that all loads applied in a sequence have identical probability distribution, and that the material and the structure suffer no degradation.

Then the probability distribution of the maximum $S^*(n)$ of these $n$ loads is represented by

$$F_{S^*}(x) = \{F_S(x)\}^n \qquad \cdots\cdots(6)$$

In reference 2) it is stated that since the structure undergoes no degradation, Eq. (1) is applicable to this case and that by substituting Eq. (6) into Eq. (1) one can obtain the reliability function $L_N(n)$ of structures subjected to $n$ load applications as

$$L_N(n) = \int_0^\infty F_{S^*}(y) f_R(y) dy = \int_0^\infty \{F_S(y)\}^n f_R(y) dy \qquad \cdots\cdots(7)$$

Then by virtue of Eqs. (3) and (7), we have

$$\begin{aligned} F_N(n) &= \int_0^\infty [1 - \{F_S(y)\}^n] f_R(y) dy \\ &= \int_0^\infty F_R(x) f_{S^*}(x) dx \qquad \cdots\cdots(8) \end{aligned}$$

where $f_{S^*}(x) = \dfrac{dF_{S^*}(x)}{dx} = n\{F_S(x)\}^{n-1} f_S(x)$

denotes the probability density of the maximum load.

Expressions for $f_N(n)$ and $h_N(n)$ are reduced from Eqs. (4)-(8); i.e.,

$$f_N(n) = \int_0^\infty \{F_S(x)\}^{n-1} \bar{F}_S(x) f_R(x) dx \qquad \cdots\cdots(9)$$

and

$$h_N(n) = \frac{\int_0^\infty \{F_S(x)\}^{n-1} \bar{F}_S(x) f_R(x) dx}{\int_0^\infty \{F_S(x)\}^{n-1} f_R(x) dx} \qquad \cdots\cdots(10)$$

where $\bar{F}_S(x) = 1 - F_S(x)$

Since the degradation effect is not considered in Eq. 7, the reliability function is dominated by the maximum of the repeated loads regardless of their relative intensities or of the order of their occurrences. Hence in this case, the non-failure effect need not be considered.

2.2.2. Structure with Degradation Dependent on Number of Load Applications

In the following, there are presented results on the degradation effect dependent exclusively on the number of load applications.

Let

$$R(k) = R(1)\psi(k) \qquad \cdots\cdots\cdots\cdots\cdots(11)$$

in which $\psi(k)$ is usually a non-increasing positive function of $k$ because in most cases, material degrades as a result of load applications.

Under these conditions, the reliability function is obtained in the same manner as the deduction of Eq. (7); i.e.,

$$L_N(n) = \int_0^\infty [\prod_{k=1}^{n} F_{S(k)}\{y(1)\psi(k)\}] f_{R(1)}\{y(1)\} dy(1) \qquad \cdots\cdots\cdots\cdots\cdots(12)$$

Then other reliability parameters introduced in 2.1. are given by

$$F_N(n) = \int_0^\infty [1-\prod_{k=1}^{n} F_{S(k)}\{y(k)\}] f_{R(1)}\{y(1)\} dy(1) \qquad \cdots\cdots\cdots\cdots\cdots(13)$$

$$f_N(n) = \int_0^\infty [\prod_{k=1}^{n-1} F_{S(k)}\{y(k)\}] \bar{F}_{S(n)}\{y(n)\} f_{R(1)}\{y(1)\} dy(1) \qquad \cdots\cdots\cdots(14)$$

$$h_N(n) = \frac{f_N(n)}{\int_0^\infty [\prod_{k=1}^{n-1} F_{S(k)}\{y(k)\}] f_{R(1)}\{y(1)\} dy(1)} \qquad \cdots\cdots\cdots\cdots\cdots(15)$$

The type of degradation characterized by Eq. (11) would be appropriate when the successive loads have a constant intensity or are distributed in a narrow range. However, when the loads are more dispersive, this assumption is no longer appropriate, since the rate of strength degradation will depend upon the intensities of loads. In the latter case, the results in Eqs. (12)-(15) would underestimate the degradation effect for strong loads and overestimate it for weak loads.

## 2.3. Reliability with Strength Degradation Dependent on the Load Intensity

In this section, we discuss the strength degradation dependent on the load intensity, the structural reliability based on such a degradation behaviour and also on the non-failure effect. These cases have not been considered in the previous section.

2.3.1. Assumption on Strength Degradation

Some structural materials may undergo no strength degradation against load applications, while others may be sensitively affected by them. Materials science points out that the failure process of a material is intuitively stochastic. We shall consider, however, that the strength of structural materials subjected to a load $S$ degrades in proportion to a deterministic function $\varphi(S)$ dependent on $S$.

The initial strength of the structural material shall be represented by $R_0$, and the residual strength after its survival in $N$ load applications $S_1, S_2, \cdots, S_N$, by $R_N(S_1, S_2, \cdots, S_N)$, or simply $R_N$.

Modes of strength degradation throughout the present study shall be treated under the following assumption.

ASSUMPTION The rate of strength degradation due to a load $S$ is given by a strength degradation factor $\varphi(S)$ or in short, a degradation factor which depends only upon the intensity of the load and varies in the range $0<\varphi(S)\leq 1$.

By using the above assumption, the following relation can be given:

$$R_N = \varphi(S_N)R_{N-1} \quad \cdots\cdots\cdots\cdots\cdots (16)$$

or

$$R_N(S_1, S_2, \cdots, S_N) = R_0 \prod_{j=1}^{N} \varphi(S_j) \quad \cdots\cdots\cdots\cdots (16a)$$

where $R_N$ is the residual strength after survival in $N$ load applications, and $S_N$ is the $N$ th load. It should be noted that the strength degradation is related to a conditional probability distribution of the residual strength on the phypothesis of survival in the previous load.

We consider the following two typical modes of strength degradation.

[A] Poisson Pattern Degradation

The strength degradation factor can be presumed from the relation between the internal failure (or damage) of materials and the number of occurence of "cracks" whose random process is assumed to be of a Posison pattern. Then the analytical procedure shown in Appendix results in

$$e^{-\lambda V} \lesssim \frac{r_0}{r} \lesssim 1 \quad \cdots\cdots\cdots\cdots\cdots (17)$$

as the ratio of the expected residual strength $r_1/r_0$ where $r_0$ is the initial strength, and $r_1$ is the expected residual strength, and both $\lambda$ and $V$ are constants.

[B] Degradation Mode based on the Experimental Results

The characteristics of strength degradation of concrete have been investigated by Y. Niwa, W. Koyanagi and K. Nakagawa[4] by means of experiments on failure process of concrete. They explain the relation between the stress history and the residual strength as follows.

"The change in the characteristics of material strength is represented in term of the ratio of the residual strength $\gamma$ defined by

$$r = \frac{\text{Uniaxial compressive strength after survival in the initial load}}{\text{Uniaxial compressive strength under the initial load}}$$

The strength ratio $r$ shows the behaviour as shown in Fig. 1. This figure shows that the residual strength in a uniaxial compressive load is hardly changed when the initial load is below 60% of the ultimate strength of the specimen. However, in a higher initial load the strength ratio $r$ reaches about 0.8 when the initial load is nearly the ultimate strength."

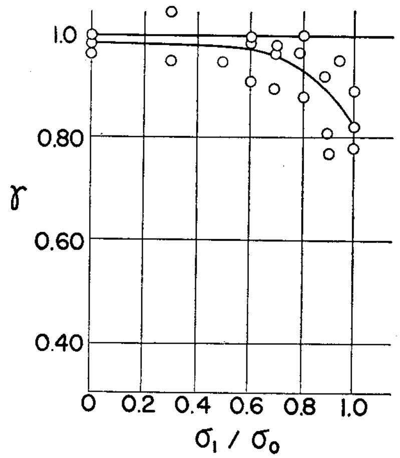


Fig. 1. Experimental Result on the Residual Strength[4]
$\sigma_1/\sigma_0$=uniaxial compressive stress history.

The foregoing discussions lead us to two typical types of the strength degradation factor $\varphi(x)$. One has an exponential relation between the applied load $x$ and $\varphi(x)$; and the other for which the strength degradatoin can be neglected when the applied load is small in comparison with its ultimate strength, but when the load exceeds a certain definite level, the strength decreases by a considerable extent.

These two types of $\varphi(x)$ are formulated as follows:

[A]
$$\varphi_A(x) = \exp\left(-c_A \cdot \frac{x}{r_m}\right) \qquad \cdots\cdots\cdots\cdots\cdots(18)$$

$c_A$: non-dimensional parameter

[B]
$$\varphi_B(x) = \frac{1}{2}\left\{1 - \mathrm{erf}\left(\frac{\frac{x}{r_m} - \xi}{\sqrt{2}\, c_B}\right)\right\} \qquad \cdots\cdots\cdots\cdots\cdots(19)$$

$r_m$: mean value of the initial strength

$c_B$, $\xi$: non-dimensional parameters

The expression for $\varphi_A(x)$ in Eq. (18) has been determined by reference to Eq. (A–1), in which the most severe degradation mode has been adapted ($\mu=1$); and $\lambda$ has naturally been considered to be proportional to the applied load $x$.

The physical meaning of $\varphi_B(x)$ in Eq. (19) is that the strength is assumed to reduce by half of its initial value when the applied olad $x$ is equal to $\xi r_m$. Fig. 2. illustrates the strength degradation factors $\varphi_A(x)$ and $\varphi_B(x)$.

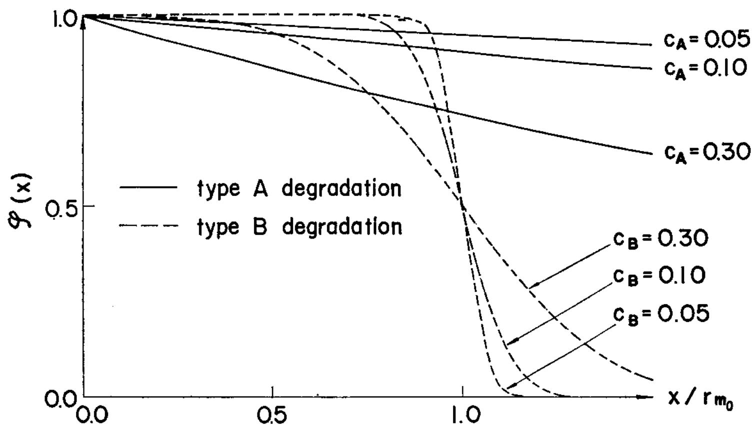


Fig. 2. The Strength Degradation Modes.

2.3.2. Reliability Formulation with Non-Failure Effect and Degradation Effect

(1) Reliability for a Sequence of Future Random Loads

Given the conditional probability distribution $F_{R_{n-1}}(x)$ of the residual strength of the structure on the hypothesis of its survival in the foregoing $n-1$ loads, the failure rate $h_N(n)$ defined by Eq. (5) is represented by

$$h_N(n) = \int_0^\infty F_{R_{n-1}}(x) f_{S_n}(x) dx \qquad \cdots\cdots(20)$$

where $f_{S_n}(x)$ is the probability density of the $n$ th load.

$$L_N(n) = \prod_{k=1}^{n} \{1-h_N(k)\} \qquad \cdots\cdots(21)$$

$$F_N(n) = 1-\prod_{k=1}^{n} \{1-h_N(k)\} \qquad \cdots\cdots(22)$$

$$f_N(n) = h_N(n) \prod_{k=1}^{n-1} \{1-h_N(k)\} \qquad \cdots\cdots(23)$$

These results have been derived on the assumption that a sequence of the loads applied are mutually independent random variables which are also independent of the structural strength. Derivation of the probability distribution of the residual strength is made in the next chapter.

(2) Reliability for a Sequence of Future Loads with Past Loads of Known Intensities

When a structure has survived with strength degradation in a sequence of past loads $S_1, S_2, \cdots, S_l$ of known intensities $s_1, s_2, \cdots, s_l$, and its reliability for the first future load $S_1$ is in question, the problem involves the conditional probability distribution, $F^{(l)}_{R_0^*}(x; s_1, s_2, \cdots, s_l)$, of the residual strength on the hypothesis of structural survivals under $l$ past loads described above. Then the "conditional reliability" and the "conditional failure rate" are to be defined as such that

$$L_N^{(l)}(1; s_1, s_2, \cdots, s_l) = 1-h_N^{(l)}(1; s_1, s_2, \cdots, s_l) \quad \cdots\cdots\cdots\cdots\cdots(25)$$

$$h_N^{(l)}(1; s_1, s_2, \cdots, s_l) = \int_0^\infty F^{(l)}_{R_0^*}(x; s_1, s_2, \cdots, s_l) f_{S_1}(x)dx \quad \cdots\cdots\cdots\cdots(24)$$

The conditional probability distribution $F^{(l)}_{R_0^*}(x; s_1, s_2, \cdots, s_l)$ is also discussed in the next chapter.

The reliability for a sequence of future loads is obtained with $h_N^{(l)}(1; s_1, s_2, \cdots, s_l)$ which can be given by substituting Eq. (25) into Eq. (21).

## 3. Probability Distribution of Residual Strength

### 3.1. Definitions

Probabilistic parameters related to residual strength and load histories involved in the analysis in this chapter are indicated below.

| | |
|---|---|
| $S_1, S_2, \cdots, S_N$ | =sequence of loads. |
| $R_0$ | =initial strength of structure. |
| $R_N$, $R_N(S_1, S_2, \cdots, S_N)$ | =residual strength after survival in $N$ load applications. |
| $\tilde{R}_N$ | =residual strength after survival in $N$ load applications on the hypothesis that the structure has withstood the $N+1$ th load. |
| $F_{R_0}(x)$ | =probability distribution of $R_0$. |
| $F_{R_k}(x)$ | =conditional probability distribution of $R_k$. |
| $F_{\tilde{R}_k}(x)$ | =conditional probability distribution of $\tilde{R}_k$. |
| $F_{R_k}(x; s_1, s_2, \cdots, s_k)$ | =conditional probability distribution of $R_k$ under the condition that $S_1=s_1, S_2=s_2, \cdots, S_k=s_k$. |
| $F_{\tilde{R}_k}(x; s_1, s_2, \cdots, s_{k+1})$ | =conditional probability distribution of $\tilde{R}_k$ under the condition that $S_1=s_1, S_2=s_2, \cdots, S_{k+1}=s_{k+1}$. |
| $F_{S_k}(x)$ | =probability distribution of the $k$ th load. |

$f_{R_0}(x)$ = probability density of $R_0$.
$f_{R_k}(x)$ = conditional probability density of $R_k$.
$f_{\tilde{R}_k}(x)$ = conditional probability density of $\tilde{R}_k$.
$f_{R_k}(x;\, s_1, s_2, \cdots, s_k)$ = conditional probability density of $R_k$ under the condition that $S_1=s_1$, $S_2=s_2$, $\cdots$, $S_k=s_k$.
$f_{\tilde{R}_k}(x;\, s_1, s_2, \cdots, s_{k+1})$ = conditional probability density of $\tilde{R}_k$ under the condition that $S_1=s_1$, $S_2=s_2$, $\cdots$, $S_{k+1}=s_{k+1}$.
$f_{S_k}(x)$ = probability density of the $k$ th load.
$f_{S_1,S_2,\cdots,S_k}(s_1, s_2, \cdots, s_k)$ = joint probability density of $k$ loads.
$\mathfrak{D}$ = domain of variation of loads.

The load can vary in compressive or tensile fields. When the characteristics of compressive and tensile strengths are symmetric, it suffices to deal with the absolute value of the load, which shall be the case treated in the subsequent discussions.

Then the domain of load $s$ is given as $0 \leqslant s < \infty$

It is noticed that $S$ and $R$ represent random variables of load and strength, while $s$ and $r$ represent deterministic variables associated with them.

### 3.2. Analytical Procedure

3.2.1. Case with a Sequence of Future Loads

The conditional probability distribution, $F_{R_1}(x)$, of the residual strength conditional on the survival in the first possible load in future is expressed as

$$F_{R_1}(x) = \boldsymbol{P}[R_1 \leqslant x \mid \text{ no failure in } S_1] = \frac{\boldsymbol{P}[(R_1 \leqslant x) \cap (\text{no failure in } S_1)]}{\boldsymbol{P}[\text{no failure in } S_1]} \quad \cdots\cdots\cdots\cdots\cdots (26)$$

But

$$\begin{aligned} &\boldsymbol{P}([R_1 \leqslant x) \cap (\text{no failure in } S_1)] \\ &= \boldsymbol{P}[\bigcup_{s_1 \in \mathfrak{D}} \{(R_1 \leqslant x) \cap (R_0 > s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1)\}] \\ &= \boldsymbol{P}[\bigcup_{s_1 \in \mathfrak{D}} \{(R_0 \varphi(s_1) \leqslant x) \cap (R_0 > s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1)\}] \\ &= \sum_{s_1 \in \mathfrak{D}} \boldsymbol{P}[(R_0 \varphi(s_1) \leqslant x) \cap (R_0 > s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1)] \\ &= \sum_{s_1 \in \mathfrak{D}} \boldsymbol{P}\left[\left(s_1 < R_0 \leqslant \frac{x}{\varphi(s_1)}\right) \cap (s_1 < S_1 \leqslant s_1 + ds_1)\right] \\ &= \int_0^\infty \left[F_{R_0}\left(\frac{x}{\varphi(s_1)}\right) - F_{R_0}(s_1)\right] f_{S_1}(s_1) H(x - s_1 \varphi(s_1)) ds_1 \quad \cdots\cdots\cdots\cdots (27) \end{aligned}$$

and

$$
\begin{aligned}
&\boldsymbol{P}[\text{no failure in } S_1] \\
&= \boldsymbol{P}[\bigcup_{s_1 \in \mathfrak{D}} (R_0 > s_1) \cap (s_1 < S_1 \leq s_1 + ds_1)] \\
&= \sum_{s_1 \in \mathfrak{D}} \boldsymbol{P}[(R_0 > s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1)] \\
&= \int_0^\infty [1 - F_{R_0}(s_1)] f_{S_1}(s_1)] ds_1 \\
&= 1 - \int_0^\infty F_{R_0}(s) f_{S_1}(s_1) ds_1 \qquad \cdots\cdots\cdots\cdots\cdots (28)
\end{aligned}
$$

where $H(x)$ is the Heaviside step function.
Then Eq. (26) yields

$$
F_{R_1}(x) = \frac{\int_0^\infty \left[ F_{R_0}\left(\frac{x}{\varphi(s_1)}\right) - F_{R_0}(s_1) \right] f_{S_1}(s_1) H(x - s_1 \varphi(s_1)) ds_1}{1 - \int_0^\infty F_{R_0}(s_1) f_{S_1}(s_1) ds_1} \qquad \cdots\cdots\cdots (29)
$$

Eq. (29) accounts for the union of all possible cases illustrated in Fig. 3.

The foregoing method can be extended to the discussion of the conditional probability distribution, $F_{R_2}(x)$, of the residual strength after survival in two future load applications, which leads to

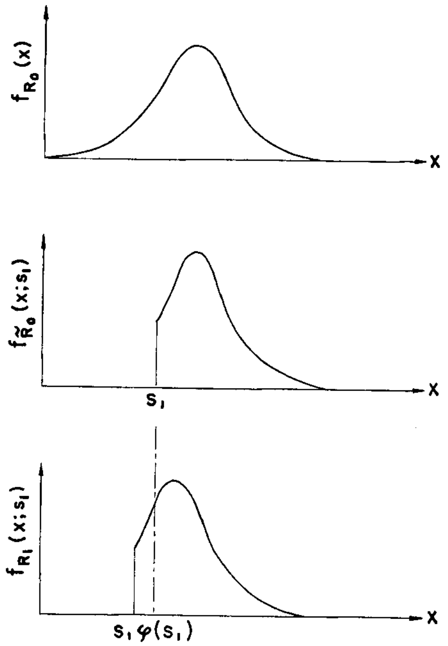


Fig. 3. Illustration of the Probability Distribution of the Residual Strength.

$$F_{R_2}(x) = \boldsymbol{P}[R_2 \leqslant x \mid \text{no failure in } S_1 \text{ and } S_2] = \frac{\boldsymbol{P}[(R_2 \leqslant x) \cap (\text{no failure in } S_1 \text{ and } S_2)]}{\boldsymbol{P}[\text{no failure in } S_1 \text{ and } S_2]} \quad \cdots\cdots(30)$$

where

$$\begin{aligned}
&\boldsymbol{P}[(R_2 \leqslant x) \cap (\text{no failure in } S_1 \text{ and } S_2)] \\
&= \boldsymbol{P}[\bigcup_{\substack{s_1 \in \mathfrak{D} \\ s_2 \in \mathfrak{D}}} \{(R_2 \leqslant x) \cap (R_0 < s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap (R_1 > s_2) \cap (s_2 < S_2 \leqslant s_2 + ds_2)\}] \\
&= \boldsymbol{P}[\bigcup_{\substack{s_1 \in \mathfrak{D} \\ s_2 \in \mathfrak{D}}} \{(R_1 \varphi(s_2) \leqslant x) \cap (R_0 > s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap (R_1 > s_2) \\
&\qquad \cap (s_2 < S_2 \leqslant s_2 + ds_2)\}] \\
&= \boldsymbol{P}\left[\bigcup_{\substack{s_1 \in \mathfrak{D} \\ s_2 \in \mathfrak{D}}} \left\{\left(s_2 < R_1 \leq \frac{x}{\varphi(s_2)}\right) \cap (R_0 > s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap (s_2 < S_2 \leqslant s_2 + ds_2)\right\}\right] \\
&= \boldsymbol{P}\left[\bigcup_{\substack{s_1 \in \mathfrak{D} \\ s_2 \in \mathfrak{D}}} \left\{\left(\frac{s_2}{\varphi(s_1)} < R_0 \leqslant \frac{x}{\varphi(s_2)\varphi(s_1)}\right) \cap (R_0 < s_1) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \right.\right. \\
&\qquad \left.\left. \cap (s_2 < S_2 \leqslant s_2 + ds_2)\right\}\right] \\
&= \boldsymbol{P}\left[\bigcup_{s_1 < \frac{s_2}{\varphi(s_1)}} \left\{\left(\frac{s_2}{\varphi(s_1)} < R_0 \leq \frac{x}{\varphi(s_1)\varphi(s_2)}\right) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap (s_2 < S_2 \leq s_2 + ds_2)\right\}\right. \\
&\qquad \left. + \bigcup_{\frac{s_2}{\varphi(s_1)} < s_1} \left\{\left(s_1 < R_0 \leq \frac{x}{\varphi(s_1)\varphi(s_2)}\right) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap (s_2 < S_2 \leqslant s_2 + ds_2)\right\}\right] \\
&= \sum_{s_1 < \frac{s_2}{\varphi(s_1)}} \boldsymbol{P}\left[\frac{s_2}{\varphi(s_1)} < R_0 \leq \frac{x}{\varphi(s_1)\varphi(s_2)}\right] \cdot \boldsymbol{P}[(s_1 < S_1 \leq s_1 + ds_1) \cap (s_2 \leq S_2 \leq s_2 + ds_2)] \\
&\qquad + \sum_{\frac{s_2}{\varphi(s_1)} < s_1} \boldsymbol{P}\left[s_1 < R_0 \leqslant \frac{x}{\varphi(s_1)\varphi_2(s)}\right] \cdot \boldsymbol{P}[(s_1 < S_1 \leqslant s_1 + ds_1) \cap (s_2 < S_2 \leq s_2 + ds_2)] \\
&= \iint_{\mathfrak{D}_B} \left[F_{R_0}\left(\frac{x}{\varphi(s_1)\varphi(s_2)}\right) - F_{R_0}\left(\frac{s_2}{\varphi(s_1)}\right)\right] f_{S_1S_2}(s_1, s_2) H(x - s_2\varphi(s_2)) ds_1 ds_2 \\
&\qquad + \iint_{\mathfrak{D}_C} \left[F_{R_0}\left(\frac{x}{\varphi(s_1)\varphi(s_2)}\right) - F_{R_0}(s_1)\right] f_{S_1S_2}(s_1, s_2) H(x - s_1\varphi(s_1)\varphi(s_2)) ds_1 ds_2 \quad \cdots\cdots(31)
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{P}[\text{no failure in } S_1 \text{ and } S_2] \\
&= \iint_{\mathfrak{D}_B} \left[1 - F_{R_0}\left(\frac{s_2}{\varphi(s_1)}\right)\right] f_{S_1S_2}(s_1, s_2) ds_1 ds_2 + \iint_{\mathfrak{D}_C} [1 - F_{R_0}(s_1)] f_{S_1S_2}(s_1, s_2) ds_1 ds_2 \quad \cdots(32)
\end{aligned}$$

in which

$$\mathfrak{D}_B = [s_1\varphi(s_1) < s_2]$$
$$\mathfrak{D}_C = [s_1\varphi(s_1) \geq s_2]$$

The domains of integration $\mathfrak{D}_B$ and $\mathfrak{D}_C$ corresponds, respectively, to (B) and (C) in Fig. 4.

Thus $F_{R_2}(x)$ is represented by

$$F_{R_2}(x) = \frac{\displaystyle\iint_{\mathfrak{D}_B} \left[F_{R_0}\left(\frac{x}{\varphi(s_1)\varphi(s_2)}\right) - F_{R_0}\left(\frac{s_2}{\varphi(s_1)}\right)\right] f_{S_1S_2}(s_1, s_2) H(x - s_2\varphi(s_2)) ds_1 ds_2}{\displaystyle\iint_{\mathfrak{D}_B} \left[1 - F_{R_0}\left(\frac{s_2}{\varphi(s_1)}\right)\right] f_{S_1S_2}(s_1 s_2) ds_1 ds_2}$$

$$\frac{\displaystyle + \iint_{\mathfrak{D}_C} \left[F_{R_0}\left(\frac{x}{\varphi(s_1)\varphi(s_2)}\right) - F_{R_0}(s_1)\right] f_{S_1S_2}(s_1, s_2) H(x - s_1\varphi(s_1)\varphi(s_2)) ds_1 ds_2}{\displaystyle + \iint_{\mathfrak{D}_C} [1 - F_{R_0}(s_1)] f_{S_1S_2}(s_1, s_2) ds_1 ds_2} \qquad \cdots\cdots\cdots\cdots\cdots(33)$$

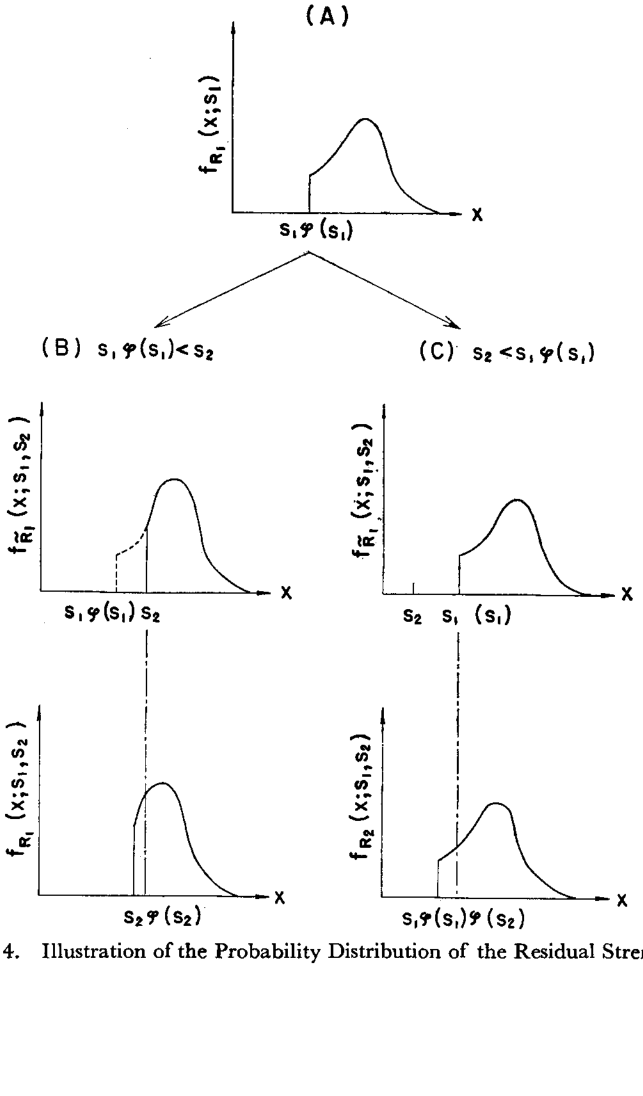


Fig. 4. Illustration of the Probability Distribution of the Residual Strength.

Likeweise, the conditional probability distribution, $F_{R_n}(x)$, of the residual strength after survival in $n$ future load applications can be expressed as

$$F_{R_n}(x) = \boldsymbol{P}[R_n \leq x \mid \text{no failure in } S_1, S_2, \cdots, S_n] = \frac{\boldsymbol{P}[(R_n \leq x) \cap (\text{no failure in } S_1, S_2, \cdots, S_n)]}{\boldsymbol{P}[\text{no failure in } S_1, S_2, \cdots, S_n]} \qquad \cdots\cdots(34)$$

where

$$\begin{aligned}
&\boldsymbol{P}[(R_n \leq x) \cap (\text{no failure in } S_1, S_2, \cdots, S_n)] \\
&= \boldsymbol{P}\Big[\bigcup_{\substack{s_i \in \mathfrak{D} \\ i=1,2,\cdots,n}} \{(R_n \leq x) \cap (R_0 > s_1) \cap (R_1 > s_2) \cap \cdots \cap (R_{n-1} > s_n) \\
&\qquad \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap \cdots \cap (s_n < S_n \leqslant s_n + ds_n)\}\Big] \\
&= \boldsymbol{P}\Big[\bigcup_{s_i \in \mathfrak{D}} \Big\{\Big(R_0 \leq \frac{x}{\prod_{j=1}^{n} \varphi(s_j)}\Big) \cap (R_0 > s_1) \cap \Big(R_0 > \frac{s_2}{\varphi(s_1)}\Big) \cap \cdots \\
&\qquad \cdots \cap \Big(R_0 > \frac{s_n}{\prod_{j=1}^{n-1} \varphi(s_j)}\Big) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap \cdots \\
&\qquad \cdots \cap (s_n < S_n \leqslant s_n + ds_n)\Big\}\Big] \\
&= \boldsymbol{P}\Big[\sum_{k=1}^{2^{n-1}} \bigcup_{s_i \in \mathfrak{D}} \Big\{\Big(\frac{\sigma_{nk}}{\prod_{j=1}^{n} \varphi(s_j)} < R_0 \leqslant \frac{x}{\prod_{j=1}^{n} \varphi(s_j)}\Big) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap \cdots \\
&\qquad \cdots \cap (s_n < S_n \leqslant s_n + ds_n)\Big\}\Big] \\
&= \sum_{k=1}^{2^{n-1}} \int_{D_{1k}} ds_1 \int_{D_{2k}} ds_2 \cdots \int_{D_{nk}} \Big\{F_{R_0}\Big(\frac{x}{\prod_{j=1}^{n} \varphi(s_j)}\Big) - F_{R_0}\Big(\frac{\sigma_{nk}}{\prod_{j=1}^{n} \varphi(s_j)}\Big)\Big\} H(x - \sigma_{nk}) \\
&\qquad \cdot f_{S_1 S_2 \cdots S_n}(s_1, s_2, \cdots, s_n) ds_n \qquad \cdots\cdots(35)
\end{aligned}$$

and

$$\begin{aligned}
&\boldsymbol{P}[\text{no failure in } S_1, S_2, \cdots, S_n] \\
&= \sum_{k=1}^{2^{n-1}} \boldsymbol{P}\Big[\bigcup_{\substack{s_i \in \mathfrak{D} \\ i=1,2,\cdots,n}} \Big\{\Big(\frac{\sigma_{nk}}{\prod_{j=1}^{n} \varphi(s_j)} < R_0\Big) \cap (s_1 < S_1 \leqslant s_1 + ds_1) \cap \cdots \\
&\qquad \cdots \cap (s_r < S_n \leqslant s_n + ds_n)\Big\}\Big] \qquad \cdots\cdots(36)
\end{aligned}$$

$\sigma_{nk}$ is the marginal strength shown in Table 1 following the $k$-th of the $2^{n-1}$ possible paths varying with the relations among the intensities of $n$ loads. $D_{ik}$ is the domain of integration for the $i$-th load shown in Table 2 following the $k$-th path constructed in the same manner.

Thus,

Table. 1. Parameters Involved in Residual Strength.

| n | $\sigma_{nk}$ : marginal strength | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | $s_1\phi(s_1)$ | | | | | | | |
| 2 | $s_1\phi(s_1)\phi(s_2)$ | | | | $s_2\phi(s_2)$ | | | |
| 3 | $s_1\prod_{j=1}^{3}\phi(s_j)$ | | $s_3\phi(s_3)$ | | $s_2\phi(s_2)\phi(s_3)$ | | $s_3\phi(s_3)$ | |
| 4 | $s_1\prod_{j=1}^{4}\phi(s_j)$ | $s_4\phi(s_4)$ | $s_3\phi(s_3)\phi(s_4)$ | $s_4\phi(s_4)$ | $s_2\prod_{j=2}^{4}\phi(s_j)$ | $s_4\phi(s_4)$ | $s_3\phi(s_3)\phi(s_4)$ | $s_4\phi(s_4)$ |

Table 2. Domains of Integration in Eqs. (35), (38).

| $D_{nk}$ | Partition of load domain | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $D_{1k}$ | $0 \sim \infty$ | | | | | | | |
| $D_{2k}$ | $0 \sim \sigma_{11}$ | | | | $\sigma_{11} \sim \infty$ | | | |
| $D_{3k}$ | $0 \sim \sigma_{21}$ | | $\sigma_{21} \sim \infty$ | | $0 \sim \sigma_{22}$ | | $\sigma_{22} \sim \infty$ | |
| $D_{4k}$ | $0 \sim \sigma_{31}$ | $\sigma_{31} \sim \infty$ | $0 \sim \sigma_{32}$ | $\sigma_{32} \sim \infty$ | $0 \sim \sigma_{33}$ | $\sigma_{33} \sim \infty$ | $0 \sim \sigma_{34}$ | $\sigma_{34} \sim \infty$ |

$$F_{R_n}(x) = B_{R_n}(x)/B_{R_n}(\infty) \qquad \cdots\cdots\cdots\cdots\cdots (37)$$

where

$$B_{R_n}(x) = \sum_{k=1}^{2^n-1}\int_{D_{1k}} ds_1 \int_{D_{2k}} ds_2 \cdots \int_{D_{nk}} \left\{F_{R_0}\left(\frac{x}{\prod_{i=1}^{n}\varphi(s_i)}\right) - F_{R_0}\left(\frac{\sigma_{nk}}{\prod_{i=1}^{n}\varphi(s_i)}\right)\right\}$$
$$\cdot H(x-\sigma_{nk}) \cdot f_{S_1S_2\cdots S_n}(s_1, s_2, \cdots, s_n)ds_n \qquad \cdots\cdots\cdots\cdots\cdots (38)$$

Given the probability distirubtion, $F_{R_0}(x)$ of the initial strength, the joint probability distribution $f_{S_1S_2\cdots S_n}(s_1, s_2, \cdots, s_n)$ of the loads $S_1, S_2, \cdots, S_n$ and the degradation factor $\varphi(x)$, numerical value of $F_{R_n}(x)$ can be obtained from Eq. (29), (33) or (37).

3.2.2. Case with Past Loads of Known Intensities

The conditional probability distribution, $F^{(1)}_{R_0^*}(x; s_1)$, of the residual strength on the hypothesis of survival in the first load $S_1 = s_1$ is given by the specific case of Fig. 3 with $s_1$ assuming fixed value. Then it is derived as follows:

$$
\begin{aligned}
F^{(1)}_{R_0^*}(x;s_1) &= \boldsymbol{P}[R_0^* \leq x \mid S_1 = s_1] \\
&= \boldsymbol{P}[\tilde{R}_0\varphi(s_1) \leq x \mid S_1 = s_1] \\
&= \boldsymbol{P}[R_0\varphi(s_1) \leq x \mid R_0 > s_1] \\
&= \boldsymbol{P}\left[R_0 \leq \frac{x}{\varphi(s_1)} \mid R_0 > s_1\right] \\
&= \frac{\boldsymbol{P}\left[\left(R_0 \leq \frac{x}{\varphi(s_1)}\right) \cap (R_0 > s_1)\right]}{\boldsymbol{P}[R_0 > s_1]} \\
&= \frac{\int_{s_1}^{\frac{x}{\varphi(s_1)}} f_{R_0}(r)\,dr}{1-F_{R_0}(s_1)} H(x - s_1\varphi(s_1)) \\
&= \frac{F_{R_0}\left(\frac{x}{\varphi(s_1)}\right) - F_{R_0}(s_1)}{1-F_{R_0}(s_1)} H(x - s_1\varphi(s_1)) \quad \cdots\cdots\cdots\cdots\cdots (39)
\end{aligned}
$$

Likewise, the conditional probability distribution, $F^{(2)}_{R_0^*}(x;s_1,s_2)$, of the residual strength after survival in two loads such that $S_1 = s_1$ and $S_2 = s_2$, can be given by given by

$$
\begin{aligned}
F^{(2)}_{R_0^*}(x;s_1,s_2) &= \boldsymbol{P}[R_0^* \leq x \mid (R_1 > s_2) \cap (R_0 > s_1)] \\
&= \boldsymbol{P}[\tilde{R}_1\varphi(s_2) \leq x \mid S_1 = s_1, S_2 = s_2] \quad \cdots\cdots\cdots\cdots\cdots (40)
\end{aligned}
$$

In passing $R_1$ and $\tilde{R}_1$ shall be represented by $R_1(s_1)$ and $\tilde{R}_1(s_1)$, respectively, in order to note that $R_1$ is based on the condition that $S_1 = s_1$. Then Eq. (40) can be rewritten as follows

$$
\begin{aligned}
F^{(2)}_{R_0^*}(x;s_1,s_2) &= \boldsymbol{P}[\tilde{R}_1(s_1)\varphi(s_2) \leq x \mid S_2 = s_2, S_1 = s_1] \\
&= \boldsymbol{P}[R_1(s_1)\varphi(s_2) \leq x \mid R_1(s_1) > s_2] \\
&= \boldsymbol{P}\left[R_1(s_1) \leq \frac{x}{\varphi(s_2)} \mid R_1(s_1) > s_2\right] \\
&= \frac{\boldsymbol{P}\left[\left(R_1(s_1) \leq \frac{x}{\varphi(s_1)}\right) \cap (R_1(s_1) > s_1)\right]}{\boldsymbol{P}[R_1(s_1) > s_2]} \\
&= \frac{\int_{s_2}^{\frac{x}{\varphi(s_2)}} f_{R_1}(r;s_1)\,dr}{1-F_{R_1}(s_2;s_1)} H(x - s_2\varphi(s_2)) \\
&= \frac{\int_{s_2}^{\frac{x}{\varphi(s_2)}} f^{(1)}_{R_0^*}(r;s_1)\,dr}{1-F^{(1)}_{R_0^*}(s_2;s_1)} H(x - s_2\varphi(s_2))
\end{aligned}
$$

$$= \frac{F^{(1)}_{R_0^*}\left(\dfrac{x}{\varphi(s_2)}; s_1\right) - F^{(1)}_{R_0^*}(s_2; s_1)}{1 - F^{(1)}_{R_0^*}(s_2; s_1)} H(x - s_2\varphi(s_2)) \quad \cdots\cdots(41)$$

By substituting from Eq. (39) into Eq. (40), $F^{(2)}_{R_0^*}(x; s_1, s_2)$ can also be represented in terms of $F_{R_0}(x)$ by

$$F^{(2)}_{R_0^*}(x; s_1, s_2) = \frac{F_{R_0}\left(\dfrac{x}{\varphi(s_1)\varphi(s_2)}\right) - F_{R_0}\left(\dfrac{s_2}{\varphi(s_1)}\right)}{1 - F_{R_0}\left(\dfrac{s_2}{\varphi(s_2)}\right)} H(x - s_2\varphi(s_2)) \cdot H(s_2 - s_1\varphi(s_1))$$
$$+ \frac{F_{R_0}\left(\dfrac{x}{\varphi(s_1)\varphi(s_2)}\right) - F_{R_0}(s_1)}{1 - F_{R_0}(s_1)} H(x - s_1\varphi(s_1)\varphi(s_2)) \cdot H(s_1\varphi(s_1) - s_2) \quad \cdots\cdots\cdots\cdots\cdots(42)$$

The first term on the right-hand side of Eq. (42) corresponds to case (B) of Fig. 4, and the second term to case (C). Depending on which case may be realized, the relevant term of the two remains significant and the other vanishes by virtue of the step functions involved in them.

In the same manner, the conditional probability distribution, $F^{(3)}_{R_0^*}(x; s_1, s_2, s_3)$ of the residual strenth after survival in three load applications with intensities $s_1, s_2, s_3$ is developed, the result of which is shown below:

$$F^{(3)}_{R_0^*}(x; s_1, s_2, s_3)$$
$$= \frac{F_{R_0}\left(\dfrac{x}{\varphi(s_1)\varphi(s_2)\varphi(s_3)}\right) - F_{R_0}\left(\dfrac{s_3}{\varphi(s_1)\varphi(s_2)}\right)}{1 - F_{R_0}\left(\dfrac{s_3}{\varphi(s_1)\varphi(s_2)}\right)} H(x - s_3\varphi(s_3)) \cdot H(s_3 - s_2\varphi(s_2)) \cdot H(s_2 - s_1\varphi(s_1))$$
$$+ \frac{F_{R_0}\left(\dfrac{x}{\varphi(s_1)\varphi(s_2)\varphi(s_3)}\right) - F_{R_0}\left(\dfrac{s_2}{\varphi(s_1)}\right)}{1 - F_{R_0}\left(\dfrac{s_2}{\varphi(s_2)}\right)} H(x - s_2\varphi(s_2)\varphi(s_3)) \cdot H(s_2\varphi(s_2) - s_3) \cdot H(s_2 - s_1\varphi(s_1))$$
$$+ \frac{F_{R_0}\left(\dfrac{x}{\varphi(s_1)\varphi(s_2)\varphi(s_3)}\right) - F_{R_0}\left(\dfrac{s_3}{\varphi(s_1)\varphi(s_2)}\right)}{1 - F_{R_0}\left(\dfrac{s_3}{\varphi(s_1)\varphi(s_2)}\right)} H(x - s_3\varphi(s_3)) \cdot H(s_3 - s_1\varphi(s_1)\varphi(s_2)) \cdot H(s_1\varphi(s_1) - s_2)$$
$$+ \frac{F_{R_0}\left(\dfrac{x}{\varphi(s_1)\varphi(s_2)\varphi(s_3)}\right) - F_{R_0}(s_1)}{1 - F_{R_0}(s_1)} H(x - s_1\varphi(s_1)\varphi(s_2)\varphi(s_3)) \cdot H(s_1\varphi(s_1)\varphi(s_2) - s_3) \cdot H(s_1\varphi(s_1) - s_2)$$
$$\cdots\cdots\cdots\cdots\cdots(43)$$

The terms on the righ side of Eq. (43) correspond in order to the cases, respectively, where (i) $s_3 > s_2\varphi(s_2) > s_1\varphi(s_1)\varphi(s_2)$, (ii) $s_2\varphi(s_2) > s_3$, $s_2 > s_1\varphi(s_1)$, (iii) $s_3 > s_1\varphi(s_1)\varphi(s_2) > s_2\varphi(s_2)$ and (iv) $s_1\varphi(s_1)\varphi(s_2) > s_3$, $s_1\varphi(s_1) > s_2$.

For an arbitrary case indicated above, the terms of the four in Eq. (43), except that relevant to the case, vanish just like in Eq. (42).

In conclusion the conditional probability distribution, $F_{R_0^*}^{(l)}(x; s_1, s_2, \cdots, s_l)$, of the residual strength after survival in $l$ past loads can be given by

$$F_{R_0^*}^{(l)}(x; s_1, s_2, \cdots, s_l) = \frac{F_{R_0}\left(\dfrac{x}{\prod_{i=1}^{l}\varphi(s_i)}\right) - F_{R_0}\left(\dfrac{\sigma_{lk}}{\prod_{i=1}^{l}\varphi(s_i)}\right)}{1 - F_{R_0}\left(\dfrac{\sigma_{lk}}{\prod_{i=1}^{l}\varphi(s_i)}\right)} H(x - \sigma_{lk}) \qquad \cdots\cdots(44)$$

and its probability density, given by

$$f_{R_0^*}^{(l)}(x; s_1, s_2, \cdots, s_l) = \frac{f_{R_0}\left(\dfrac{x}{\prod_{i=1}^{l}\varphi(s_i)}\right) H(x - \sigma_{lk})}{1 - F_{R_0}\left(\dfrac{\sigma_{lk}}{\prod_{i=1}^{l}\varphi(s_i)}\right)} \cdot \frac{1}{\prod_{i=1}^{l}\varphi(s_i)} \qquad \cdots\cdots(45)$$

## Numerical Results and Discussions

### 4.1. General Remarks

Numerical computations have been made on the analytical results obtained in the preceding chapters. The sequence of random lodads $S_1, S_2, \cdots, S_N$ and the initial resisting strength $R_0$ have been treated as independent normal random variables, modified so that their probability densities vanish in the negative range and their ordinates are adjusted to have the total area of unity.

To the parameters in order to characterize the distribution of these random variables, the following values have been assigned:

Central safety factor = (the initial mean resisting strength)/(mean intensity of a load)

$$r = r_m/s_m = 1.2 \sim 4.0$$

Coefficient of variation of load intensity[5)]

$$c_S = 0.2 \sim 0.6$$

Coefficient of variation of the initial resisting strength[6)]

$$c_R = 0.05 \sim 0.2$$

The parameters which determine the degradation factor in Eq. (16) have been given by the following numerical values:

Type [A], Eq. (18)

$$c_A = 0.1$$

Type [B], Eq. (19)

$$c_B = 0.1$$
$$\xi = 0.1$$

For the parameter values indicated above, the numerical results of the present study have been obtained. These will be discussed in the subsequent sections.

Along with those for the present analysis, values for the classical theory discussed in 2.2 have also been obtained for reference, which are discussed in

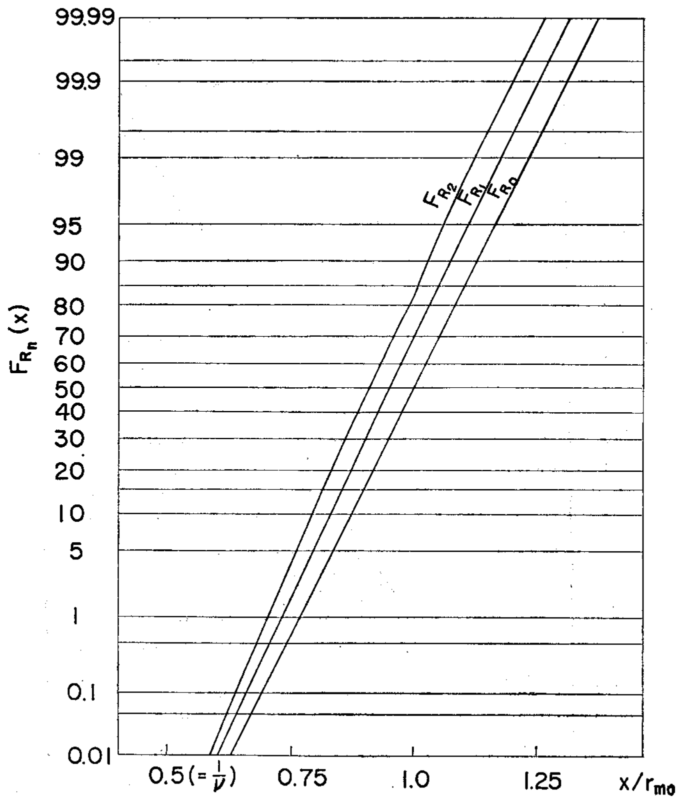


Fig. 5. **Probability Distribution of the Residual Strength with the Type A Degradation**
($\nu=2.0$, $c_R=0.10$, $c_S=0.4$, $c_A=0.10$).

comparison in later sections. In their comuptation, the degradation function $\psi(k)$ in Eq. (11) was assumed to take the form

$$\psi(k)=\left\{\exp\left(-c_A\cdot\frac{s_m}{r_m}\right)\right\}^k=\exp\left(-\frac{c_A k}{\nu}\right) \quad \cdots\cdots\cdots\cdots\cdots(46)$$

i.e., in Eq. (44), the rate of degradation in each load was assumed to be equal to that of the type [A] in the analysis of the present study if the load intensity is fixed to its mean value $s_m$ in Eq. (11).

### 4.2. Probability Distribution of the Residual Strength

The probability distribution of the initial strength $R_0$ and those of the residual strength $R_1$ and $R_2$ are shown in Figs. 5 and 6 for the type A and B degradations, respectively. Since these figures are drawn on nomral probability papers, $F_{R_0}$ forms a straight line. $F_{R_1}$ and $F_{R_2}$ curves are nearly straight lines except for lower

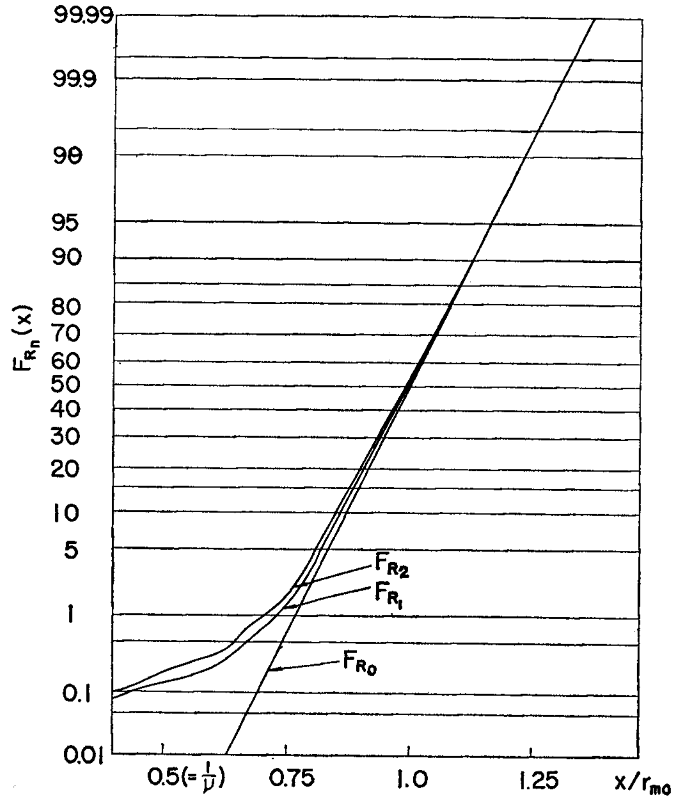


Fig. 6. Probability Distribution of the Residual Strength with the Type B Degradation ($\nu=2.0$, $c_R=0.10$, $c_S=0.4$, $c_B=0.10$).

values on the abscissa thus showing that $R_1$ and $R_2$ are also almost normal.

## 4.3. Mean Value and Coefficient of Variation of the Residual Strength

### 4.3.1. Mean Value

Fig. 7 shows the mean value $r_{m_1}$ and $r_{m_2}$ of the residual strength after structural survivals, respectively, in the first and the second loads relative to the mean initial strength $r_{m_0}$.

Fig. 7(a) is for the type A degradation, and (b), for the type B degradation.

It is noted first that the ordinates of the curves in this figure are on the whole lower than unity, which implies that the degradation effect dominates these results. It is natural that $r_{m_1}/r_{m_0}$ and $r_{m_2}/r_{m_0}$ take on lower values for small $\nu$ so that the reduction of the mean strength through load applications is remarkable for a low central safety factor.

It is also noted that $r_{m_2}/r_{m_0}$ necessarily assumes smaller values than $r_{m_1}/r_{m_0}$ thus demonstrating that the degradation effect is accumulated through repeated loads.

A great value of the coefficient of variation $c_S$ of the load would imply a severe load condition and hence would have a greater degradation effect than a

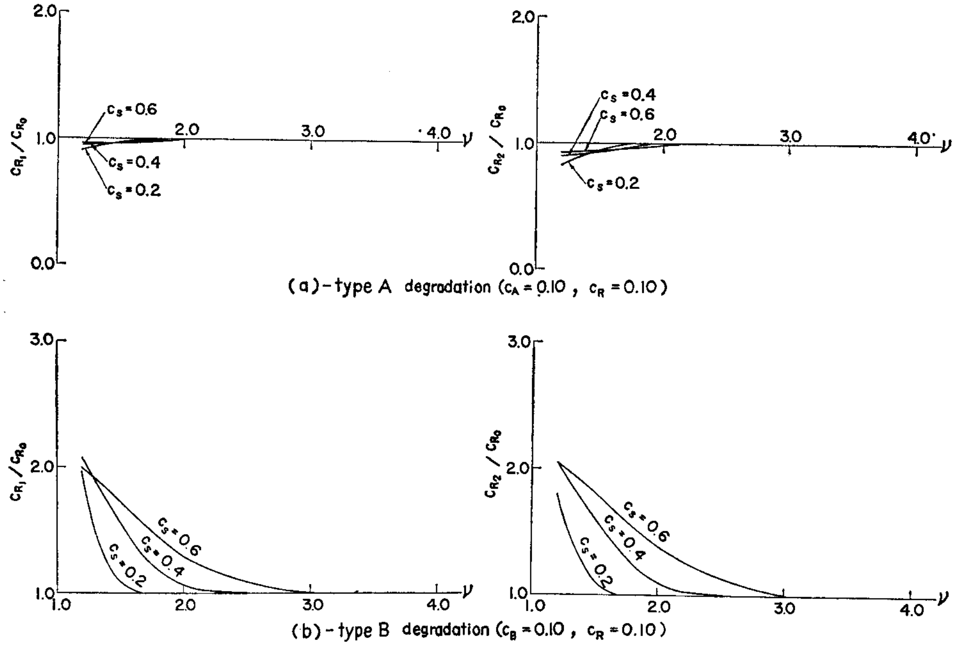


Fig. 7. The Mean Value Ratio of the Residual Strength.
(a)–Type A degradation ($c_A$=0.10, $c_R$=0.10)
(b)–Type B degradation ($c_B$=0.10, $c_R$=0.10)

small $c_S$. Most of the results in Fig. 7 are consistent with this argument. However, there are portions in a range of small $\nu$ where the greater $c_S$ is, the greater $r_{m_1}$ and $r_{m_2}$ are. This is considered to be the non-failure effect which has become conceivable through the structural survival in the most severe load condition under a low safety factor.

The characteristics of the types A and B of degradation are explicit in Fig. 7. In the type A, Fig. (a), the degradation effect is appreciable over a wide range of $\nu$, as a consequence of an exponential form of $\varphi_A(S)$ which causes a considerable degradation even in a load of relatively low intensity. The type B, on the other hand, exhibits, Fig. 7(b), an excessive degradation in the vicinity of $\nu=1.0$ while the degradation effect disappears quickly as $\nu$ increases. This results from the form of $\varphi_B(S)$ for which degradation takes place exclusively for loads of intensities close to the mean strength of the structure.

4.3.2. Coefficient of Variation

In Fig. 8 are shown the coefficients of variation $c_{R_1}$ and $c_{R_2}$ of the residual strength after survivals, respectively, in the first and the second loads represented by their ratios to that $c_{R_0}$ of the initial strength.

Fig. 8(a) shows the results for the type A degradation and Fig. (b), for the type B. It is remarkable that $c_{R_n}$ tends to decrease, but only by a small amount, through the repetition of loads for the type A, while the type B results in a considerable amount of increase in $c_{R_n}$ in repeated loads especially for lower value of $\nu$. This contrast could be reasoned with the aid of Figs. 5 and 6 showing the probability distribution of the residual strength.

From Fig. 5, it can be stated that the strength degradation of the type A tends to set the probability distribution of the residual strength at a position parallel to that of the initial strength as a consequence of the degradation effect in a wide range of the load intensity as discussed in the previous section. Hence, even though the mean value of the residual strength is strongly affected by the degradation effect as in Fig. 7(a), its coefficient of variation does not greatly change from $c_{R_0}$. Values of $c_{R_n}$ slightly less than unity in Fig. 8 (a) are considered to be results of the non-failure effect, since cutting off of the lower-range tail of the probability density of the resisting strength, as illustrated in Fig. 3 (b), reduces its variance.

As to the type B degradation, its effect is remarkable exclusively in the range where the load and the resisitance take on relatively close values, which is manifest in the $F_{R_n}$ curves for the lower range of abscissa in Fig. 6. It is clear that such results make $c_{R_n}$ assume that greater values than unity in Fig. 8 (b) are consequences of the degradation effect, and it is natural that they increase with decreasing $\nu$.

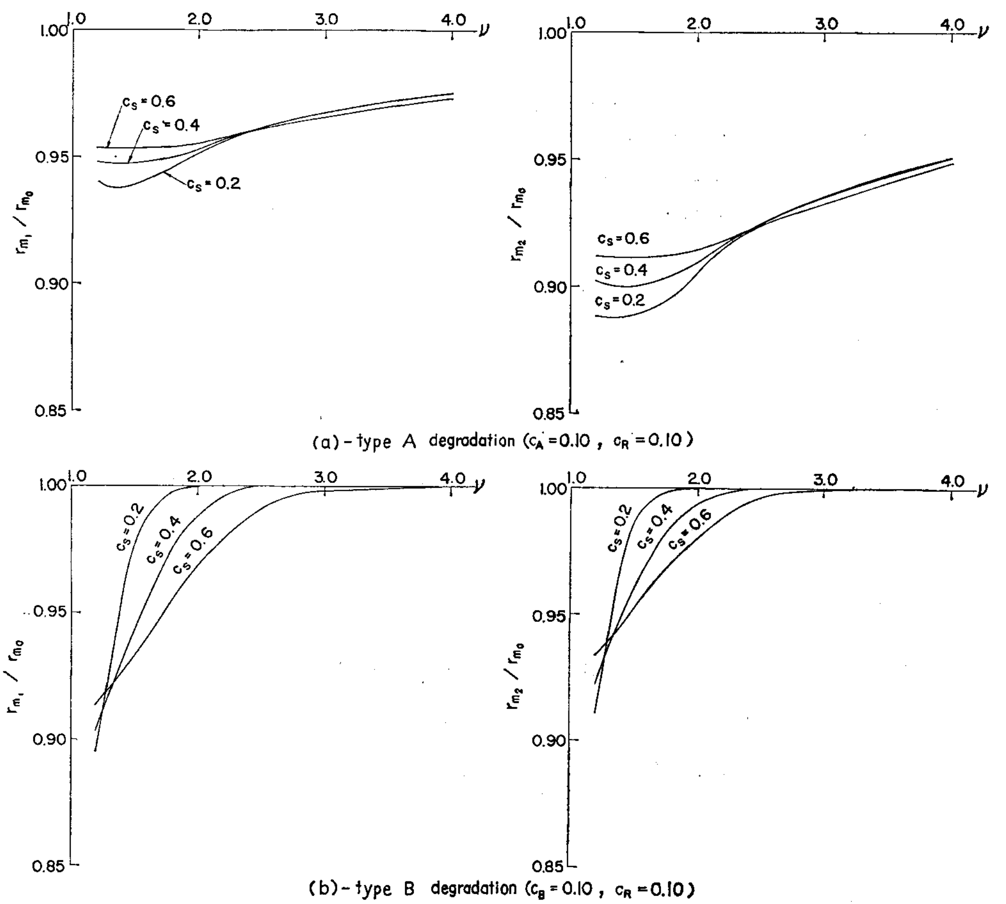


Fig. 8. The Ratio of the Coefficient of Variation of the Residual Strength.
(a)–type A degradation ($c_A=0.10$, $c_R=0.10$)
(b)–type B degradation ($c_B=0.10$, $c_R=0.10$)

## 4.4. Reliability of Structures against a Sequence of Future Loads

### 4.4.1. Failure Rate

Fig. 9 shows variations of the failure rate $h_N(n)$, defined by Eq. (20), for the type A and the type B degradations as lodas are repeated. It is noted first that the failure rate for the type A increases in every load for any values of the central safety factor $\nu$, thus emphasizing the presence of the degradation effect for a wide range of $\nu$, whereas that for the type B behaves in the same manner only for small $\nu$.

This result completely agrees with the characteristics of the two types of degradation described in preceding sections.

In Fig. 9, also shown are the numerical results for the classical theory discussed in 2.2. The curves for the constant resistance have been computed for Eq. (5).

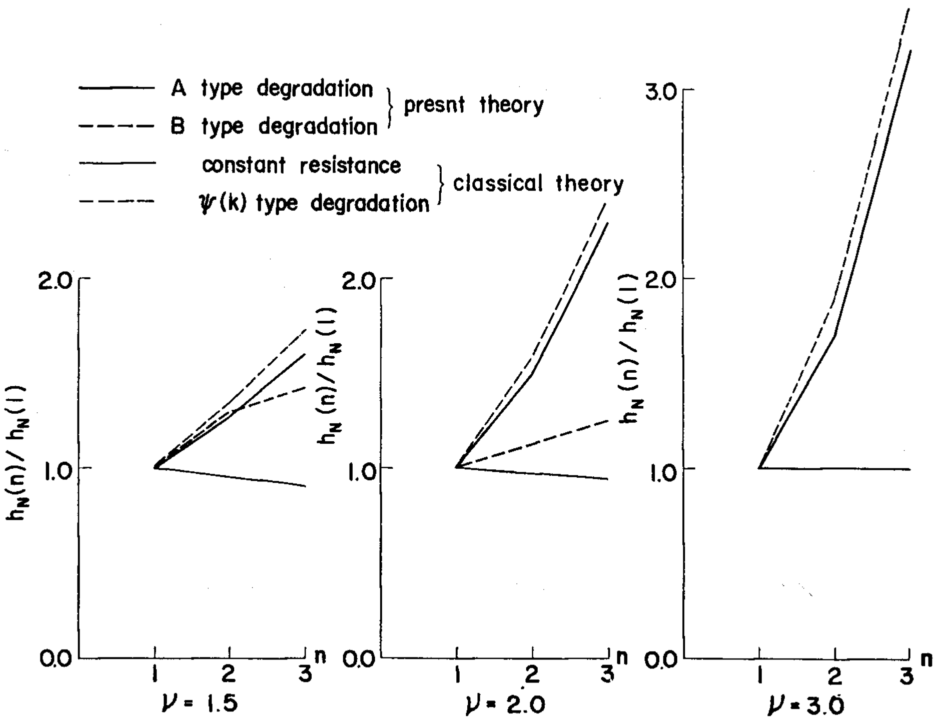


Fig. 9. The Failure Rate Ratio
($c_R$=0.10, $c_S$=0.4, $c_A$=0.10, $c_B$=0.10, $\xi$=1.0).

They exhibit monotonically decreasing characteristics as pointed out by A.H.-S.Ang and M. Amin[3]. However, in the case of $\nu = 3.0$, the result for the present study for the type B degradation approaches the classical theory for a constant resistance.

It is noted in Fig. 9 that the failure rate based on this study for the type A degradation and that based on the classical theory for the $\psi(k)$ type degradation assume fairly close values. Hence, from Fig. 9 and the arguments on the form of $\psi(k)$ defined by Eq. (44), it can be stated that for the type A degradation the mean load intensity $s_m$ can be a representative of the random load S in obtaining the failure rate function. It should, however, be kept in mind that this conclusion would not be valid if $\psi(k)$ takes a form similar to the type B. For, in the range $\nu > 1.5$, loads of its mean intensity $s_m$ cause little degradation as we shall see later in Fig. 12. Hence, the results for $\psi(k)$ type degradation similar to the type B will almost coincide with those for a constant resistance in Fig. 9 which differ considerablly from the results based on the analysis in the present study.

It is noted, in passing, that the failure rate in Fig. 9 is given as a ratio to the initial failure rate $h_N(1)$, so that it may give an impression that structures with a large safety factor $\nu$ become more dangerous through repeated loads than those with a small $\nu$. However, if $\nu$ is large, $h_N(1)$ itself is very small as Fig. 10 may imply. Hence structures with a large $\nu$ will keep failure rates much lower than the case of a small $\nu$.

4.4.2. Reliability Function

Fig. 10 shows the numerical results for the reliability function obtained from Eq. 21 with the aid of the numerical values of the failure rate described in the previous subsection.

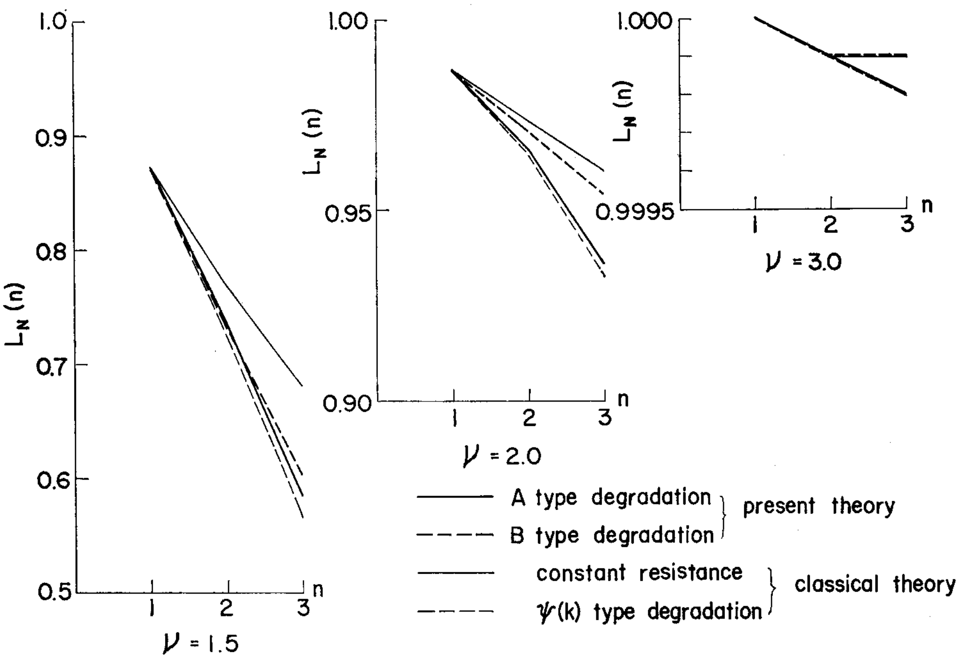


Fig. 10. The Reliability Function
($c_R$=0.10, $c_S$=0.4, $c_A$=0.10, $c_B$=0.10, $\xi$=1.0).

It is considered appropriate in this figure that the initial reliability $L_N(1)$ is asymptotic to unity with an increase in $\nu$. The behaviour of $L_N(2)$ and $L_N(3)$ are under a direct influence of the failure rate $h_N(n)$ by virtue of Eq. (20). The results for the type B degradation approach more quickly to those of the classical theory with constant resistance as $\nu$ increases. The results for the type A are in fairly good agreement with the classical theory for the $\psi(k)$ type degradation.

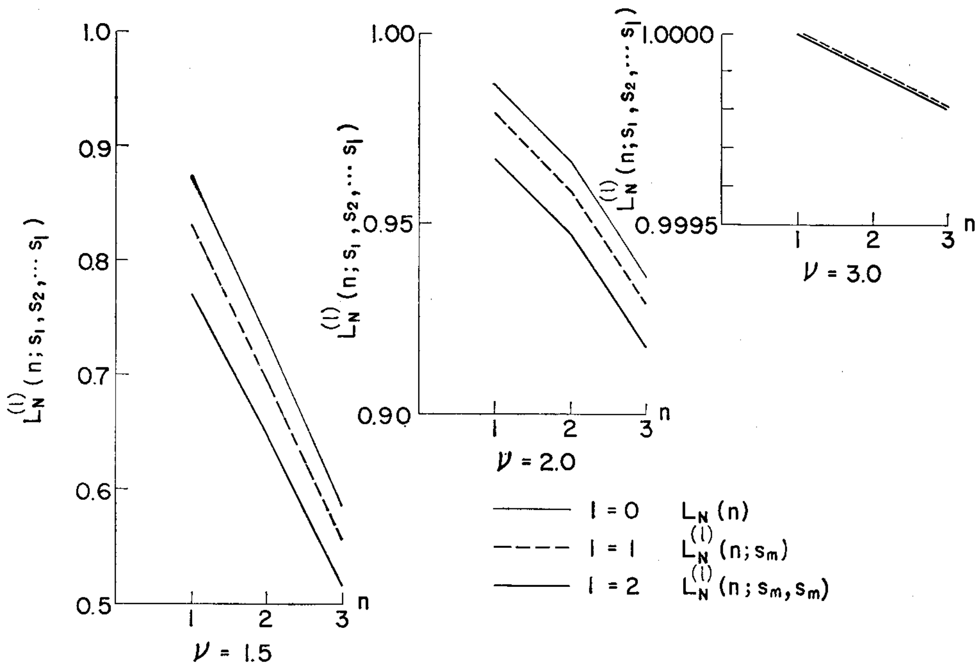


Fig. 11. The Conditional Reliability with Past Load Experiances in Type A Degradation Mode ($c_R=0.10$, $c_S=0.4$, $c_A=0.10$).

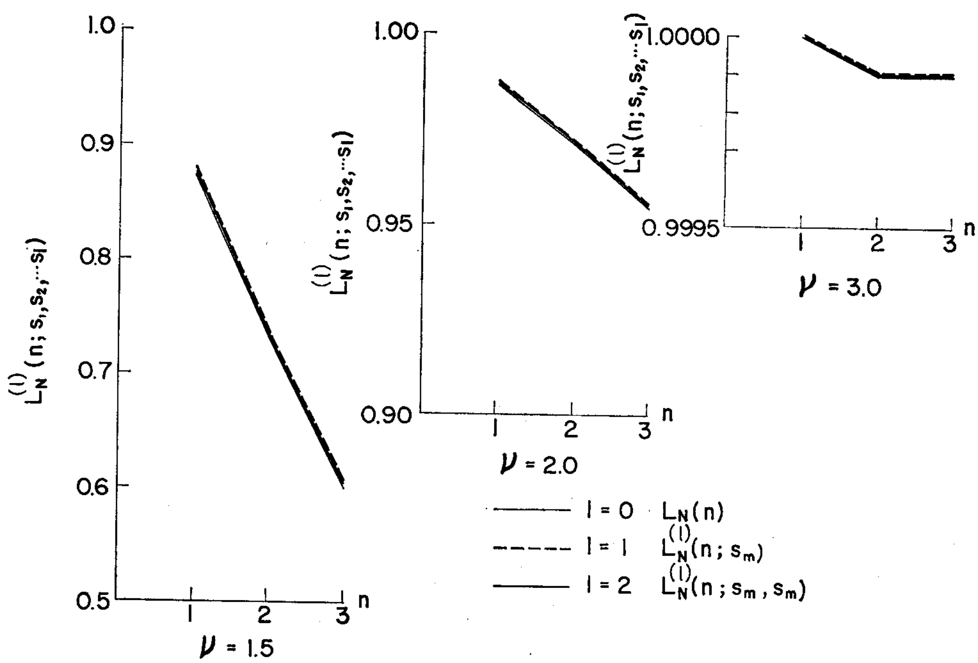


Fig. 12. The Conditional Reliability with Past Load Experiances in Type B Degradation Mode ($c_R=0.10$, $c_S=0.4$, $c_B=0.10$, $\xi=1.0$).

### 4.5. Future Reliability of Existing Structures with Experiences of Past Loads

Figs. 11 and 12 are plots of the conditional reliability discussed in 2.3.2 (2) with experiences of past loads which are equal to the mean load intensity $s_m$. Fig. 11, which accounts for the type A degradation, demonstrates that the influence of the past load is appreciable in the range of $\nu$ somewhat less than 2.0. In Fig. 12 for the type B degradation, however, the past loads hardly affect the future reliability. Such a contrast between the two types can again be explained from their characteristics. The past loads of the intensity equal to its mean value were not enough to make an appreciable degradation take place in type B structures; while in type A structures degradation was appreciable even under loads of mean intensities.

## 5. Conclusions

The present study has developed a new branch of the reliability theory for repeated loads by introducing the concept of the strength degradation dependent on the load intensity and by taking the non-failure effect into account. Also, it gives us some essential and interesting results both in the theoretical procedure and in the numerical survey. The conclusions obtained in this study are summarized below.

(1) The structural reliability with strength degradation dependent on the load intensity can be obtained by the method developed in the present theory, in which the probability distribution of the residual strength and the strength degradation factor take an important role.

(2) The analytical procedure for the structural reliability for repeated loads has become more general than the classical theory in the sense that the non-failure effect is considered in this study.

(3) The structural behaviour related to the reliability in repeated loads can be explained in terms of the non-failure effect and the strength degradation effect. The analytical method thus worked out in this study provides more general and more convincing conclusions than those so far developed as to the variation of reliability in a sequence of loads.

(4) The discussions have been extended to the reliability in future repeated loads of existing structures which have withstood past loads of known intensities with a considerable strength degradation.

(5) The reliability theory developed in this study would be a powerful means in the estimation of structural reliability against loads with a relatively small number of applications and severe intensity.

## Acknowledgments

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## Appendix "Poisson pattern degradation"

A heterogeneous material with a random disposition of its elements, such as concrete, subjected to a triaxial compressive load or a uniaxial tensile load will generate many "cracks".

The random process of the occurence of "cracks" as a Poisson pattern based on the following assumption provides a probability of its occurence.

### Assumptions

1. The material has no "crack" in the initial state.
2. The material is statistically homogeneous and isotropic.
3. "Cracks" occur in the volume element $\Delta v$.
4. Total volume $V$ of material is very large in comparison with small volume element $\Delta v$.
5. Cracks occurring at different volume elements are statistically independent; i.e., the occurrence forms a Poisson process.
6. The cracks in each element degrade its strength by the factor $1-\mu$ in which $\mu$ assumes a value between 0 and 1 dependent on stress conditions, material properties, loading velocities, and so on.

Under these assumptions, the residual strength of the structural members shall be analysed below.

$\boldsymbol{P}(1, \Delta v)$: the probability of the existence of a "crack" in a small volume element $\Delta v$.

$\boldsymbol{P}(0, \Delta v)$: the probability of the existence of no "crack" in a small volume element $\Delta v$.

From the assumption of the Poisson pattern

$$\boldsymbol{P}(1, \Delta v) = \lambda \Delta v$$

where $\lambda$ is a constant. Then the following relation is given

$$\boldsymbol{P}(0, \Delta v) + \boldsymbol{P}(1, \Delta v) = 1$$

We introduce the following assumption,

$$\boldsymbol{P}(0, v + \Delta v) = \boldsymbol{P}(0, \Delta v) \cdot \boldsymbol{P}(0, v)$$

Some manipulations on these relations lead to

$$\frac{\boldsymbol{P}(0, v+\Delta v) - \boldsymbol{P}(0, v)}{\Delta v} = -\lambda \cdot \boldsymbol{P}(0, v)$$

By letting $\Delta v$ shrink to an infinitesimal order, we get the following relation

$$\frac{d\boldsymbol{P}(0, v)}{dv} = -\lambda \boldsymbol{P}(0, v)$$

and solving it under the condition that $\boldsymbol{P}(0, 0) = 1$, we obtain

$$\boldsymbol{P}(0, v) = e^{-\lambda V}, \qquad \boldsymbol{P}(1, v) = 1 - e^{-\lambda V}$$

The above-mentioned probabilities can be given only in the homogeneous stress field all over the material.

The constant $\lambda$ should be treated as a parameter dependent on the stress intensity; i.e.,

$$\lambda = \lambda(\sigma_1, \sigma_2, \sigma_3)$$

where $\sigma_j$ is a principal stress $j = 1, 2, 3$.

The strength of material is generally represented by $r$. When a specific material undergoes local damages due to occurence of cracks, the strength of that part of the material is changed into $(1-\mu)r$.

Then the expected residual strength $r_1$ is given by

$$r_1 = r_0 \boldsymbol{P}(0, V) + (1-\mu) r_0 \boldsymbol{P}(1, V)$$

using the initial strength $r_0$, and so

$$\frac{r_1}{r_0} = \boldsymbol{P}(0,\ V) + (1-\mu)\boldsymbol{P}(1,\ V)$$
$$= 1-\mu(1-e^{-\lambda V})$$

Then the ratio of the expected residual strength $r_1/r_0$ is given as follows.

$$e^{-\lambda V} \lesssim \frac{r_1}{r_0} \lesssim 1 \qquad \cdots\cdots\cdots\cdots\cdots(\text{A–1})$$

This inquality gives upper and lower bounds of the ratio of residual strength when the random process of the occurence of cracks is assumed to be of a Poisson pattern.

The assumption of the Poisson pattern, however, restricts the range of its application to the level of a load much lower than the ultimate strength of the material. We must notice that there will appear complex phenomena, as the influence of the crack interaction and the growth from the local failure to rupture take place when the level of load intensity approaches the ultimate strength.