$\frac{1}{3}$-harmonic Oscillation in Three-phase Circuit with Series Condensers

By

Kōshi OKUMURA* and Akira KISHIMA*

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The $\frac{1}{3}$-harmonic oscillation originated in the three phase circuit with series condensers is treated. The system equation is reduced to the nonautonomous type of nonlinear differential equation

$$\frac{dx_k}{d\tau}=\sum_{i=1}^{5}a_{ki}x_i+\varepsilon f_k(x_1, x_2, \cdots, x_5, \tau) \qquad k=1, 2, \cdots, 5 \quad \varepsilon\text{: small parameter}$$

First by means of analog computer the $\frac{1}{3}$-harmonic oscillation is investigated and then the extended form of Bogoliubov and Mitropolski's asymptotic method for the system with some-degrees of freedom is used for obtaining the periodic solution.

1. Introduction

We have encountered the phenomena where nonlinear oscillation occurs in a three-phase circuit with series condensers. This kind of nonlinear oscillation, for example, $\frac{1}{3}$-harmonic oscillation results from the nonlinearity of the no-load characteristics of the transformer.

The analytical treatment of the nonlinear three-phase circuit is finally reduced to the solution of the nonlinear differential equation with some degrees of freedom, so that it is rather labourious. In the case of neglecting the zero sequence component, the analysis of the subharmonic oscillation has been reported[1),2)], where the system becomes an autonomous type after some transformation process. In this paper, considering the zero sequence flux interlinkage, we shall analize $\frac{1}{3}$-harmonic oscillation originating in the circuit. The system equation in our case is reduced to non-autonomous type whose solution is made by the extended form of Bogoliubov and Mitropolski's asymptotic method.[3)]

2. Fundamental equation and its solution by analog computer

The three-phase circuit treated here is shown in Fig. 1, where generator voltages are balanced and circuit elements (line resistance R in the primary winding and series condenser C) in each phases are also balanced. The transformer is in the

* Department of Electrical Engineering II

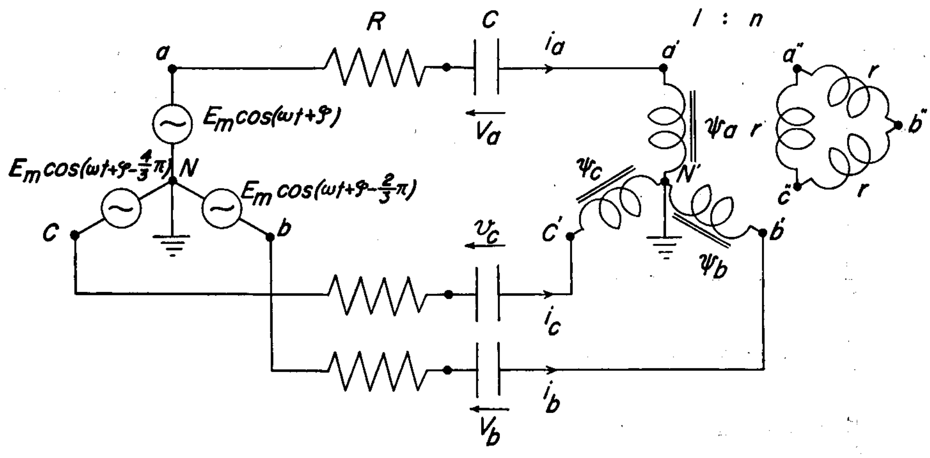

Fig. 1. Three phase circuit with series condensers.

star-delta connection and the secondary windings contain the small resistance r. The primary neutral of the transformer and the generator neutral are both grounded. Our problem is to analyse the $\frac{1}{3}$-harmonic oscillation in this circuit. If the characteristic of no load transformer is assumed to be cubic, the fundamental equations are given by the system of nonlinear differential equations of non-autonomous type,

$$\left.\begin{aligned}
\frac{d\psi_d}{d\tau} &= E+\psi_q-v_d-\xi\Phi_d(\psi_d, \psi_q, \psi_0, \tau) \\
\frac{d\psi_q}{d\tau} &= -\psi_d-v_q-\xi\Phi_q(\psi_d, \psi_q, \psi_0, \tau) \\
\frac{dv_d}{d\tau} &= v_q+\eta\Phi_d(\psi_d, \psi_q, \psi_0, \tau) \\
\frac{dv_q}{d\tau} &= -v_d+\eta\Phi_q(\psi_d, \psi_q, \psi_0, \tau) \\
\frac{d\psi_0}{d\tau} &= -\zeta\Phi_0(\psi_d, \psi_q, \psi_0, \tau)
\end{aligned}\right\} \qquad (1)$$

where

$$\left.\begin{aligned}
\Phi_d(\psi_d, \psi_q, \psi_0, \tau) &= (\psi_d^2+\psi_q^2)\psi_d+2\{(\psi_d^2-\psi_q^2)\cos(3\tau) \\
&\quad -2\psi_d\psi_q\sin(3\tau)\}\psi_0+4\psi_d\psi_0^2 \\
\Phi_q(\psi_d, \psi_q, \psi_0, \tau) &= (\psi_d^2+\psi_q^2)\psi_q-2\{(\psi_d^2-\psi_q^2)\sin(3\tau) \\
&\quad +2\psi_d\psi_q\cos(3\tau)\}\psi_0+4\psi_q\psi_0^2 \\
\Phi_0(\psi_d, \psi_q, \psi_0, \tau) &= (\psi_d^2-3\psi_q^2)\psi_d\cos(3\tau)+(\psi_q^2-3\psi_d^2)\psi_q\sin(3\tau) \\
&\quad +6(\psi_d^2+\psi_q^2)\psi_0+4\psi_0^3
\end{aligned}\right\} \qquad (2)$$

and ξ, η and ζ are normalized values of the resistance of the lines, the elastance of the series condensers and the resistance of the secondary windings of the transformer, respectively. (See Appendix 1). Here we represent Eq. (1) by new coordinate. (See Appendix 2). Thus Eq. (1) becomes

$$\left.\begin{aligned}
\frac{dx_1}{d\tau} &= x_2-x_3+\varepsilon X_1(x_1, x_2, x_5, \tau) \\
\frac{dx_2}{d\tau} &= -x_1-x_4+\varepsilon X_2(x_1, x_2, x_5, \tau) \\
\frac{dx_3}{d\tau} &= h_3x_1+x_4+\varepsilon X_3(x_1, x_2, x_5, \tau) \\
\frac{dx_4}{d\tau} &= h_1x_2-x_3+\varepsilon X_4(x_1, x_2, x_5, \tau) \\
\frac{dx_5}{d\tau} &= \varepsilon X_5(x_1, x_2, x_5, \tau)
\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}
\varepsilon X_1(x_1, x_2, x_5, \tau) &= -\xi m_3x_1-\xi\{f_1(x_1, x_2)+g_1(x_1, x_2, x_5, \tau)\} \\
\varepsilon X_2(x_1, x_2, x_5, \tau) &= -\xi m_1x_2-\xi\{f_2(x_1, x_2)-g_2(x_1, x_2, x_5, \tau)\} \\
\varepsilon X_3(x_1, x_2, x_5, \tau) &= (\eta m_3-h_3)x_1+\eta\{f_1(x_1, x_2)+g_1(x_1, x_2, x_5, \tau)\} \\
\varepsilon X_4(x_1, x_2, x_5, \tau) &= (\eta m_1-h_1)x_2+\eta\{f_1(x_1, x_2)-g_2(x_1, x_2, x_5, \tau)\} \\
\varepsilon X_5(x_1, x_2, x_5, \tau) &= -\zeta\{k(x_1, x_2, \tau)+h(x_1, x_2)x_5+4x_5^3\}
\end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned}
f_1(x_1, x_2) &= \rho_0(3x_1^2+x_2^2)+x_1(x_1^2+x_2^2) \\
f_2(x_1, x_2) &= 2\rho_0x_1x_2+x_2(x_1^2+x_2^2) \\
g_1(x_1, x_2, x_5, \tau) &= 2[\{(\rho_0+x_1)^2-x_2^2\}\cos(3\tau+3\theta_0) \\
&\qquad -2(\rho_0+x_1)x_2\sin(3\tau+3\theta_0)]x_5+2(\rho_0+x_1)x_5^2 \\
g_2(x_1, x_2, x_5, \tau) &= 2[\{(\rho_0+x_1)^2-x_2^2\}\sin(3\tau+3\theta_0) \\
&\qquad +2(\rho_0+x_1)x_2\cos(3\tau+3\theta_0)]x_5-2x_2x_5^2 \\
h(x_1, x_2) &= 6\{(\rho_0+x_1)^2+x_2^2\} \\
k(x_1, x_2, \tau) &= \{x_2^2-3(\rho_0+x_1)^2\}x_2\sin(3\tau+3\theta_0) \\
&\qquad +\{(\rho_0+x_1)^2-3x_2^2\}(\rho_0+x_1)\cos(3\tau+3\theta_0) \\
m_1 &= \rho_0^2 \\
m_3 &= 3\rho_0^2
\end{aligned}\right\} \tag{5}$$

Before we deal with the solution of Eq. (3), we show some results obtained by means of analog computation. Instead of using Eq. (1), we made use of the equations represented by 0-, α-, β-components for analog computation (See Appendix 2). Fig. 2 shows the region where $\frac{1}{3}$-harmonic oscillation is sustained for certain parameters. Fig. 3 shows the typical wave forms in the region of Fig. 2.

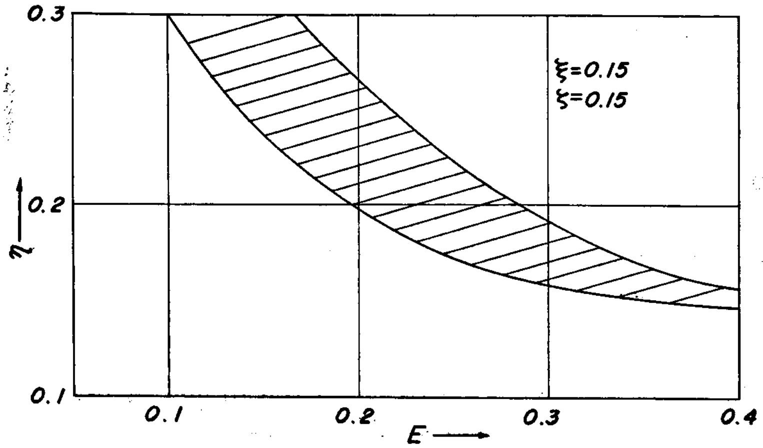

Fig. 2. The region where $\frac{1}{3}$ harmonic oscillation is sustained (Analog computer)

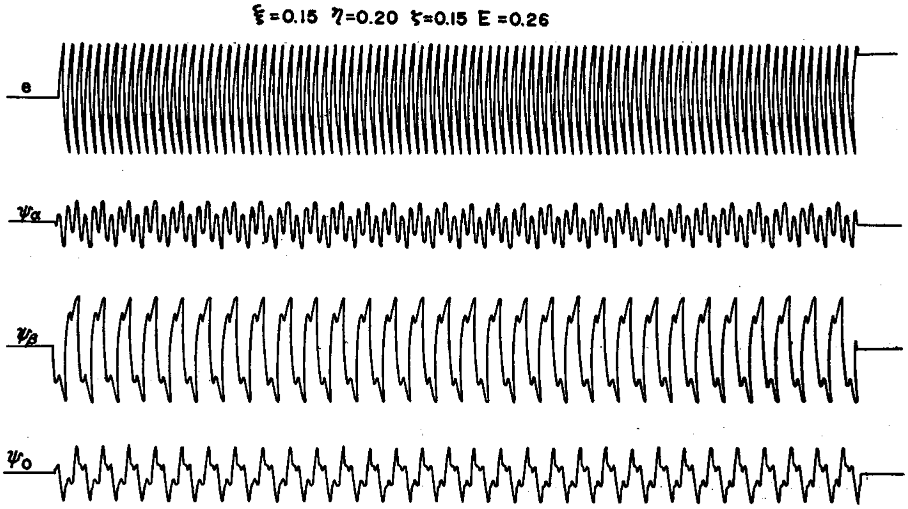

Fig. 3. Typical wave form of $\frac{1}{3}$ harmonic oscillation in the three phase circuit.

3. The analysis of fundamental equation

Here, we are in a position to obtain the first approximate solution of system (3) by the extended form of asymptotic method.

In Eq. (3), putting $\varepsilon=0$, we have linear equation

$$\left.\begin{aligned}\frac{dx_1}{d\tau} &= x_2-x_3\\ \frac{dx_2}{d\tau} &= -x_1-x_4\end{aligned}\right|$$

$$\left.\begin{aligned}\frac{dx_3}{d\tau} &= h_3x_1+x_4\\ \frac{dx_4}{d\tau} &= h_1x_2-x_3\\ \frac{dx_5}{d\tau} &= 0\end{aligned}\right\} \tag{6}$$

Eq. (6) is called the unpurturbed system of Eq. (3). If we assume no permanent magnetization, we have

$$x_5 = 0 \tag{7}$$

We denote the natural frequencies of Eq. (6), ω_1 and ω_2 ($\omega_1 < \omega_2$). The parameters h_1 and h_3 are chosen so as to hold the relation

$$2\omega_1 = \omega_2 = \frac{4}{3} \tag{8}$$

Following the extended form of the asymptotic method, we may write the solution of Eq. (6) as

$$x_k^{(0)} = (x+jy)\varphi_k e^{j\omega_1\tau}+(x-jy)\varphi_k^* e^{-j\omega_1\tau}+(u+jv)\chi_k e^{j\omega_2\tau}+(u-jv)\chi_k^* e^{-j\omega_2\tau} \tag{9}$$
$$k=1, 2, 3, 4$$

where φ_k and χ_k are the eigen functions for eigen values, $j\omega_1$ and $j\omega_2$ respectively and asterisk indicates the complex conjugate. We shall obtain the approximate solution of Eq. (3) using the expansion

$$x_k = x_k^{(0)}(x, y, u, v)+\varepsilon x_k^{(1)}(x, y, u, v)+\varepsilon^2 x_k^{(2)}(x, y, u, v)+\cdots \tag{10}$$
$$k=1, 2, 3, 4$$

where real variables x, y, u and v are assumed to be determined by the equation

$$\left.\begin{aligned}\frac{dx}{d\tau} &= \varepsilon A_1(x, y, u, v)+\varepsilon^2 A_2(x, y, u, v)+\cdots\\ \frac{dy}{d\tau} &= \varepsilon B_1(x, u, y, v)+\varepsilon^2 B_2(x, y, u, v)+\cdots\\ \frac{du}{d\tau} &= \varepsilon C_1(x, y, u, v)+\varepsilon^2 C_2(x, y, u, v)+\cdots\\ \frac{dv}{d\tau} &= \varepsilon D_1(x, y, u, v)+\varepsilon^2 D_2(x, y, u, v)+\cdots\end{aligned}\right\} \tag{11}$$

Here, we need the zero-seqeuence component x_5. Variable x_5 is considered to be smaller than other variables x_i ($i=1, 2, 3, 4$) and there is little difference in neglecting higher powers of x_5 than the first in Eq. (3). We rewrite the last equation of Eq. (3) as follows.

$$\frac{dx_5}{d\tau} = -\zeta\{k(x_1, x_2, \tau)+h(x_1, x_2)x_5\} \tag{12}$$

Making use of the method of harmonic balance, we shall have the stationary solution of zero-sequence component x_5. The periodic solution of Eq. (12) may be assumed to be of the form

$$x_5 = \sum_{l=1}^{L} (Z_l e^{j\Omega_l \tau}+Z_l^* e^{-j\Omega_l \tau}) \tag{13}$$

where Z_l is complex function of real variables $x, y, u,$ and v and is written as

$$Z_l = Z_l(x, y, u, v) = P_l(x, y, u, v)+jQ_l(x, y, u, v) \tag{14}$$

On the other hand, the substitution of Eq. (9) into $k(x_1, x_2, \tau)$ and $h(x_1, x_2)$ gives

$$\left.\begin{aligned} k(x_1^{(0)}, x_2^{(0)}, \tau) &= k(x, y, u, v, \tau) = \sum_{n=1}^{N} \{K_n e^{j\Omega_n \tau}+K_n^* e^{-j\Omega_n \tau}\} \\ h(x_1^{(0)}, x_2^{(0)}, \tau) &= h(x, y, u, v, \tau) = H_0+\sum_{m=1}^{M} (H_m e^{j\omega_m \tau}+H_m^* e^{-j\omega_m \tau}) \end{aligned}\right\} \tag{15}$$

where K_n, H_m are complex function written as

$$\left.\begin{aligned} K_n &= K_n(x, y, u, v) \\ &= u_n(x, y, u, v)+jv_n(x, y, u, v) \qquad n=1, 2, \cdots, N \\ H_m &= H_m(x, y, u, v) \\ &= p_m(x, y, u, v)+jq_m(x, y, u, v) \qquad m=1, 2, \cdots, M \end{aligned}\right\} \tag{16}$$

and H_0 is real function written as

$$H_0 = H_0(x, y, u, v) \tag{17}$$

and the values L, M and N are positive integers.

If the value L is given, the values M snd N are determined by the value L. Substituting Eq. (13) and (15) into Eq. (12) and equating the coefficeint of each frequency component, we have

$$\boldsymbol{\Psi} \boldsymbol{Z} = \boldsymbol{u} \tag{18}$$

where $\boldsymbol{\Psi}$ is $2L\times 2L$ matrix whose elements are the function of variables x, y, u and v and $\boldsymbol{Z}$ and $\boldsymbol{u}$ are $2L$ column real vectors written as

$$\left.\begin{aligned} \boldsymbol{Z} &= {}^t(u_1, v_1, \cdots, u_L, v_L) \\ \boldsymbol{u} &= {}^t(P_1, Q_1, \cdots, P_L, Q_L) \end{aligned}\right\} \tag{19}$$

Under the assumption that $\boldsymbol{\Psi}$ is nonsingular we have

$$\boldsymbol{Z} = \boldsymbol{\Psi}^{-1}\boldsymbol{u} \tag{20}$$

In the zero-sequence flux interlinkages there exist frequency components of order $\frac{1}{3}, \frac{3}{3}, \frac{5}{3}, \cdots$ but terms of harmonics higher than order $\frac{3}{3}$ are ignored. Consequently we may preferably put

$$\left.\begin{aligned} L &= M = N = 2 \\ \Omega_l &= \frac{1}{3}(2l-1) \\ \omega_m &= \frac{2}{3}m \end{aligned}\right\} \tag{21}$$

Note that the frequencies of zero-sequence component are invariant by the transformation defined in Appendix 2. From the above procedure, we obtain the expression of $\boldsymbol{\Psi}$ as follows.

$$\boldsymbol{\Psi} = \begin{pmatrix} H_0+p_1 & q_1-\frac{1}{3} & p_1+p_2 & q_1+q_2 \\ q_1+\frac{1}{3} & H_0-p_1 & q_2-q_1 & p_1-p_2 \\ p_1+p_2 & q_2-q_1 & H_0 & -1 \\ q_1+q_2 & p_1-p_2 & 1 & H_0 \end{pmatrix} \tag{22}$$

where each component for $\boldsymbol{\Psi}$ is

$$\left.\begin{aligned} H_0 &= 6\zeta\{\rho_0^2+4(x^2+y^2+u^2+v^2)\} \\ p_1 &= 6\zeta(2\rho_0 x+4xu+4yv) \\ p_2 &= 12\zeta\rho_0 u \\ q_1 &= 6\zeta(2\rho_0 y+4xv-4yu) \\ q_2 &= 12\zeta\rho_0 v \end{aligned}\right\} \tag{23}$$

and components of $\boldsymbol{u}$ are

$$\left.\begin{aligned} u_1 = &-\zeta[6\rho_0(u^2-v^2)\cos(3\theta_0)+12\rho_0 uv\sin(3\theta_0)+12(u^2-v^2)\{x\cos(3\theta_0) \\ &+y\sin(3\theta_0)\}-24uv\{y\cos(3\theta_0)-x\sin(3\theta_0)\}+12(x^2-y^2)\{u\cos(3\theta_0) \\ &+v\sin(3\theta_0)\}+24xy\{u\sin(3\theta_0)-v\cos(3\theta_0)\}] \\ v_1 = &-\zeta[6\rho_0(u^2-v^2)\sin(3\theta_0)-12\rho_0 uv\cos(3\theta_0)+12(u^2-v^2)\{y\cos(3\theta_0) \\ &-x\sin(3\theta_0)\}+24uv\{x\cos(3\theta_0)+y\sin(3\theta_0)\}+12(x^2-y^2)\{u\sin(3\theta_0) \\ &-v\cos(3\theta_0)\}-24xy\{u\cos(3\theta_0)+v\sin(3\theta_0)\}] \\ u_2 = &-\zeta\{12\rho_0(ux-vy)\cos(3\theta_0)+12\rho_0(uy+vx)\sin(3\theta_0) \\ &+4(u^2-3v^2)u\cos(3\theta_0)+4(3u^2-v^2)v\sin(3\theta_0)+4(3x^2-y^2)y\sin(3\theta_0) \\ &+4(x^2-3y^2)x\cos(3\theta_0)\} \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} v_2 = -\zeta\{&-12\rho_0(vx+uy)\cos(3\theta_0)-12\rho_0(vy-ux)\sin(3\theta_0) \\ &-4(v^2-3u^2)v\cos(3\theta_0)-4(u^2-3v^2)u\sin(3\theta_0) \\ &-4(3x^2-y^2)y\cos(3\theta_0)-4(3y^2-x^2)x\sin(3\theta_0)\} \end{aligned}\right\}$$

Substituting the variable x_5 into the first four equations of Eq. (3) and making use of the extended form of asymptotic method, we have

$$\left.\begin{aligned} \frac{dx}{d\tau} &= \varepsilon A_1(x, y, u, v) \\ &= -a_{11}x-\left(\omega_1-\frac{2}{3}-b_{11}\right)y-(\xi a_{13}x-\eta b_{13}y)(x^2+y^2) \\ &\quad -(\xi a_{14}x-\eta b_{14}y)(u^2+v^2)-\{\xi a_{12}\rho_0(xu+yv)-\eta b_{12}\rho_0(xv-yu)\} \\ &\quad +R_e(U) \\ \frac{dy}{d\tau} &= \varepsilon B_1(x, y, u, v) \\ &= -a_{11}y+\left(\omega_1-\frac{2}{3}-b_{11}\right)x-(\xi a_{13}y+\eta b_{13}x)(x^2+y^2) \\ &\quad -(\xi a_{14}y+\eta b_{14}x)(u^2+v^2)-\{\xi a_{12}\rho_0(xv-yu)+\eta b_{12}\rho_0(xu+yv)\} \\ &\quad +I_m(U) \\ \frac{du}{d\tau} &= \varepsilon C_1(x, y, u, v) \\ &= -a_{21}u-\left(\omega_2-\frac{4}{3}+b_{21}\right)v-(\xi a_{23}u+\eta b_{23}v)(u^2+v^2) \\ &\quad -(\xi a_{24}u+\eta b_{24}v)(x^2+y^2)-\{\xi a_{22}\rho_0(x^2-y^2)+2\eta b_{22}\rho_0 xy\} \\ &\quad +R_e(V) \\ \frac{dv}{d\tau} &= \varepsilon D_1(x, y, u, v) \\ &= -a_{21}v+\left(\omega_2-\frac{4}{3}+b_{21}\right)u-(\xi a_{23}v-\eta b_{23}u)(u^2+v^2) \\ &\quad -(\xi a_{24}v-\eta b_{24}u)(x^2+y^2)-\{2\xi a_{22}\rho_0 xy-\eta b_{22}\rho_0(x^2-y^2)\} \\ &\quad +I_m(V) \end{aligned}\right\} \quad (25)$$

a_{11}, b_{11}, $\cdots$ being constants and $R_e(\ )$, $I_m(\ )$ indicating the real part and the imginary part of the complex functions U and V respectively
where

$$\left.\begin{aligned} U &= U(x, y, u, v) \\ &= -(\xi+j3\eta)(S_0Z_1+S_1Z_1^*+S_2Z_2^*) \\ V &= V(x, y, u, v) \\ &= -(\xi-j3\eta)(S_1Z_1+S_0Z_2+S_2Z_1^*+S_3Z_2^*) \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned}
S_0 &= S_0(x, y, u, v)\\
&= 2(u^2-v^2)\cos(3\theta_0)+4uv\sin(3\theta_0)+j\{2(u^2-v^2)\sin(3\theta_0)\\
&\quad -4uv\cos(3\theta_0)\}\\
S_1 &= S_1(x, y, u, v)\\
&= 4(ux-vy)\cos(3\theta_0)+4(uy+vx)\sin(3\theta_0)\\
&\quad +j\{4(ux-vy)\sin(3\theta_0)-4(uy+vx)\cos(3\theta_0)\}\\
S_2 &= S_2(x, y, u, v)\\
&= 2(x^2-y^2)\cos(3\theta_0)+4xy\sin(3\theta_0)\\
&\quad +2\rho_0\{u\cos(3\theta_0)+v\sin(3\theta_0)\}+j[\{2(x^2-y^2)\sin(3\theta_0)\\
&\quad -4xy\cos(3\theta_0)+2\rho_0\{u\sin(3\theta_0)-v\cos(3\theta_0)\}\}]\\
S_3 &= 2\rho_0\{x\cos(3\theta_0)+y\sin(3\theta_0)\}+j2\rho_0\{x\sin(3\theta_0)-y\cos(3\theta_0)\}
\end{aligned}\right\} \tag{27}$$

In the nonlinear equation (25), two sorts of steady states are considered: one corresponds to singular point and another to periodic solution. We deal with the former case. The singular points of Eq. (25) are obtained by the solutions of simultaneous nonlinear algebraic equation

$$\left.\begin{aligned}
\varepsilon A_1(x, y, u, v) &= 0\\
\varepsilon B_1(x, y, u, v) &= 0\\
\varepsilon C_1(x, y, u, v) &= 0\\
\varepsilon D_1(x, y, u, v) &= 0
\end{aligned}\right\} \tag{28}$$

4. The stability of singular points

We must investigate the stability of singular points $\boldsymbol{x}=(x_0, y_0, u_0, v_0)$ (we use vector notation hereafter) of Eq. (25). These singular points are determined by Newton method which is often effective for the solution of nonlinear algebraic equation. Considering the variation $\delta\boldsymbol{x}$ from $\boldsymbol{x}_0$, we have the variational equation of Eq. (25).

$$\frac{d\delta\boldsymbol{x}}{d\tau} = \boldsymbol{J}_0\delta\boldsymbol{x} \tag{29}$$

where $\boldsymbol{J}_0$ is Jacobi matrix.

Note that it is rather difficult to obtain explicitly the components of $\boldsymbol{J}_0$ since Eq. (25) includes functions U and V, which, as is seen from Eq. (26), are represented by the sum of the products of complex function S_i and Z_j. Function S_i is explicitly expressible as the value of real function of x, y, u and v but function Z_j is not so. The components of $\boldsymbol{J}_0$ includes terms, for example,

$$\frac{\partial S_i Z_j}{\partial x} = \frac{\partial S_i}{\partial x} Z_j + S_i \frac{\partial Z_j}{\partial x} \qquad \begin{matrix} i=0,\ 1,\ 2,\ 3 \\ j=1,\ 2 \end{matrix} \tag{30}$$

The terms $\frac{\partial S_i}{\partial x}$, S_i are easily obtained but Z_j, $\frac{\partial Z_j}{\partial x}$ are not explicitly expressible as the real function of variables x, y, u and v. Then we need to devise any useful way in order to have term $\frac{\partial Z_j}{\partial x}$. Column vector $\boldsymbol{Z}$ which gives the solution of Eq. (18) is represented as the product of the square matrix $\boldsymbol{\Psi}^{-1}$ and column vector $\boldsymbol{u}$. Making the partial derivatives of column vector $\boldsymbol{Z}$ by x, we have

$$\frac{\partial \boldsymbol{Z}}{\partial x} = -\boldsymbol{\Psi}^{-1}\left(\frac{\partial \boldsymbol{\Psi}}{\partial x}\boldsymbol{\Psi}^{-1}\boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial x}\right) \tag{31}$$

and $\frac{\partial \boldsymbol{Z}}{\partial y}$, $\frac{\partial \boldsymbol{Z}}{\partial u}$ and $\frac{\partial \boldsymbol{Z}}{\partial v}$ are similar to Eq. (31). As is easily seen from Eq. (22), $\frac{\partial \boldsymbol{\Psi}}{\partial x}$ and $\frac{\partial \boldsymbol{u}}{\partial x}$ are easily obtained since $\boldsymbol{\Psi}$ and $\boldsymbol{u}$ are explicitly expressible as the function of x, y, u and v, and matrix $\boldsymbol{\Psi}^{-1}$ is obtained numerically by what we call Sweep-out method. If Jacobi matrix $\boldsymbol{J}_0$ is obtained by the above procedure, then the characteristic equation of Eq. (29) is written as

$$\det(\lambda \boldsymbol{1} - \boldsymbol{J}_0) = 0 \qquad (\boldsymbol{1}\text{: unit matrix}) \tag{32}$$

If the coefficients and Hurwitz determinants of Eq. (32) are all positive, then the singular points are stable and $\frac{1}{3}$-harmonic oscillations are sustained for parameters ξ, η, ζ and E which give stable singular points.

5. Numerical examples

In this section we show some numerical examples in certain parameters. We consider the case where $E=0.20$, $\xi=0.15$, $\eta=0.24$, $\zeta=0.15$. For these parameters $\rho_0=0.2020$, $\theta_0=-1.56$ are obtained. The periodic solutions are shown in Table 1.

The four stable solutions M_1, M_2, M_3 and M_4 predict the physical existence of four modes of $\frac{1}{3}$-hamonoic oscillation in the original three phase circuit.

6. Conclusion

Making use of analog computer, we show the existence of $\frac{1}{3}$-harmonic oscillations in the three phase circuit with series condensers and have made their analysis by the extended asymptotic method of Bogoliubov and Mitropolski. The results are shown by numerical examples for certain parameters.

Table 1. Periodic solution for parameters E=0.20 ξ=0.15 η=0.24 ζ=0.15.

Mode	x_k	$(x+jy)\ \varphi_k$	$(u+jv)\ \chi_k$	z_1	z_2	Stability
	x_1	0.0281+j0.2013	0.0660−j0.1721			
	x_2	−0.2013+j0.0281	0.1721+j0.0660			
M_1	x_3	−0.0671+j0.0094	−0.0574−j0.0220			Stable
	x_4	−0.0094−j0.0671	0.0220−j0.0574			
	x_5			0.0176+j0.0119	−0.0103−j0.0055	
	1	−0.1079+j0.1723	0.1810+j0.0351			
	x_2	−0.1723−j0.1079	−0.0351+j0.1810			
M_2	x_3	−0.0577−j0.0360	0.0117−j0.0603			Stable
	4	0.0357−j0.0574	0.0603+j0.0117			
	x_5			−0.0125−j0.0172	0.0004+j0.0117	
	x_1	−0.1934+j0.0626	−0.0032+j0.1843			
	x_2	−0.0626−j0.1934	−0.1843−j0.0032			
M_3	x_3	−0.0209−j0.0645	0.0614+j0.0010			Stable
	x_4	0.0645−j0.0209	−0.0010+j0.0614			
	x_5			0.0058+j0.0204	0.0099−j0.0062	
	x_1	0.1603−j0.1250	0.1160+j0.1432			
	x_2	0.1250+j0.1603	−0.1432+j0.1160			
M_4	x_3	0.0417+j0.0534	0.0477−j0.0387			Stable
	x_4	−0.0534+j0.0417	0.0387+j0.0477			
	x_5			−0.0191+j0.0093	−0.0103−j0.0055	
	x_1	0.0700−j0.1575	0.0704−j0.1415			
	x_2	0.1575+j0.0700	0.1415+j0.0704			
M_5	x_3	0.0525+j0.0233	0.0527+j0.0472			Unstable
	x_4	−0.0233+j0.0525	−0.0472+j0.0527			
	x_5			0.0080+j0.0130	0.0073−j0.0031	
	x_1	−0.1196+j0.1240	0.1449−j0.0631			
	x_2	−0.1240−j0.1196	0.0631+j0.1449			
M_6	x_3	−0.0413−j0.0399	−0.0210−j0.0483			Unstable
	x_4	0.0399−j0.0413	0.0483−j0.0210			
	x_5			0.0142−j0.0056	−0.0010+j0.0079	
	x_1	−0.1714+j0.0181	0.0873+j0.1318			
	x_2	−0.0181−j0.1714	−0.1318+j0.0873			
M_7	x_3	−0.0060−j0.0571	0.0439−j0.0291			Unstable
	x_4	0.0571−j0.0060	0.0291+j0.0439			
	x_5			−0.0153+j0.0004	0.0073−j0.0031	
	x_1	−0.1429−j0.0963	−0.1146+j0.1089			
	x_2	0.0963−j0.1429	−0.1089−j0.1146			
M_8	x_3	0.0321−j0.0476	0.0363+j0.0382			Unstable
	x_4	0.0476+j0.0321	−0.0382+j0.0363			
	x_5			0.0145+i0.0048	−0.0063−j0.0048	

7. Acknowledgement

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Appendix 1.

The fundamental equation of the nonlinear three phase circuit shown in Fig. 1 is obtained by the following graphical procedure. If the voltage sources are short, the circuit of Fig. 1 is represented in Fig. 4 by linear graph. The circuit under consideration with arbitrary node numbering and arbitrary branch numbering and orientation is shown in Fig. 4.

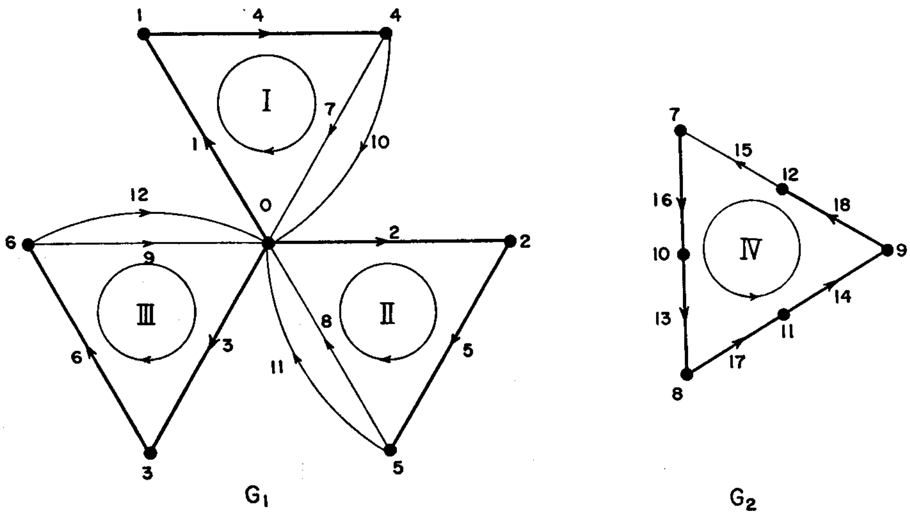

Fig. 4. Linear graph for the three phase circuit.

Bryant[4] has shown that his method of tree construction always leads to a fundamental loop matrix $\boldsymbol{B}$ of the form

$$\boldsymbol{B}=[\boldsymbol{1},\boldsymbol{F}]=\left(\begin{array}{ccc:ccc}\boldsymbol{1}_{\alpha\alpha} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{F}_{\alpha\delta} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{1}_{\beta\beta} & \boldsymbol{0} & \boldsymbol{F}_{\beta\delta} & \boldsymbol{F}_{\beta\varepsilon} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1}_{\gamma\gamma} & \boldsymbol{F}_{\gamma\delta} & \boldsymbol{F}_{\gamma\varepsilon} & \boldsymbol{F}_{\gamma\zeta}\end{array}\right)$$

We now define as our state variables, $\boldsymbol{\Psi}$ chord flux-linkage and $\boldsymbol{v}$ tree capacitor voltages.

In our circuit, we may write

$$\boldsymbol{\Psi}={}^t(\psi_a,\psi_b,\psi_c)$$
$$\boldsymbol{v}={}^t(v_a,v_b,v_c)$$
$$\boldsymbol{E}={}^t\left(E_m\cos(\omega t+\varphi),\ E_m\cos\left(\omega t+\varphi-\frac{2}{3}\pi\right),\ E_m\cos\left(\omega t+\varphi-\frac{4}{3}\pi\right)\right)$$
$$\boldsymbol{J}_\gamma={}^t(J_a,J_b,J_c)$$
$$\boldsymbol{f}_\gamma(\boldsymbol{\Psi})={}^t(c_3\psi_a^3,c_3\psi_b^3,c_3\psi_c^3)\qquad c_3\text{: positive constant}$$
$$\boldsymbol{R}_{\varepsilon\varepsilon}=\text{diag}\,(R,R,R)$$
$$\boldsymbol{C}=\text{diag}\,(C,C,C)$$
$$\boldsymbol{r}=\text{diag}\,(r,r,r)$$

where

- $\boldsymbol{\Psi}$: column vector of the flux-interlinkages for branch (7, 8, 9)
- $\boldsymbol{v}$: column vector of the voltage across capacitor branch (4, 5, 6)
- $\boldsymbol{E}$: voltage source column vector for branch (1, 2, 3)
- $\boldsymbol{J}_\gamma$: current column vector for branch (10, 11, 12)
- $\boldsymbol{f}_\gamma$: current column vector for branch (7, 8, 9)
- $\boldsymbol{R}_{\varepsilon\varepsilon}$: diagonal matrix for resistive branch (1, 2, 3)
- $\boldsymbol{C}$: diagonal matrix for capacitive branch (4, 5, 6)
- $\boldsymbol{r}$: diagonal matrix for resistive branch (16, 17, 18)

Let us select the tree of G_1 and G_2 shown as thick line in Fig. 4 and the set of fundamental loops defined by these trees is shown as loop I, II, III, IV.

After some elimination processes, we have the state equation for G_1

$$\left.\begin{aligned}\frac{d\boldsymbol{\Psi}}{dt}&=-\boldsymbol{F}_{\gamma\delta}\boldsymbol{v}-\boldsymbol{F}_{\gamma\varepsilon}\boldsymbol{R}_{\varepsilon\varepsilon}{}^t\boldsymbol{F}_{\gamma\varepsilon}\{\boldsymbol{f}_\gamma(\boldsymbol{\Psi})+\boldsymbol{J}_\gamma\}+\boldsymbol{E}\\ \frac{d\boldsymbol{v}}{dt}&=\boldsymbol{C}^{-1}{}^t\boldsymbol{F}_{\gamma\delta}\{\boldsymbol{f}_\gamma(\boldsymbol{\Psi})+\boldsymbol{J}_\gamma\}\end{aligned}\right\}\tag{34}$$

For G_2, the fundamental loop matrix $\boldsymbol{B}_2$ of chord (15) is given by

$$\left.\begin{aligned}\boldsymbol{B}_2&=(\boldsymbol{b}_w,\boldsymbol{b}_2)\\ \boldsymbol{b}_w&=(1,1,1)\\ \boldsymbol{b}_2&=(1,1)\end{aligned}\right\}\tag{35}$$

The tree voltage vector $\boldsymbol{V}_2$, tree current vector $\boldsymbol{I}_2$, chord voltage vector $\boldsymbol{v}_1$ and chord current vector $\boldsymbol{i}_1$ are written as

$$\left.\begin{aligned} \boldsymbol{V}_2 &= {}^t(\boldsymbol{v}_w, \boldsymbol{v}_2) \\ \boldsymbol{I}_2 &= {}^t(\boldsymbol{i}_w, \boldsymbol{i}_2) \\ \boldsymbol{v}_1 &= (v_0) \\ \boldsymbol{i}_1 &= (i_0) \end{aligned}\right\} \tag{36}$$

where

$\boldsymbol{v}_w$: voltage column vector of secondary windings branch (16, 17, 18)
$\boldsymbol{v}_2$: voltage column vector of resistive tree branch (13, 14)
$\boldsymbol{i}_w$: current column vector of secondary windings branch (16, 17, 18)
$\boldsymbol{i}_2$: current column vector of resistive tree branch (13, 14)

The diagonal matrix $\boldsymbol{r}$ is decomposed as

$$\boldsymbol{r} = \boldsymbol{r}_1 \dotplus \boldsymbol{r}_2 \tag{37}$$

and combined relations between $\boldsymbol{v}_1$, $\boldsymbol{i}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{i}_2$ are

$$\left.\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{r}_1\boldsymbol{i}_1 \\ \boldsymbol{v}_2 &= \boldsymbol{r}_2\boldsymbol{i}_2 \end{aligned}\right\} \tag{38}$$

After some elimination process, we have

$$(\boldsymbol{r}_1+\boldsymbol{b}_2\boldsymbol{r}_2{}^t\boldsymbol{b}_2)\boldsymbol{b}_w\boldsymbol{i}_w/(\boldsymbol{b}_w, \boldsymbol{b}_w)+\boldsymbol{b}_w\boldsymbol{v}_w = 0 \tag{39}$$

where $(\boldsymbol{b}_w, \boldsymbol{b}_w)$ is the inner product of $\boldsymbol{b}_w$ and $\boldsymbol{b}_w$. The relations between $\boldsymbol{J}_\gamma$, $\frac{d\boldsymbol{\Psi}}{dt}$, $\boldsymbol{v}_w$ and $\boldsymbol{i}_w$ are held by equations

$$\left.\begin{aligned} \boldsymbol{v}_w &= n\frac{d\boldsymbol{\Psi}}{dt} \\ \boldsymbol{i}_w &= -\frac{1}{n}\boldsymbol{J}_\gamma \end{aligned}\right\} \tag{40}$$

Substitution of Eq. (40) into Eq. (34) and (39) gives finally

$$\left.\begin{aligned} \frac{d\boldsymbol{\Psi}}{dt} &= -\boldsymbol{F}_{\gamma\delta}\boldsymbol{v}-\boldsymbol{F}_{\gamma\varepsilon}\boldsymbol{R}_{\varepsilon\varepsilon}{}^t\boldsymbol{F}_{\gamma\varepsilon}\{\boldsymbol{f}_\gamma(\boldsymbol{\Psi})-n\boldsymbol{i}_w\}+\boldsymbol{E} \\ \frac{d\boldsymbol{v}}{dt} &= \boldsymbol{C}^{-1}\boldsymbol{F}_{\gamma\delta}\{\boldsymbol{f}_\gamma(\boldsymbol{\Psi})-n\boldsymbol{i}_w\} \\ \boldsymbol{b}_w\frac{d\boldsymbol{\Psi}}{dt} &= -\frac{1}{n^2}(\boldsymbol{r}_1+\boldsymbol{b}_2\boldsymbol{r}_2{}^t\boldsymbol{b}_2)\boldsymbol{b}_w\boldsymbol{i}_w/(\boldsymbol{b}_w, \boldsymbol{b}_w) \end{aligned}\right\} \tag{41}$$

In our case, $\boldsymbol{F}_{\gamma\delta}$ and $\boldsymbol{F}_{\gamma\varepsilon}$ are 3×3 unit matrix. Then we have as the state equation for the three phase circuit

$$\left.\begin{aligned}
\frac{d\psi_a}{dt} &= -v_a - R(c_3\psi_a^3 - ni_0) + E\cos(\omega t+\varphi)\\
\frac{d\psi_b}{dt} &= -v_b - R(c_3\psi_b^3 - ni_0) + E\cos\left(\omega t+\varphi-\frac{2}{3}\pi\right)\\
\frac{d\psi_c}{dt} &= -v_c - R(c_3\psi_c^3 - ni_0) + E\cos\left(\omega t+\varphi-\frac{4}{3}\pi\right)\\
\frac{dv_a}{dt} &= \frac{1}{C}(c_3\psi_a^3 - ni_0)\\
\frac{dv_b}{dt} &= \frac{1}{C}(c_3\psi_b^3 - ni_0)\\
\frac{dv_c}{dt} &= \frac{1}{C}(c_3\psi_c^3 - ni_0)\\
\frac{d\psi_a}{dt}+\frac{d\psi_b}{dt}+\frac{d\psi_c}{dt} &= -\frac{3}{n^2}r\dot{i}_0
\end{aligned}\right\} \qquad (42)$$

Appendix 2.

The equation (42) is written by the 0-α-β-components for analog computer use. At first, Eq. (42) is normalized by putting

$$\left.\begin{aligned}
&\omega t+\varphi \rightarrow \tau \quad \alpha_\psi \boldsymbol{\Psi} \rightarrow \boldsymbol{\Psi} \quad \alpha_v \boldsymbol{v} \rightarrow \boldsymbol{v} \quad \alpha_v E_m \rightarrow E\\
&R\frac{3c_3}{4\omega\alpha_\psi^2} \rightarrow \xi \quad \frac{1}{\omega C}\frac{3c_3}{4\omega\alpha_\psi^2} \rightarrow \eta \quad r\frac{c_3}{4n^2\alpha_\psi^2\omega} \rightarrow \zeta
\end{aligned}\right\} \qquad (43)$$

where

$$\alpha_\psi = \omega\alpha_v$$

Eq. (42) is represented by 0-, α-, β-components using transformation matrix $\boldsymbol{C}_{0\alpha\beta}$ defined as

$$\boldsymbol{C}_{0\alpha\beta} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3}\\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \qquad \boldsymbol{C}_{0\alpha\beta}^{-1} = \begin{pmatrix} 1 & 1 & 0\\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2}\\ 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \qquad (44)$$

If we assume the case $r \ll n^2R$, we have the following set of equations

$$\left.\begin{aligned}
\frac{d\psi_\alpha}{d\tau} &= -v_\alpha-\xi\{(\psi_\alpha^2+\psi_\beta^2)\psi_\alpha+2(\psi_\alpha^2-\psi_\beta^2)\psi_0+4\psi_\alpha\psi_0^2\}+E\cos(\tau)\\
\frac{d\psi_\beta}{d\tau} &= -v_\beta-\xi\{(\psi_\alpha^2+\psi_\beta^2)\psi_\beta-4\psi_\alpha\psi_\beta\psi_0+4\psi_\beta\psi_0^2\}+E\sin(\tau)\\
\frac{dv_\alpha}{d\tau} &= \eta\{(\psi_\alpha^2+\psi_\beta^2)\psi_\alpha+2(\psi_\alpha^2-\psi_\beta^2)\psi_0+4\psi_\alpha\psi_0^2\}\\
\frac{dv_\beta}{d\tau} &= \eta\{(\psi_\alpha^2+\psi_\beta^2)\psi_\beta-4\psi_\alpha\psi_\beta\psi_0+4\psi_\beta\psi_0^2\}\\
\frac{d\psi_0}{d\tau} &= -\zeta\{(\psi_\alpha^2-3\psi_\beta^2)\psi_\alpha+6(\psi_\alpha^2-\psi_\beta^2)\psi_0+4\psi_0^3\}
\end{aligned}\right\} \tag{45}$$

Furthermore, the transformation matrix defined by

$$C_{0dq}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\tau) & \sin(\tau)\\ 0 & -\sin(\tau) & \cos(\tau)\end{pmatrix}\qquad C_{0dq}^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\tau) & -\sin(\tau)\\ 0 & \sin(\tau) & \cos(\tau)\end{pmatrix} \tag{46}$$

leads Eq. (45) to Eq. (1) in section 1.

Appendix 3.

If $\zeta=0$ and $\psi_0=0$, Eq. (1) becomes

$$\left.\begin{aligned}
\frac{d\psi_d}{d\tau} &= E+\psi_q-v_d-\xi(\psi_d^2+\psi_q^2)\psi_d\\
\frac{d\psi_q}{d\tau} &= -\psi_d-v_q-\xi(\psi_d^2+\psi_q^2)\psi_q\\
\frac{dv_d}{d\tau} &= v_q+\eta(\psi_d^2+\psi_q^2)\psi_d\\
\frac{dv_q}{d\tau} &= -v_d+\eta(\psi_d^2+\psi_q^2))\psi_q
\end{aligned}\right\} \tag{47}$$

which is an autonomous system.
The singular points of this system are given by

$$\left.\begin{aligned}
\psi_{d0} &= \rho_0\cos(\theta_0)\\
\psi_{q0} &= \rho_0\sin(\theta_0)\\
v_{d0} &= \eta\rho_0^3\sin(\theta_0)\\
v_{q0} &= -\eta\rho_0^3\cos(\theta_0)
\end{aligned}\right\} \tag{48}$$

where ρ_0 and θ_0 satisfy the equation

$$\left.\begin{aligned}(\xi^2+\eta^2)\rho_0^6-2\eta\rho_0^4+\rho_0^2-E^2 &= 0\\ \theta_0 &= \tan^{-1}\left\{\frac{1}{\xi}\left(\eta-\frac{1}{\rho_0^2}\right)\right\}\end{aligned}\right\} \tag{49}$$

The stability of the singular points is investigated in Refference (1). Putting

$$\left.\begin{aligned}\Delta\psi_d &= \psi_d-\psi_{d0}\\ \Delta\psi_q &= \psi_q-\psi_{q0}\\ \Delta v_d &= v_d-v_{d0}\\ \Delta v_q &= v_q-v_{q0}\end{aligned}\right\} \tag{50}$$

we define as our new variables x_1, x_2, x_3, x_4, and x_5

$$\left.\begin{aligned}x_1+jx_2 &= (\Delta\psi_d+j\Delta\psi_q)e^{-j\theta_0}\\ x_3+jx_4 &= (\Delta v_d+j\Delta v_q)e^{-j\theta_0}\\ x_5 &= \psi_0\end{aligned}\right\} \tag{51}$$

We have Eq. (3) represented by new coordinate whose origin is given by Eq. (48).