Some Contributions to Optimization of Faraday MHD Generator Duct

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By

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For the purpose of contributing to the optimum design of the Faraday MHD generator duct, the authors derive a numerical calculation from the basic quasi one-dimensional MHD equations of the diverging rectangular duct and the integrals which express duct size, viz. length, surface area or volume. The calculation is intended to minimize the integrals under the condition of extracting a required output power from the duct, when the applied magnetic flux density, the mass flow rate and the duct inlet or outlet total pressure and temperature of the working gas are held constant.

1. Introduction

In designing an MHD generator, it is very important that the duct is constructed in optimum form, for example, the duct size, namely length, surface area or volume, is minimized under the condition of extracting a needed output power, when the applied magnetic flux density, the mass flow, the total pressure and temperature in the duct inlet or outlet are kept constant. This is accomplished by means of optimizing the distributions of the gas pressure, temperature, velocity, the loading parameter and etc. along the flow. Using such an idea and applying calculus of variations to the quasi one-dimensional MHD equations and an integral, which expresses the duct size, Carter[1),2)] and others[3)~6)] have proposed a new optimization theory for the ideally segmented electrode Faraday generator duct of constant velocity, constant or distributed Mach number.

But Carter and others have treated the case where the electrical conductivity of the working gas is governed by a power law of the gas pressure and temperature. So in this paper, the authors shall study the minization of duct size in the case where the conductivity is represented by a power-exponential formula, by means of the quasi one-dimentional MHD equations and the duct size integrals. In this connection, they point out that Carter's integral doesn't give the correct expression for the duct surface area, though it does the correct one for the duct length

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or volume. And they obtain the accurate expression for the area. Then they discuss the optimization of the conventional diverging rectangular duct of constant velocity, constant Mach number, constant loading parameter etc. Further they investigate the optimization of the prearranged cross-sectional area duct[7] of the constant velocity, constant Mach and constant loading parameter.

Finally in this paper, the authors assume that the working gas obeys the perfect gas law and neglect the thermal and frictional losses.

2. Basic Equations

As is well known, the quasi one-dimentional magnetohydrodynamic equations pertaining to the ideally segmented electrode generator duct (Fig. 1) are given by

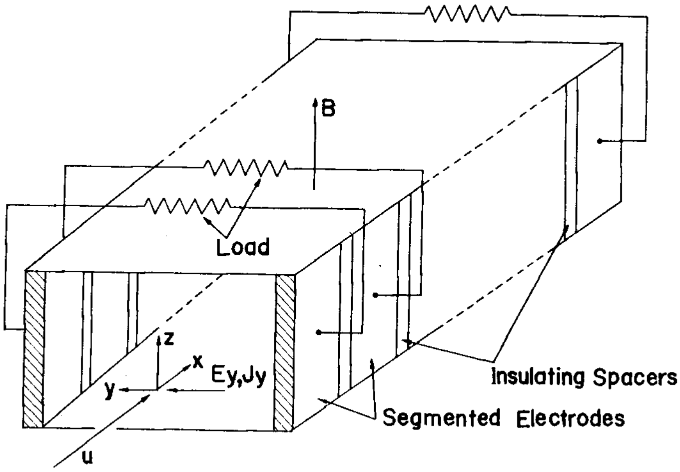

Fig. 1. Sketch of segmented electrode Faraday generator duct.

$\rho uA = \rho_0 u_0 A_0 = m_0$: continuity equation, (1)

$\rho u \frac{du}{dx} + \frac{dp}{dx} = J_y B$: momentum equation, (2)

$\rho u \frac{d}{dx}(c_p T + u^2/2) = J_y E_y$: energy equation, (3)

$p = \rho RT$: state equation, (4)

and the Ohm's law is given by

$$J_y = \sigma(E_y - uB) = -\sigma uB(1-\kappa). \quad (5)$$

In these equations and Fig. 1

A: cross-sectional area of duct,

B: magnetic flux density*,

$c_p = \alpha R$: specific heat* at constant pressure,

c_v: specific heat* at constant volume,
E_y: electric field intensity in y-direction,
J_y: current density in y-direction, which is equal to total one,
m_0: mass flow rate*,
p: gas pressure,
R: gas constant*,
T: gas temperature,
u: gas velocity,
$\alpha=\gamma/(\gamma-1)$*,
$\gamma=c_p/c_v$: specific heat ratio*,
$\kappa=E_y/uB$: loading parameter,
σ: electrical conductivity,
suffixes 0 and 1: denote the quantities in duct inlet and outlet respectively,
*: shows the quantities which are assumed constant in analysis. (6)

Now many pioneers in this field have shown that σ depends on p and T with respect to the working gases with no elevation of electron temperature. So we assume that σ is represented by the following expression

$$\sigma=cp^m T^n \exp(-T_i/T), \tag{7}$$

where

c, m, n and T_i: constants. (7)′

Next the integral which expresses the duct size has been given by Carter as follows:

$$I_N=\int_0^l A^N dx=\begin{cases} l \text{ for } N=0, \\ V \text{ for } N=1, \end{cases} \tag{8}$$

where

l : duct length,
V: duct volume. (8)′

Though Carter has described that four times of $I_N(N=1/2)$ in Eq. (8) give the duct surface area, the result doesn't give the correct area, because dA/dx is not considered in Eq.s (8). The accurate expression of $I_{1/2}=S$ is given by

$$S=I_{1/2}=2\int_0^l\left[\left\{\frac{A}{k}+\left(\frac{A'}{4}\right)^2\right\}^{1/2}+\left\{kA+\left(\frac{A'}{4}\right)^2\right\}^{1/2}\right]dx \qquad (9)_1$$

for the diverging rectangular duct, where

$A'=dA/dx$,
k: ratio of duct width and height. $(9)'_1$.

In this connection, as $k=1$ for the square duct, we have

$$S=I_{1/2}=4\int_0^l \{A+(A'/4)^2\}^{1/2}dx. \qquad (9)_2$$

Moreover the value of S becomes minimum in the case of the square duct.

3. Optimization Theory of Diverging Rectangular Duct

3.1 Constant velocity duct

First let us discuss the optimization in the case of constant velocity duct, viz.

$$u=u_0=\text{constant}. \tag{10}$$

Then using Eq.s (2) to (5) and the transformation of variable of $x=l-x'$, we obtain

$$\frac{dp}{dx'}=\sigma u_0 B^2(1-\kappa), \tag{11}$$

$$\frac{dT}{dx'}=\frac{\sigma u_0 B^2(1-\kappa)\kappa T}{\alpha p}. \tag{12}$$

Here putting

$$\log(p/p_1)=\alpha(\xi+\zeta), \tag{13}$$

$$\log(T/T_1)=\zeta, \tag{14}$$

Eq.s (11) and (12) are transformed as follows:

$$\frac{d\xi}{dx'}=\frac{\sigma u_0 B^2(1-\kappa)^2}{\alpha p}, \tag{15}$$

$$\frac{d\zeta}{dx'}=\frac{\sigma u_0 B^2(1-\kappa)\kappa}{\alpha p}. \tag{16}$$

From these two equations, it follows that

$$\frac{d\xi}{d\zeta}=\frac{1-\kappa}{\kappa}. \tag{17}$$

Transforming I_N in Eq. (8) with Eq.s (10), (13), (14) and (16), we arrive at

$$I_N=\int_0^{\zeta_0} A^N \frac{dx'}{d\zeta}d\zeta=C_u I_{Nu}, \tag{18}$$

where

$$\left.\begin{aligned}
&I_{Nu}=\int_0^{\zeta_0}\frac{\exp\{-r\zeta+q\xi+T_t^*\exp(-\zeta)\}}{(1-\kappa)\kappa}d\zeta,\\
&C_u=C_{t1}(1-\delta_{t1}u_0^2)^{-r}u_0^{-(N+1)},\\
&C_{t1}=\{\alpha p'_{t1}/(\sigma'_{t1}B^2)\}(m_0 R T_{t1}/p'_{t1})^N,\\
&q=-(m+N-1)\alpha,\\
&r=-N+n-q,\\
&T_t^*=T_t/T_1,\\
&T_0=T_{t0}(1-\delta_{t0}u_0^2),\\
&T_1=T_{t1}(1-\delta_{t1}u_0^2),\\
&T_{t1}=T_{t0}-W/(c_p m_0),\\
&W\text{: output power},\\
&\delta_{t0}=1/(2c_pT_{t0}),\qquad \delta_{t1}=1/(2c_pT_{t1}),\\
&\zeta_0=\log(T_0/T_1),
\end{aligned}\right\} \tag{18)'$$

$$\left.\begin{array}{l}\sigma_{t1}'=cp'_{t1}{}^{m}T_{t1}{}^{n},\\ p_{t1}'=p_{t1}\{1+u_0^2/(2c_pT_1)\}^{\alpha}\{1+\alpha u_0^2/(2c_pT_1)\}^{-1},\\ p_t:\ \text{total pressure,}\\ T_t:\ \text{total temperature.}\end{array}\right)$$

Elimination of κ from Eq. (17) and the first equation of (18)′ yields

$$I_{Nu}=\int_0^{\zeta_0}F_{Nu}(\zeta,\xi,\xi')\,d\zeta, \tag{19}$$

where

$$F_{Nu}(\zeta,\xi,\xi')=\frac{(\xi'+1)^2}{\xi'}\exp\{-r\zeta+q\xi+T_t{}^*\exp(-\zeta)\},\ \xi'=d\xi/d\zeta. \quad (19)'$$

From Eq. (17) and the Euler differential equation about ξ and ζ for the minimization of I_N, which can be derived from Eq.s (19) and (19)′ with the aid of calculus of variations, we can obtain the following one

$$\frac{d\kappa}{d\zeta}=-(q+r_\zeta)\{\lambda(\zeta)-\kappa\}(1-\kappa)/\kappa, \tag{20}$$

where

$$\left.\begin{array}{l}\lambda(\zeta)=(q+r_\zeta/2)/(q+r_\zeta),\\ r_\zeta=r+T_t{}^*\exp(-\zeta).\end{array}\right\} \quad (20)'$$

When m_0, T_{t0} and a required output power W are assumed constant in a constant velocity duct, T_0 and T_1 or ζ_0 become constant. Therefore we see that I_N in Eq.s (18) is the integral which is intended to minimize the duct size under the condition of extracting W from the duct.

Next if we are able to solve Eq.s (17) and (20) simultaneously by a suitable numerical calculation as Runge-Kutta-Gill one, we can determine the optimum relations of κ and ξ vs. ζ, consequently κ and p vs. T pertaining to the specific values of parameters Γ's, where Γ denotes one of u_0, M_0, κ_0, M_1, κ_1 and etc. Moreover when we apply the numerical solutions of Eq.s (17) and (20) with the various values of Γ's to I_N, we can find the values of Γ's which minimize still more $I_{N\min}$.

Next let us discuss the minimization of the surface area of the square duct with Eq. $(9)_2$.

From Eq.s (1), (11) and (12), we can derive

$$A'=A\left(-\frac{1}{T}\frac{dT}{dx'}+\frac{1}{p}\frac{dp}{dx'}\right),$$
$$=\frac{A\sigma u_0B^2(1-\kappa)(\alpha-\kappa)}{\alpha p}. \tag{21}$$

By use of Eq.s (1), $(9)_2$, (13), (14), (16) and (21), we obtain

$$S=I_{1/2}=4\int_0^{\zeta_0}\left\{A+\left(\frac{A'}{4}\right)^2\right\}^{1/2}\frac{dx'}{d\zeta}d\zeta=4Cu\int_0^{\zeta_0}Fu(\zeta,\xi,\kappa)\,d\zeta, \tag{22}$$

where

$$\left.\begin{aligned}&F_u(\zeta,\xi,\kappa)=\frac{\exp\{-r\zeta+q\xi+T_i^*\exp(-\zeta)\}}{\kappa(1-\kappa)}\left[1+\left\{\frac{A_1^{2N}}{4C_u}\varphi_u(\zeta,\xi)\right\}^2\right]^{1/2},\\&\varphi_u(\zeta,\xi)=(1-\kappa)(\alpha-\kappa)\exp\{r'\zeta-q'\xi-T_i^*\exp(-\zeta)\},\\&q'=-\alpha(m-1-N),\\&r'=n+N-q',\\&N=1/2,\\&C_u,\ T_i^*:\ \text{given in Eq.s (18)}'.\end{aligned}\right\}\quad(22)'$$

When Eq. (17) namely

$$G_u(\zeta,\xi',\kappa)=\xi'-(1-\kappa)/\kappa=0 \tag{23}$$

is adopted as one subsidiary condition, the problem which lets us minimize S becomes the one that we solve the following simultaneous Euler equations

$$\left.\begin{aligned}&F_{u\xi}-\frac{d\nu(\zeta)}{d\zeta}=0,\\&F_{u\kappa}+\nu(\zeta)G_{u\kappa}=0.\end{aligned}\right\}\quad(24)$$

where

$$\left.\begin{aligned}&F_u=F_u(\zeta,\xi,\kappa),\qquad G_u=G_u(\zeta,\xi',\kappa),\\&F_{u\mu}=\partial F_u/\partial\mu,\qquad G_{u\mu}=\partial G_u/\partial\mu,\qquad \mu=\xi,\kappa,\\&\nu(\zeta):\ \text{Lagrange multiplier.}\end{aligned}\right\}\quad(24)'$$

By eliminating $\nu(\zeta)$ from the two equations in Eq.s (24) and combining the result with Eq.s (23), we can obtain the simultaneous differential equations which determine the optimum distributions of κ and ξ vs. ζ for the specially-fixed values of Γ's as follows:

$$\left.\begin{aligned}&F_{u\xi}+\frac{d}{d\zeta}\left(\frac{F_{u\kappa}}{G_{u\kappa}}\right)=0,\\&\xi'-(1-\kappa)/\kappa=0.\end{aligned}\right\}\quad(25)$$

Moreover by applying the numerical solutions of Eq.s (25) for the various values of Γ's to Eq.s (22), we are able to find. the values of Γ's, which minimize the minimum value of S.

Next I_N gives the duct length l when $N=0$, as shown in Eq.s (8). Accordingly putting ζ instead of ζ_0 in Eq.s (18) and (18)′, we can obtain the relation between ζ or T and x', i.e.

$$x'=C_u\int_0^{\zeta}\frac{\exp\{-r\zeta+q\xi+T_i^*\exp(-\zeta)\}}{(1-\kappa)\kappa}d\zeta. \tag{26}$$

Therefore it is seen that the numerical solution of x' has been already acquired on the way of numerical computation of I_0 in Eq.s (18). Also Eq.s (22) occur in the case where the optimization of the duct surface area is discussed.

Moreover the duct cross-sectional area A is evaluated by the equation

$$A=m_0/\rho u_0, \tag{27}$$

which is derived from Eq.s (1) and (10).

The other quantities, for example, σ can be digitally computed by substituting the numerical values of p and T, which are obtained by the above mentioned procedure, into Eq. (7).

3.2 Constant velocity and constant loading parameter duct

Concerning the constant velocity and constant loading parameter duct, we have

$$\left.\begin{array}{l} u=u_0, \\ \kappa=\kappa_0. \end{array}\right\} \tag{28}$$

Substituting Eq.s (28) in Eq. (17) give

$$\begin{array}{l} \xi'=(1-\kappa_0)/\kappa_0, \\ \therefore \xi=(1-\kappa_0)\zeta/\kappa_0, \end{array} \tag{29}$$

with the aid of $\xi_1=\zeta_1=0$.

Using Eq.s (28) and (29), I_N in Eq. (18) is reduced to

$$I_N=\frac{C_u}{(1-\kappa_0)\kappa_0}\int_0^{\zeta_0}\exp[\{q-(q+r)\kappa_0\}\zeta/\kappa_0+T_i^*\exp(-\zeta)]d\zeta. \tag{30}$$

Next $S=I_{1/2}$ becomes

$$S=I_{1/2}=4\frac{C_u}{\kappa_0(1-\kappa_0)}\int_0^{\zeta_0}F_{u\kappa}(\zeta)\,d\zeta. \tag{31}$$

where

$$F_{u\kappa}(\zeta)=\exp\left\{\frac{q-(q+r)\kappa_0}{\kappa_0}\zeta+T_i^*\exp(-\zeta)\right\}\left[1+\left\{\frac{A_1^{2N}(1-\kappa_0)(\alpha-\kappa_0)}{4C_u}\varphi_{u\kappa}(\zeta)\right\}^2\right]^{1/2},$$

$$\varphi_{u\kappa}(\zeta)=\exp\left\{\frac{-q'+(q'+r')\kappa_0}{\kappa_0}\zeta-T_i^*\exp(-\zeta)\right\}. \tag{31$'$}$$

With Eq.s (29), (30) and (31), we can numerically obtain the values of l, V and S. When we give Γ's the various values, we can find the values which make I_N minimum. In addition, as described in the preceding article, letting $N=0$ and substituting ζ in place of ζ_0 in Eq. (30), we have

$$x'=\frac{C_u}{(1-\kappa_0)\kappa_0}\int_0^{\zeta}\exp[\{q-(q+r)\kappa_0\}\zeta/\kappa_0+T_i^*\exp(-\zeta)]d\zeta. \tag{32}$$

The values of A, σ and etc. can be evaluated in the same way as in the preceding case.

3.3 Constant Mach number duct

Here let us discuss the optimization of the constant Mach number duct.

We can rewrite the basic flow equations (2) and (3) in terms of the total or stagnation quantities with the following relations

$$\left.\begin{array}{l} T_t/T=(1+X), \\ p_t'/p=(T_t/T)^{\alpha}, \end{array}\right\} \tag{33}$$

where

$$\left.\begin{array}{l} X=u^2/(2c_pT)=(\gamma-1)M^2/2, \\ M=u/\sqrt{\gamma RT}: \text{ Mach number}, \\ p_t'=p_t(1+X)^{\alpha}(1+\alpha X)^{-1} \end{array}\right\} \quad (33)'$$

as follows:

$$\frac{1}{p_t'}\frac{dp_t'}{dx'}+\frac{\alpha X}{T_t}\frac{dT_t}{dx'}=\frac{\sigma uB^2(1-\kappa)}{p}, \quad (34)$$

$$\frac{\alpha(1+X)}{T_t}\frac{dT_t}{dx'}=\frac{\sigma uB^2(1-\kappa)\kappa}{p}. \quad (35)$$

Here using

$$\log(p_t'/p_1')=\alpha(\zeta_t+\xi_t), \quad (36)$$

$$\log(T_t/T_{t1})=\zeta_t, \quad (37)$$

Eq.s (34) and (35) are transformed into

$$\frac{d\xi_t}{dx'}=\frac{\sigma uB^2(1-\kappa)^2}{\alpha p}, \quad (38)$$

$$\frac{d\zeta_t}{dx'}=\frac{\sigma uB^2(1-\kappa)\kappa}{\alpha(1+X)p}. \quad (39)$$

From these two equations, we get

$$\frac{d\xi_t}{d\zeta_t}=\frac{(1+X)(1-\kappa)}{\kappa}. \quad (40)$$

Now in the constant Mach number duct, in which $M=M_0=$constant and accordingly

$$X=X_0=\text{constant}, \quad (41)$$

Eq. (8) is transformed into

$$I_N=C_XI_{NX}, \quad (42)$$

where

$$\left.\begin{array}{l} I_{NX}=\displaystyle\int_0^{\zeta_{t0}}\frac{\exp\{-r_t\zeta_t+q\xi_t+T_{tt}^*(1+X_0)\exp(-\zeta_t)\}}{(1-\kappa)\kappa}d\zeta_t, \\ C_X=C_{t1}'X_0^{w}(1+X_0)^{1+r_t}, \\ C_{t1}'=\dfrac{\alpha p_{t1}'}{\sigma_{t1}'B^2(2c_pT_{t1})^{1/2}}\left(\dfrac{m_0RT_{t1}}{p_{t1}'\sqrt{2c_pT_{t1}}}\right)^N, \\ r_t=-N/2+n+1/2-q, \\ w=-(N+1)/2, \\ T_{tt}^*=T_t/T_{t1}, \\ q \text{ and } \sigma_{t1}': \text{ see Eq.s (18)}' \end{array}\right\} \quad (42)'$$

by the aid of Eq.s (33), (36), (37), (39) and (41). Eliminating κ from Eq.s (40), where $X=X_0$, and (42)' yields

$$I_{NX}=\int_0^{\zeta_{t0}}F_{NX}(\zeta_t,\xi_t,\xi_t')d\zeta_t, \quad (43)$$

where

$$\left.\begin{aligned}&F_{NX}(\zeta_t,\xi_t,\xi_t')=\frac{\{\xi_t'+(1+X_0)\}^2}{(1+X_0)\xi_t'}\exp\{-r_t\zeta_t+q\xi_t+T_{tt}^*(1+X_0)\exp(-\zeta_t)\},\\&\xi_t'=d\xi_t/d\zeta_t.\end{aligned}\right\}\quad(43)'$$

The Euler equation pertaining to ξ_t and ζ_t for the optimization of I_N, which is obtained from Eq.s (40), (43) and (43)', reduces to

$$\frac{d\kappa}{d\xi_t}=-\{(1+X_0)q+r_{t\zeta}\}\{\lambda_t(\zeta)-\kappa\}(1-\kappa)/\kappa,\quad(44)$$

where

$$\left.\begin{aligned}&\lambda_t(\zeta)=\{(1+X_0)q+r_{t\zeta}/2\}/\{(1+X_0)q+r_{t\zeta}\},\\&r_{t\zeta}=r_t+T_{tt}^*(1+X_0)\exp(-\zeta_t).\end{aligned}\right\}\quad(44)'$$

Next the procedure similar to the one, by which we have obtained Eq.s (22) and (22)', gives

$$S=I_{1/2}=4C_X\int_0^{\zeta_{t0}}F_X(\zeta_t,\xi_t,\kappa)\,d\zeta_t,\quad(45)$$

where

$$F_X(\zeta_t,\xi_t,\kappa)=\frac{\exp\{-r_t\zeta_t+q\xi_t+T_{tt}^*(1+X_0)\exp(-\zeta_t)\}}{\kappa(1-\kappa)}\left[1+\left\{\frac{A_1^{2N}}{4C_X}\varphi_X(\zeta_t,\xi_t,\kappa)\right\}^2\right]^{1/2},$$

$$\varphi_X(\zeta_t,\xi_t,\kappa)=(1-\kappa)\{\kappa(1/2+X_0)-\alpha(1+X_0)\}\exp\{r'\zeta_t-q'\xi_t-T_{tt}^*\exp(-\zeta_t)\},$$

$$N=1/2.\quad(45)'$$

The simultaneous Euler equations for the minimization of S, which can be derived from Eq.s (45), reduce to

$$F_X+\frac{d}{d\zeta_t}\left(\frac{F_{X\kappa}}{G_{X\kappa}}\right)=0,\quad(46)$$

where

$$G_X=\xi_t'-(1+X_0)(1-\kappa)/\kappa=0,\quad(46)'$$

through the procedure similar to the one by which the first equation of Eq.s (25) has been obtained.

By using Eq.s (33), (40), (42), (44), (45) and (46) and carrying out the same computation as described in Article 3.1, we can minimize the duct size.

3.4 Constant loading parameter duct

In the constant loading prameter duct, we have

$$\kappa=\kappa_0=\text{constant.}\quad(47)$$

Substituting Eq.s (47) in Eq.s (40) yields

$$\xi_t'=\frac{(1-\kappa_0)(1+X)}{\kappa_0}.\quad(48)$$

Next transforming Eq. (8) as in the preceding case, we obtain

$$I_N=C_\kappa I_{N\kappa},\quad(49)$$

where

$$I_{N\kappa}=\int_0^{\zeta_{t0}}X^w(1+X)^{rt+1}\exp\{-r_t\zeta_t+q\xi_t+T_{tt}^*(1+X)\exp(-\zeta_t)\}d\zeta_t,$$

$$C_x = C_{t1}'/\{(1-\kappa_0)\kappa_0\}. \tag{49'}$$

Eliminating X from Eq.s (48) and (49)′ yields

$$I_{Nk} = \int_0^{\zeta_{t0}} F_{Nk}(\zeta_t, \xi_t, \xi_t')\, d\zeta_t, \tag{50}$$

where

$$F_{Nk}(\zeta_t, \xi_t, \xi_t') = \frac{\{\xi_t' - (1-\kappa_0)/\kappa_0\}^w \xi_t'^{r_t+1}}{\{(1-\kappa_0)/\kappa_0\}^{w+r_t+1}} \exp\{-r_t\zeta_t + q\xi_t + T_{tt}^* \kappa_0 \xi_t' \exp(-\zeta_t)/(1-\kappa_0)\}. \tag{50'}$$

The Euler equation pertaining to X and ζ_t for the minimization of I_N, which is derived from Eq.s (48), (50) and (50)′, reduces to

$$\frac{dX}{d\zeta_t} = -\Big[\frac{1-\kappa_0}{\kappa_0} q + T_{tt}^* \exp(-\zeta_t) + \{r_t + \Big(T_{tt}^* \exp(-\zeta_t) - \frac{1-\kappa_0}{\kappa_0} q\Big)(1+X)\}\theta(X,\zeta_t)\Big] \times \Big\{\frac{w}{X^2} + \frac{r_t+1}{(1+X)^2} - \theta^2(X,\zeta_t)\Big\}^{-1}, \tag{51}$$

where

$$\theta(X,\zeta_t) = \frac{w}{X} + \frac{r_t+1}{1+X} + T_{tt}^* \exp(-\zeta_t). \tag{51'}$$

Here, too, by using Eq.s (48) to (51)′ and carrying out the same calculation as described in Article 3.1, we can accomplish the minimization of the duct size.

3.5 Constant Mach number and constant loading parameter duct

In this case, we have

$$\left.\begin{aligned} X &= X_0, \\ \kappa &= \kappa_0. \end{aligned}\right\} \tag{52}$$

Substituting Eq.s (52) in Eq. (40) gives

$$\xi_t' = (1+X_0)(1-\kappa_0)/\kappa_0,$$
$$\therefore \xi_t = (1+X_0)(1-\kappa_0)\zeta_t/\kappa_0 \tag{53}$$

with the aid of $\xi_{t1} = \zeta_{t1} = 0$.

Using Eq.s (52) and (53), I_N in Eq. (42) or (49) is transformed into

$$I_N = \frac{C_X}{(1-\kappa_0)\kappa_0} \int_0^{\zeta_{t0}} \exp[\{-r_t + (1+X_0)(1-\kappa_0)q/\kappa_0\}\zeta_t + T_{tt}^*(1+X_0)\exp(-\zeta_t)]\, d\zeta_t. \tag{54}$$

Also $I_{1/2}$ in Eq. (45) is transformed into

$$S = I_{1/2} = \frac{4C_X}{\kappa_0(1-\kappa_0)} \int_0^{\zeta_{t0}} F_{kX}(\zeta_t)\, d\zeta_t, \tag{55}$$

where

$$\left.\begin{aligned} F_{kX}(\zeta_t) &= \exp\Big[\Big\{-r_t + q(1+X_0)\frac{(1-\kappa_0)}{\kappa_0}\Big\}\zeta_t + T_{tt}^*(1+X_0)\exp(-\zeta_t)\Big]\Big[1 + \Big\{\frac{A_1^{2N}}{4C_X}\varphi_{Xk}(\zeta_t)\Big\}^2\Big]^{1/2}, \\ \varphi_{Xk}(\zeta_t) &= (1-\kappa_0)\{\kappa_0(1/2+X_0) - \alpha(1+X_0)\} \end{aligned}\right\} \tag{55'}$$

$$\times\exp\left[\left\{r'-q'(1+X_0)\frac{1-\kappa_0}{\kappa_0}\right\}\zeta_t-T_{tt}{}^*\exp(-\zeta_t)\right],$$

$$N=1/2.$$

As described in Article 3.2, by use of Eq.s (54) or (55) we can numerically determin the duct size. When we give Γ's the various values in the calculation of I_N, we can find the optimum values of Γ's which minimize I_N.

4. Optimization Theory of Prearranged Cross-Sectional Area Duct

In this section, we introduce an optimization theory of the duct, whose cross-sectional area is assumed to vary according to a predetermined function of x' from outlet to inlet. Now let us represent the cross-sectional area by

$$A=A_1A^*, \tag{56}$$

where

$$A^*=A^*(x'): \text{ a predetermined function of } x'. \tag{56$'$}$$

4.1 Constant velocity duct

First let us discuss the optimization of constant velocity duct. From Eq.s (1), (10) and (56), we get

$$A^*=A/A_1=p_1T/(pT_1). \tag{57}$$

By use of ζ and ξ, which are defined in Eq.s (13) and (14), Eq.s (57) are transformed as follow:

$$\frac{d\xi}{d\zeta}=\frac{1-\alpha}{\alpha}-\frac{A^{*\prime}}{\alpha A^*}\frac{dx'}{d\zeta}, \tag{58}$$

where

$$A^{*\prime}=dA^*/dx'. \tag{58$'$}$$

From Eq.s (16), (17) and (58), we can derive

$$\kappa^2-(\alpha+1)\kappa+\alpha\varphi_u(\zeta,\xi)=0, \tag{59}$$

where

$$\left.\begin{aligned}&\varphi_u(\zeta,\xi)=1+\frac{C_u}{\alpha A_1{}^N}\frac{A^{*\prime}}{A^*}\exp\{-r'\zeta+q'\xi+T_t{}^*\exp(-\zeta)\},\\&q'=\alpha(1-m),\\&r'=n-q',\end{aligned}\right\} \tag{59$'$}$$

C_u, A_1, $T_t{}^*$: given in Eq. (18),

in which the loading parameter κ must also satisfy the condition $0\leqq\kappa\leqq1$. And Eq.s (15) and (16) are rewritten as follows:

$$\frac{d\xi}{dx'}=\frac{A_1{}^N}{C_u}(1-\kappa)^2\exp\{r'\zeta-q'\xi-T_t{}^*\exp(-\zeta)\}, \tag{60}$$

$$\frac{d\zeta}{dx'}=\frac{\kappa}{1-\kappa}\frac{d\xi}{dx'}, \tag{61}$$

with the aid of Eq.s (13) and (14), respectively.

When Eq.s (59), (60) and (61) can be solved by a appropriate digital calculation for the specific values of the parameters Γ's, we can find the numerical relations of ζ, ξ, and κ vs. x' or p, T, and κ vs. x'. While the digital calculations is carried on, we can evaluate x' for T_0, which is determined by m_0, W and T_{t0}, namely the duct length l for the specified values of Γ's. When l is obtained in such a way, the surface area S and volume V are able to be numerically calculated by the following equations

$$S=I_{1/2}=4A_1^{1/2}\int_0^l\left\{A^*+A_1\left(\frac{A^{*\prime}}{4}\right)^2\right\}^{1/2}dx', \tag{62}$$

$$V=I_1=A_1\int_0^l A^*dx', \tag{63}$$

which are obtained by substituting Eq. (56) into Eq.s $(9)_2$ and (8) respectively.

When we give Γ's the various values to solve Eq.s (59), (60) and (61), we can find a set of values of Γ's, which minimize l and consequently S and V.

Next as one practical example of the prearranged cross-sectional area duct, we discuss the one whose cross-sectional area A is given by

$$A=A_1(1-gx')^2, \tag{64}$$

where g is the coefficient corresponding to the duct gradient. Then A^* and $A^{*\prime}$ are expressed by

$$\left.\begin{aligned} A^* &= (1-gx')^2, \\ A^{*\prime} &= -2g(1-gx'), \end{aligned}\right\} \tag{65}$$

and Eq.s (62) and (63) are rewriten as follows:

$$S=(2A_1^{1/2}/g)(1+A_1g^2/4)^{1/2}\{1-(1-gl)^2\}, \tag{66}$$

$$V=(A_1/3g)\{1-(1-gl)^3\}. \tag{67}$$

By numerically solving Eq.s (59), (60), (61), (66) and (67) for the various values of Γ's, where Γ's contain g, we can find a set of values of Γ's, which make duct size minimum.

4.2 Constant Mach number duct

In this case, Eq.s (1) and (56) give

$$A^*=\frac{A}{A_1}=\frac{Tp_1u_1}{T_1pu} \tag{68}$$

Now, by use of ζ_t and ξ_t which are defined in Eq.s (36) and (37), Eq.s (68) are transformed as follows:

$$\frac{d\xi_t}{d\zeta_t}=\frac{1}{2\alpha}-1-\frac{A^{*\prime}}{\alpha A^*}\frac{dx'}{d\zeta_t} \tag{69}$$

From Eq.s (39), (40), (41) and (69), we obtain the following relation

$$\left(X_0+\frac{1}{2\alpha}\right)\kappa^2-\left(2X_0+1+\frac{1}{2\alpha}\right)\kappa+(1+X_0)\varphi_X(\xi_t,\zeta_t)=0, \tag{70}$$

where

$$\left.\begin{aligned}&\varphi_X(\zeta_t,\xi_t)=1+\frac{C_X}{\alpha(1+X_0)A_1{}^N}\frac{A^{*\prime}}{A^*}\exp\{-r_t'\zeta_t+q'\xi_t+T_{tt}{}^*(1+X_0)\exp(-\zeta_t)\},\\&q'=\alpha(1-m),\\&r_t'=n+1/2-q',\\&C_X,\ T_{tt}{}^*\text{: defined in Eq. (42)}'.\end{aligned}\right\}\quad(70)'$$

Eqs. (38) and (39) are rewritten as follows:

$$\frac{d\xi_t}{dx'}=\frac{(1+X_0)A_1{}^N}{C_X}(1-\kappa)^2\exp\{r_t'\zeta_t-q'\xi_t-T_{tt}{}^*(1+X_0)\exp(-\zeta_t)\},\qquad(71)$$

$$\frac{d\zeta_t}{dx'}=\frac{1}{(1+X_0)}\frac{\kappa}{1-\kappa}\frac{d\xi_t}{dx'},\qquad(72)$$

with the aid of Eq.s (36) and (37).

By procedure similar to the one in the preceding article, we can find a set of values of Γ's, which makes I_N minimum. In this connection, the values of S and V are calculated by the Eq. (62) and (63) for $A=A_1A^*(x')$ or (66) and (67) for $A=A_1(1-gx')^2$.

4.3 Constant loading parameter duct

In the case of constant loading parameter duct, the following relations

$$\frac{d\xi_t}{dx'}=\frac{A_{t1}{}^N}{C_k}\frac{1-\kappa_0}{\kappa_0}X^{1/2}(1+X)^{-r_{t'}}\exp\{r_t'\zeta_t-q\xi_t-T_{tt}{}^*(1+X)\exp(-\zeta_t)\},\qquad(73)$$

$$\frac{d\zeta_t}{dx'}=\frac{\kappa_0}{1-\kappa_0}\frac{1}{1+X}\frac{d\xi_t}{dx'},\qquad(74)$$

$$\frac{d\xi_t}{d\zeta_t}=\frac{1-\kappa_0}{\kappa_0}(1+X),\qquad(75)$$

are derived from Eq.s (38), (39), (40) and (48). Using Eq.s (1), (36), (37) and (56), we get

$$A^*=\left(\frac{X_1}{X}\right)^{1/2}\left(\frac{1+X}{1+X_1}\right)^{\alpha-1/2}\exp\{-(\alpha-1/2)\zeta_t-\alpha\xi_t\}.\qquad(76)$$

By the same procedure as described in Article 4.1, we can look for a set of values of Γ's, which makes the duct size minimum.

4. Conclusion

In Section 3, concerning constant velocity duct, constant Mach number one and constant loading parameter one, the authors could derive the simultaneous differential equations for minimization of duct size from the basic ones with the aid of calculus of variations. If the differential equations can be numerically solved under suitable boundary conditions, we can determine the optimum relations to κ and p vs. T pertaining to the specific values of Γ's. When the differential equations are able to be digitally solved for the various values of Γ's and the results are put in I_N, we can find a set of values of Γ's which minimizes I_N. In addition, the expressions have been derived by which l, σ, A and the others can be evaluated. Moreover, with respect to constant velocity and constant loading parameter duct or constant Mach number and constant loading parameter one, they have shown that a set of values of Γ's to minimize I_N is obtained by numerically integrating I_N for the various values of Γ's.

Next in Section 4, from the basic equations the authors have introduced the simultaneous differential equations among κ, ξ, ζ and x', κ, ξ_t, ζ_t and x' or X, ξ_t, ζ_t and x' for the prearranged cross-sectional area duct of constant velocity, constant Mach number or constant loading parameter. Also they have indicated that we can get a set of optimum values of Γ's by the same procedure as mentioned above.

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