Low Order Optimal Filters for Linear Discrete-Time Systems

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This paper treats of the linear discrete-time filtering problem where some elements of the measurement signal are free from noise. The main results are as follows:

(i) A low order optimal filter is obtained by modifying the ordinary Kalman filter.

(ii) Based upon the result (i), the structure of Tse-Athans' optimal minimal-order filter is made clear.

(iii) Utilizing the result (i), the filtering problem with colored measurement noise is solved. This solution contains Bryson-Henrikson's result as a special case.

I. Introduction

The state variable approach to the filtering problem of linear discrete-time systems is developed by Kalman [1,2]. For the case in which the covariance matrix of the measurement noise is singular, Kalman [1] points out that the order of the filter can be reduced. Brammer [3] obtained a low order filter by solving the Wiener-Hopf equation; and recently Tse and Athans [4] investigated the problem of constructing the minimal-order filter. On the other hand, for the case in which measurement signals contain colored noise, Bryson and Henrikson [5] derived the filter equation by their measurement differencing approach, while Rishel [6] proved the optimality of this filter.

In this paper, we consider the linear discrete-time filtering problem in which some of the measurement elements are free from noise. First of all, a low order optimal filter is obtained by modifying the ordinary Kalman filter [2] slightly. Secondly, it is shown that the above result, although quite simple, is useful not only for the unification of several previous results [3]–[5] and for an easy understanding of them, but also for extending these results further.

The next section contains the formulation of the problem and the definition of the order of the filter. In Section 3, a filter is derived whose order is lower than the ordinary Kalman filter by the number of noise-free elements in the measurement signals. This low order filter is essentially the same as that which was

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derived by Brammer [3]. However, our method of derivation will be simpler and clearer than that of Brammer. In Section 4, it is shown that, using the result of Section 3, the structure of Tse and Athans' optimal minimal-order filter can be expressed analytically. This makes the structure of the filter very clear. In Section 5, it is shown that by applying the result of Section 3, the filtering problem with colored measurement noise can be solved easily. Furthermore, as a special case of this solution, Bryson-Henrikson's result [5] is obtained not by using their measurement differencing approach, but by using a state augmentation approach.

The main idea in this paper has arisen from the author's previous work [7] and part of this paper overlaps it. The author believes however, that this overlapping is necessary to show the effectiveness of our approach, and to make clear the relation between the Kalman filter and Tse-Athans' estimator.

II. Problem Formulation

Consider a linear discrete-time system described by

$$\boldsymbol{x}(k+1) = \Phi(k)\boldsymbol{x}(k)+\boldsymbol{w}(k),\ k = 1, 2, \cdots \tag{1}$$

$$\begin{aligned}\boldsymbol{y}(k) &= H(k)\boldsymbol{x}(k)+\boldsymbol{v}(k)\\ &= H(k)\boldsymbol{x}(k)+\begin{bmatrix}\boldsymbol{v}_1(k)\\ \cdots\cdots \\ \boldsymbol{0}\end{bmatrix}\end{aligned} \tag{2}$$

where $\boldsymbol{x}(k)\in R^n$* is the state vector at the k-th sampling instant, $\boldsymbol{y}(k)\in R^m$ is the measurement vector, $\boldsymbol{w}(k)\in R^n$, $\boldsymbol{v}(k)\in R^m$ and $\boldsymbol{v}_1(k)\in R^{m-l}$ are the random vectors, and $\boldsymbol{0}$ is the appropriate dimensional zero matrix or zero vector. Let M_{nm} denote the set of all $n\times m$ matrices, then $\Phi(k)\in M_{nn}$ and $H(k)\in M_{mn}$. Let us assume that $n\geqq m\geqq l$ and that rank $H(k)=m$.

$\boldsymbol{w}(k)$, $\boldsymbol{v}(k)$, $k=1, 2, \cdots$, and $\boldsymbol{x}(1)$ are mutually uncorrelated and their probability density functions (p.d.f.'s) are given by

$$p(\boldsymbol{w}(k)) = N(\boldsymbol{0}, Q(k)) \tag{3}$$

$$p(\boldsymbol{v}(k)) = N(\boldsymbol{0}, R(k)),\ R(k) = \begin{bmatrix} R_1(k) & \vdots & \boldsymbol{0} \\ \cdots & & \cdots \\ \boldsymbol{0} & \vdots & \boldsymbol{0}\end{bmatrix} \tag{4}$$

$$p(\boldsymbol{x}(1)) = N(\hat{\boldsymbol{x}}(1\,|\,0), P(1\,|\,0)) \tag{5}$$

where $N(\hat{\boldsymbol{x}}, P)$ denotes the normal p.d.f. with a mean vector $\hat{\boldsymbol{x}}$ and a covariance matrix P. $P(1\,|\,0)\in M_{nn}$ and $Q(k)\in M_{nn}$ are positive definite**; and $R(k)\in M_{mm}$,

* R^n denotes n-dimensional Euclidian space.

** If $Q(k)$ is non-negative, $S(k)$ in (15) may become singular. For this case, it is shown by Kalman [2] that (14) is valid by replacing $S^{-1}(k)$ with a pseudo-inverse $S^\sharp(k)$. Also see the reference [9].

$R_1(k) \in M_{(m-l)(m-l)}$, are non-negative definite.

Now some concepts about filters will be made clear. For the system described by (1) and (2), a filter is defined as a system which produces an estimate $\hat{x}(k) \in R^n$ of $x(k)$ given $\bar{y}(k) = \{y(1), y(2) \cdots, y(k)\}$ in the following manner:

$$z(k+1) = A(k+1)\Phi(k)\hat{x}(k) \tag{6}$$

$$\hat{x}(k) = B(k)z(k) + C(k)y(k) \tag{7}$$

where $A(k)$, $B(k)$ and $C(k)$ are arbitrary matrices. If $z(k) \in R^j$, then this filter is said to be of j-order. The optimal filter is the one which minimizes

$$J(k) = E\{(\hat{x}(k) - x(k))^T(\hat{x}(k) - x(k))\}$$

for all k, where the superscript "T" denotes transpose and $E\{\cdot\}$ denotes expectation operation. It is shown by Kalman [1] that for the optimal filter

$$\hat{x}(k) = E\{x(k) \,|\, y(k)\}\ .$$

This terminology is essentially the same as that of Tse and Athans [4] except that they used the term "estimator" instead of "filter". This can be easily seen by rewriting (6) and (7) into

$$z(k+1) = A(k+1)\Phi(k)B(k)z(k) + A(k+1)\Phi(k)C(k)y(k) \tag{8}$$

$$\hat{x}(k) = B(k)z(k) + C(k)y(k) \tag{9}$$

which correspond to (11) of Tse and Athans [4].

The form of (8) and (9) is suitable for the interpretation that $z(k)$ is the internal state vector of a dynamical system called filter. However, for the physical interpretation of coefficient matrices $A(k)$, $B(k)$ and $C(k)$, the form of (6) and (7) is better: $A(k)$ determines the way in which the information $z(k+1)$ (that must be stored in memory) is constructed from the present estimate $\hat{x}(k)$. $B(k)$ and $C(k)$ determine the way in which the estimate $\hat{x}(k)$ is constructed from the information $z(k)$ and the observed data $y(k)$.

The problem considered in this paper is to obtain the $(n-l)$-order optimal filter for the system described by (1) and (2).

III. Derivation of the Solution

Define

$$\hat{x}(k \,|\, j) = E\{x(k) \,|\, \bar{y}(j)\} \tag{10}$$

$$P(k \,|\, j) = E\{(\hat{x}(k \,|\, j) - x(k))\,(\hat{x}(k \,|\, j) - x(k))^T\}\ . \tag{11}$$

It is shown by Kalman [2] that $\hat{x}(k|k)$ satisfies

$$\hat{x}(k|k) = [I_n - K(k)H(k)]\hat{x}(k|k-1) + K(k)y(k) \tag{12}$$

$$x(k+1|k) = \Phi(k)\hat{x}(k|k) \tag{13}$$

where $I_n \in M_{nn}$ is the identity matrix and the filter gain $K(k) \in M_{nm}$ is obtained from the following equations.

$$K(k) = P(k|k-1)H^T(k)S^{-1}(k) \tag{14}$$

$$S(k) = H(k)P(k|k-1)H^T(k) + R(k) \tag{15}$$

$$P(k|k) = [I_n - K(k)H(k)]P(k|k-1) \tag{16}$$

$$P(k+1|k) = \Phi(k)P(k|k)\Phi^T(k) + Q(k)\,. \tag{17}$$

By taking $z(k) = \hat{x}(k|k-1)$, it is clear that the filter which consists of (12) and (13) is the n-order optimal filter. This is called the Kalman filter.

Now let

$$x(k) = \begin{bmatrix} x_1(k) \\ \cdots\cdots \\ x_2(k) \end{bmatrix}, \quad y(k) = \begin{bmatrix} y_1(k) \\ \cdots\cdots \\ y_2(k) \end{bmatrix} \tag{18}$$

$$H(k) = [H_1(k) \vdots H_2(k)] = \begin{bmatrix} H_{11}(k) \vdots H_{12}(k) \\ \cdots\cdots\cdots\cdots \\ H_{21}(k) \vdots H_{22}(k) \end{bmatrix} \tag{19}$$

where $x_2(k) \in R^l$, $y_2(k) \in R^l$, $H_{22}(k) \in M_{ll}$. Since rank $H(k) = m$, $H_{22}(k)$ can be assumed to be nonsingular without any loss of generality.

From (2), (18) and (19) we obtain

$$x_2(k) = H_{22}^{-1}(k)\,[y_2(k) - H_{21}(k)x_1(k)]\,. \tag{20}$$

It is possible to reduce the order of the filter from n to $(n-l)$ by modiyfing (12) and (13) through the use of (20). Its procedure is shown as follows. From (20)

$$\hat{x}_2(k|k) = H_{22}^{-1}(k)\,[y_2(k) - H_{21}(k)\hat{x}_1(k|k)]\,. \tag{21}$$

On the other hand, let

$$K(k) = \begin{bmatrix} K_1(k) \\ \cdots\cdots \\ K_2(k) \end{bmatrix}, \quad K_1(k) \in M_{(n-l)m} \tag{22}$$

then (12) is rewritten as

$$\hat{x}_1(k|k) = \{[I_{n-l} \vdots 0] - K_1(k)H(k)\}\hat{x}(k|k-1) + K_1(k)y(k) \tag{23}$$

$$x_2(k|k) = \{[0 \vdots I_l] - K_2(k)H(k)\}\hat{x}(k|k-1) + K_2(k)y(k\,)\,. \tag{24}$$

From the fact that (24) should be equal to the equation which is obtained by substituting (23) into (21), we have

$$K_2(k) = H_{22}^{-1}(k)\{[0 \vdots I_l] - H_{21}(k)K_1(k)\} . \tag{25}$$

Therefore

$$I_n - K(k)H(k) = \begin{bmatrix} I_{n-l} \\ \cdots\cdots \\ -H_{22}^{-1}(k)H_{21}(k) \end{bmatrix} [I_{n-l} - K_1(k)H_1(k) \vdots -K_1(k)H_2(k)]. \tag{26}$$

Let

$$A^*(k) = [I_{n-l} - K_1(k)H_1(k) \vdots -K_1(k)H_2(k)] \tag{27}$$

$$B^*(k) = \begin{bmatrix} I_{n-l} \\ \cdots\cdots \\ -H_{22}^{-1}(k)H_{21}(k) \end{bmatrix} \tag{28}$$

$$C^*(k) = K(k) . \tag{29}$$

Then we obtain from (12), (13) and (26),

$$\boldsymbol{z}(k+1) = A^*(k)\Phi(k)\hat{\boldsymbol{x}}(k\,|\,k) \tag{30}$$

$$\boldsymbol{z}(1) = A^*(0)\hat{\boldsymbol{x}}(1\,|\,0) \tag{31}$$

$$\hat{\boldsymbol{x}}(k\,|\,k) = B^*(k)\boldsymbol{z}(k) + C^*(k)\boldsymbol{y}(k) \tag{32}$$

Hence we have now one explicit form, (30)–(32), of the $(n-l)$-order optimal filters.

Although $K(k)$, which is necessary for the calculation of $A^*(k)$ and $C^*(k)$, can be calculated from (14)–(17), we can also rewrite (16) and (17) as follows by utilizing (20). Divide the matrixes $P(k\,|\,j)$ as

$$P(k\,|\,j) = \begin{bmatrix} P_{11}(k\,|\,j) \vdots P_{12}(k\,|\,j) \\ \cdots\cdots\cdots \\ P_{21}(k\,|\,j) \vdots P_{22}(k\,|\,j) \end{bmatrix}, \quad P_{22}(k\,|\,j) \in M_{ll}$$

Then from (20) and (21) we obtain

$$P(k\,|\,k) = B^*(k)P_{11}(k\,|\,k)B^{*T}(k) . \tag{33}$$

From (14), (16), (17) and (33)

$$K_1(k) = [I_{n-l} \vdots 0]P(k\,|\,k-1)H^T(k)S^{-1}(k) \tag{34}$$

$$P_{11}(k\,|\,k) = A^*(k)P(k\,|\,k-1)\begin{bmatrix} I_{n-l} \\ \cdots \\ \mathbf{0} \end{bmatrix} \tag{35}$$

$$P(k+1\,|\,k) = V(k)P_{11}(k\,|\,k)V^T(k) + Q(k) \tag{36}$$

$$V(k) = \Phi(k)B^*(k) . \tag{37}$$

Therefore, $K(k)$ can be calculated from (14)–(17) or from (15), (22), (25), (27), (28), (34)–(37). Which way is easier will depend on each specified case.

Brammer [3] has derived a result which is essentially the same as (30)–(32) by solving the Wiener-Hopf equation. Our method is, however, simpler and clearer than that of Brammer.

IV. Structure of the Minimal-order Optimal Filter

The problem of obtaining the minimal-order optimal filter for the system (1) and (2) is investigated by Tse and Athans [4]. Their result is summarized as follows.

If $R_1(k)$, $k=1, 2, \cdots$, is positive-definite, then the minimal-order optimal filter is of $(n-l)$-order. Among its coefficient matrices, $C(k)$ is given uniquely by (29). Other coefficient matrices $A(k)$ and $B(k)$ are given as an arbitrary pair $\{A(k), B(k)\}$ which satisfies

$$B(k)A(k)+C^*(k)H(k) = I_n \tag{38}$$

Furthermore, Tse and Athans present a method for obtaining an explicit pair $\{A(k), B(k)\}$ which satisfies (38). Their method, however, needs a solution of $(n-m)n+m_1n$ simultaneous algebraic equations (see (60 a) of the reference [4]).

Now let us show that, utilizing the result of the preceding section, the solution $\{A(k), B(k)\}$ of (38) can be expressed analytically. The result is as follows.

If $R_1(k)$, $k=1, 2, \cdots$, is positive-definite, the coefficient matrices $A(k)$, $B(k)$ and $C(k)$ of the minimal-order optimal filter are given by

$$\tilde{A}(k) = L(k)A^*(k) \tag{39}$$

$$\tilde{B}(k) = B^*(k)L^{-1}(k) \tag{40}$$

and $C^*(k)$ (see (29)), where $L(k) \in M_{(n-l)(n-l)}$, $k=1, 2, \cdots$, are arbitrary nonsingular matrices.

In order to prove this, it is enough to show that the pair $\{\tilde{A}(k), \tilde{B}(k)\}$ is a solution of (38) and that any solution of (38) can be expressed as $\{\tilde{A}(k), \tilde{B}(k)\}$ by choosing $L(k)$ adequately.

The former is obvious from (26), and the latter can be shown easily by noting that rank $[I-C^*(k)H(k)]=n-l$.

The result of this section is the same as that of [7]. It is included, however, in this paper again to make clear that the minimal-order optimal filter can be obtained easily by a slight modification of the ordinary Kalman filter and to give the following comment: Tse and Athans [8] are quite right in having the feeling that $P(k|k)$ may be obtainable through a matrix equation with lower dimension than n. Relations (34)–(37) are the very realization of this feeling.

V. The Case of Colored Measurement Noise

Consider a linear system

$$\boldsymbol{x}_s(k+1) = \Phi_s(k)\boldsymbol{x}_s(k)+\boldsymbol{w}_s(k) \tag{41}$$

$$\boldsymbol{y}(k) = H_s(k)\boldsymbol{x}_s(k)+\begin{bmatrix} \boldsymbol{v}_1(k) \\ \cdots\cdots \\ \boldsymbol{x}_n(k) \end{bmatrix} \tag{42}$$

$$\boldsymbol{x}_n(k+1) = \Phi_n(k)\boldsymbol{x}_n(k)+\boldsymbol{w}_n(k) \tag{43}$$

where $\boldsymbol{x}_s(k)\in R^{n-l}$ is the state vector, $\boldsymbol{y}(k)\in R^m$ is the measurement vector, $\boldsymbol{x}_n(k)\in R^l$ is the colored noise vector, and $\boldsymbol{w}_s(k)$, $\boldsymbol{w}_n(k)$, and $\boldsymbol{v}_1(k)$ are random vectors. (42) means that a part of the measurement noise is time-correlated (or colored).
Let

$$\boldsymbol{w}(k) = \begin{bmatrix} \boldsymbol{w}_s(k) \\ \cdots\cdots \\ \boldsymbol{w}_n(k) \end{bmatrix}, \quad \boldsymbol{x}(1) = \begin{bmatrix} \boldsymbol{x}_s(1) \\ \cdots\cdots \\ \boldsymbol{x}_n(1) \end{bmatrix}$$

and assume that $\boldsymbol{w}(k)$, $\boldsymbol{v}_1(k)$, $k=1, 2, \cdots$ and $\boldsymbol{x}(1)$ are mutually independent and their p.d.f.'s are given by

$$\begin{aligned} p(\boldsymbol{w}(k)) &= N(\mathbf{0}, Q(k)) \\ p(\boldsymbol{v}_1(k)) &= N(\mathbf{0}, R_1(k)) \\ p(\boldsymbol{x}_1(k)) &= N(\hat{\boldsymbol{x}}(1\,|\,0), P(1\,|\,0)) \,. \end{aligned}$$

We consider the problem of obtaining the maximum likelihood estimate of $\boldsymbol{x}_s(k)$, $\hat{\boldsymbol{x}}_s(k\,|\,k)$, from the measurements $\bar{\boldsymbol{y}}(k)$.

For the special case of $m=l$, that is, the case in which whole measurement signals are contaminated by colored noises, this problem has already been solved by Bryson and Henrikson [5]. They use their measurement differencing approach. They reformulate the original problem by taking as a new measurement a weighted first difference of the present and previous measurements, and then apply the filtering theory of Kalman.

In this section, the above problem is solved by using Kalman's augmented state approach and it is shown that the result of Bryson and Henrikson [5] can be derived from this solution by taking $m=l$.

Following Kalman [2], let

$$\boldsymbol{x}(k) = \begin{bmatrix} \boldsymbol{x}_s(k) \\ \cdots\cdots \\ \boldsymbol{x}_n(k) \end{bmatrix}, \quad \Phi(k) = \begin{bmatrix} \Phi_s(k) & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \Phi_n(k) \end{bmatrix}$$

$$H(k) = \begin{bmatrix} H_s(k) & \begin{matrix} \vdots\ \mathbf{0} \\ \cdots \\ \vdots\ I_l \end{matrix} \end{bmatrix} = \begin{bmatrix} H_{s1}(k) & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ H_{s2}(k) & \vdots & I_l \end{bmatrix}.$$

Then the above problem is reduced to the one in Section 2. This reduced problem can be solved in the same way as in Section 3. By eliminating $\hat{\boldsymbol{x}}_n(k\,|\,k)$, we finally obtain the following:

$$z(k+1) = [\Phi_s(k) - K_1(k+1)W(k)]\hat{x}_s(k|k) - K_1(k+1)U\Phi_n(k)U^T y(k) \tag{44}$$

$$x_s(k|k) = z(k) + K_1(k)y(k) \tag{45}$$

$$z(1) = [I_{n-l} - K_1(1)H_s(1) \vdots -K_1(1)U]\hat{x}(1|0) \tag{46}$$

where

$$K_1(k) = [I_{n-l} \vdots 0]P(k|k-1)H^T(k)S^{-1}(k) \tag{47}$$

$$S(k) = H(k)P(k|k-1)H^T(k) + R(k) \tag{48}$$

$$P_{11}(k|k) = [I_{n-l} - K_1(k)H_s(k) \vdots -K_1(k)U]P(k|k-1)\begin{bmatrix} I_{n-l} \\ \cdots \\ 0 \end{bmatrix} \tag{49}$$

$$P(k+1|k) = V(k)P_{11}(k|k)V^T(k) + Q(k) \tag{50}$$

$$V(k) = \begin{bmatrix} \Phi_s(k) \\ \cdots\cdots \\ -\Phi_n(k)H_{s2}(k) \end{bmatrix} \tag{51}$$

$$W(k) = H_s(k+1)\Phi_s(k) - U\Phi_n(k)H_{s2}(k) \tag{52}$$

$$U = \begin{bmatrix} 0 \\ \cdots \\ I_l \end{bmatrix} \in M_{ml} \tag{53}$$

For the case of $m=l$, we have only to set

$$\begin{aligned} U &= I_l \\ H_{s2}(k) &= H_s(k) \\ R(k) &= 0 \end{aligned} \tag{54}$$

in the above result. If, moreover, we eliminated $z(k)$ from (44) and (45), we have

$$\begin{aligned} \hat{x}_s(k+1|k+1) &= \Phi_s(k)\hat{x}_s(k|k) \\ &\quad - K_1(k+1)[y(k+1) - \Phi_n(k)y(k) - W(k)\hat{x}_s(k|k)] \end{aligned} \tag{55}$$

It is easy to show that (55) is equivalent to (10) of Bryson and Henrikson [5].

The continuous-time version of the problem in this section is solved by Bucy [7]. Letting $m=l$ in (44)–(54) and making the sampling interval tend to zero as in Kalman [2], we can derive (10) of Bucy [7], which expresses the filter involving no differentiation of observed data.

VI. Conclusion

The filtering problem of a linear discrete-time system is investigated where some elements of the measurement are free from noise. A low order optimal filter for this problem is obtained by modifying the ordinary Kalman filter. This result is useful not only for the unification of several previous results [3]–[5] and for an easy understanding of them, but also for obtaining extended results: Firstly, on the basis of the above result, the structure of Tse-Athans' optimal minimal-order filter is made clear. Secondly, the filtering problem with colored measurement noise is solved and it is shown that this solution contains the Bryson-Henrikson's result as a special case.

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