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By

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The oscillations of a one-dimensional anharmonic lattice is investigated sufficiently near the equilibrium state, which is described by the equation

$$d^2y_i/dt^2 = y_{i+1}-2y_i+y_{i-1}+\alpha\{(y_{i+1}-y_i)^3-(y_i-y_{i-1})^3\}, \quad -\infty<t<+\infty,$$

where $i=1,2,\cdots n$, $\alpha=$constant, boundary conditions are $y_0=y_{n+1}=0$ and initial conditions $y_i(0)$, $dy_i(0)/dt$ are given. A large part of the oscillations starting from initial data sufficiently near the origin of the phase space of the system is proved to be a conditionally periodic motion under the incommensurable condition on linear frequencies mentioned later.

1. Introduction

In 1955 Fermi, Pasta and Ulam 1) experimented numerically as an example of one-dimensional anharmonic lattices the system of ordinary differential equations:

$$(1) \quad \ddot{y}_i = y_{i+1}-2y_i+y_{i-1}+\alpha\{(y_{i+1}-y_i)^p-(y_i-y_{i-1})^p\},$$

for $-\infty<t<+\infty$, $i=1, 2,\cdots, N-1$, $y_0=y_N=0$, $y_i(0)$, $\dot{y}_i(0)$: given, where $p=2$ or 3, $N=16$, 32 or 64, $\alpha=1/4$ or 1.

First they expected that the anharmonicity of the system would cause the equipartition of energy among all the N-1 modes of the system, but they obtained "recurrences" to the initial data. On the other hand in 1954 Kolomgorov 2) proved that the large part of conditionally periodic motions of the dynamical system is conserved by the small perturbation of the Hamiltonian under some assumptions on the Hamiltonian. Then Arnol'd 3) and Moser 4) improved the result so that in the case of oscillations (a limiting degenerate case) analogous results near the equilibrium point hold under some improved assumptions. Here we consider the system (1) with $p=3$ and arbitrary N. We show that if n-frequencies of the linear system (1) with $\alpha=0$ are incommensurable in the later-mentioned sense, then the large part of the oscillations starting from initial data sufficiently near the origin of

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the phase space (the equilibrium state) of the system is conditionally periodic. The proof is composed of the reduction of the system into the normal form by Birkhoff's transofrmation 5) and the check of the condition assumed in Kolmogorov-Arnol'd-Moser's theorem. About the large nonlinearity of analogous anharmonic lattices and the equipartition of energy there is the paper 6), numerical approaches 7), 8), 9) and papers refered to in them.

2. A reduction of the system by normal modes to a canonical form

We consider the system (1) with $p=3$.

$$(2)\quad \ddot{y}_i = y_{i+1}-2y_i+y_{i-1}+2\alpha\{(y_{i+1}-y_i)^3-(y_i-y_{i-1})^3\};t\geq 0,$$
$$i=1,2,\cdots,N-1\equiv n,\quad y_0=y_N=0,\quad y_i(0),\dot{y}_i(0);\text{given.}$$

The system (2) has the solution $\{y_i(t)\}_{i=1\cdots,n,}$ in $t\geq 0$ for all initial data. The total energy has the form:

$$(3)\quad E=\frac{1}{2}\sum_{i=0}^{n}\{\dot{y}_i^2+(y_{i+1}-y_i)^2+\alpha(y_{i+1}-y_i)^4\}.$$

In order to obtain the Hamiltonian of (2) with respect to normal modes we substitute the expression $y_i=\sqrt{\frac{2}{N}}\sum_{k=1}^{n}a_k(t)\sin kih,\quad h=\frac{\pi}{N}$ into (3).

We arrive at the following:

$$\frac{1}{2}\sum_{i=0}^{n}\dot{y}_i^2=\frac{1}{4}\sum_{i=0}^{2n+1}\dot{y}_i^2=\frac{1}{2}\sum_{k=1}^{n}\dot{a}_k^2,$$

$$\frac{1}{2}\sum_{i=0}^{n}(y_{i+1}-y_i)^2=\frac{1}{4}\sum_{i=0}^{2n+1}(y_{i+1}-y_i)^2=\frac{1}{2}\sum_{k=1}^{n}2(1-\cos kh)a_k^2,$$

$$\frac{\alpha}{2}\sum_{i=0}^{n}(y_{i+1}-y_i)^4=\frac{\alpha}{4N}\{\sum_{k+l+m+j=2(n+1)}a_ka_la_ma_j\cdot$$

$$(4)\qquad \cdot(\beta_k\beta_l\beta_m\beta_j-6\alpha_k\alpha_l\beta_m\beta_j+\alpha_k\alpha_l\alpha_m\alpha_j)+$$
$$+4\sum_{k+l+m=j}a_ka_la_ma_j\,(\beta_k\beta_l\beta_m\beta_j+3\beta_k\beta_l\alpha_m\alpha_j-3\alpha_k\alpha_l\beta_m\beta_j-\alpha_k\alpha_l\alpha_m\alpha_j)+$$
$$+4\sum_{k+l+m=j+2(n+1)}a_ka_la_ma_j\,(\beta_k\beta_l\beta_m\beta_j+3\beta_k\beta_l\alpha_m\alpha_j-3\alpha_k\alpha_l\beta_m\beta_j-\alpha_k\alpha_l\alpha_m\alpha_j)+$$
$$+3\sum_{k+l=m+j}a_ka_la_ma_j\,(\beta_k\beta_l\beta_m\beta_j-2\beta_k\beta_l\alpha_m\alpha_j+4\beta_k\alpha_l\beta_m\alpha_j+\alpha_k\alpha_l\alpha_m\alpha_j)\},$$

where $1\leq k,l,m,j\leq n,\quad \alpha_k=\cos kh-1,\quad \beta_k=\sin kh.$

Then $H(\dot{a},a)=E(\dot{y},y)|_{y,\dot{y}\to a,\dot{a}}=$

$$=\frac{1}{2}\{\sum_{k=1}^{n}\dot{a}_k^2+(-2\alpha_k)a_k^2+\frac{\alpha}{2N}\sum_{(k,l,m,j)}(\cdots\cdots)a_ka_la_ma_j\},$$

where in the last term the summations and coefficients are the same as those in (4). Putting $p_k = \dot{a}_k/\sqrt{\lambda_k}$, $q_k = \sqrt{\lambda_k}a_k$, $\lambda_k = \sqrt{-2\alpha_k}$ gives the Hamiltonian with respect to variables p, q:

$$(5)\quad H(p, q) = \frac{1}{2}\{\sum_{k=1}^{n}\lambda_k(p_k^2+q_k^2)+$$
$$+\frac{\alpha}{2\mathcal{N}}\sum_{(k,l,m,j)}(\cdots\cdots)\, q_k q_l q_m q_j/\sqrt{\lambda_k\lambda_l\lambda_m\lambda_j}\},$$

where also the abbreviation of summations and coefficients is used.

3. Birkhoff's transformation

First by introduction of conjugate variables $\xi = (\xi_1,\cdots,\xi_n)$, $\eta = (\eta_1,\cdots,\eta_n)$ such that $\xi_k=\frac{1-i}{2}(p_k-iq_k), \eta_k = \frac{1-i}{2}(p_k+iq_k)$, $(k=1,\cdots, n;\ i=\sqrt{-1})$ we get the following:

$$H(\xi,\eta) = H(p, q)\Big|_{p,q\to\xi,\eta} \equiv H_2+H_4,$$

where $H_2(\xi,\eta) = \sum_{k=1}^{n}(i\lambda_k)\xi_k\eta_k$,

$$(6)\quad H_4(\xi,\eta) = -\frac{\alpha}{16\mathcal{N}}\sum_{(k,l,m,j)}(\xi_k-\eta_k)(\xi_l-\eta_l)(\xi_m-\eta_m)(\xi_j-\eta_j)\cdot$$
$$\cdot(\cdots\cdots)/\sqrt{\lambda_k\lambda_l\lambda_m\lambda_j},$$

and the abbreviation of summations and coefficients is the same as above.

Now we remember Birkhoff's transofrmation that normalizes the fourth degree term H_4; we apply to H a transformation

$$\xi_j = \bar{p}_j+\partial K/\partial\eta_j,\quad \bar{q}_j = \eta_j+\partial K/\partial\bar{p}_j,\quad (j=1,\cdots,n),$$

where $K = \sum_{k_1+\cdots+k_n+l_1+\cdots+l_n=4} a_{k_1\cdots k_n l_1\cdots l_n}\bar{p}_1^{k_1}\cdots\bar{p}_n^{k_n}\eta_1^{l_1}\cdots\eta_n^{l_n}$,

$k_j, l_j = 0, 1,\cdots,4$, $a_{k_1\cdots k_n l_1\cdots l_n} =$ constant.

If we solve explicitly for ξ, η in terms of $\bar{p}$, $\bar{q}$, we obtain

$$\xi_j = \bar{p}_j+\partial K^*/\partial\bar{q}_j+\cdots,\quad \eta_j = \bar{q}_j-\partial K^*/\partial\bar{p}_j+\cdots (j=1,\cdots,n),$$

where K^* denotes the function obtained by replacing η by $\bar{q}$ in K and the power series converge in a neighbourhood of the origin. The modified value of H, obtained by substitution is

$$H(\bar{p},\bar{q}) = H_2(\bar{p}_1+\partial K^*/\partial\bar{q}_1+\cdots,\cdots,\bar{q}_n-\partial K^*/\partial\bar{p}_n+\cdots)+H_4,$$

where the arguments of H_4 are the same as those of H_2. To terms of the fourth degree inclusive we find

$$H(\bar{p}, \bar{q}) = \sum_{j=1}^{n} i\lambda_j \bar{p}_j\bar{q}_j + \sum_{j=1}^{n} i\lambda_j\Big(\bar{q}_j\frac{\partial K^*}{\partial \bar{q}_j} - \bar{p}_j\frac{\partial K^*}{\partial \bar{p}_j}\Big) + H_4(\bar{p}_1, \cdots, \bar{q}_n) +$$

$$+\text{higher degree terms} \equiv H_2(\bar{p}, \bar{q}) + \bar{H}_4(\bar{p}, \bar{q}) + \tilde{H}(\bar{p}, \bar{q}).$$

Thus the forms of H_2 is unmodified while $\bar{H}_4$ takes the form

$$\sum_{j=1}^{n} i\lambda_j(\bar{q}_j\cdot\partial K^*/\partial\bar{q}_j - \bar{p}_j\cdot\partial K^*/\partial\bar{p}_j) + H_4(\bar{p}_1, \cdots, \bar{q}_n) =$$

$$= \sum_{k_1+\cdots+l_n=4} [a_{k_1\cdots l_n}\{\sum_{j=1}^{n} i\lambda_j\,(l_j - k_j)\} + h_{k_1\cdots k_n l_1\cdots l_n}]\bar{p}_1^{k_1}\cdots\bar{p}_n^{k_n}\bar{q}_1^{l_1}\cdots\bar{q}_n^{l_n},$$

where $h_{k_1\cdots k_n l_1\cdots l_n}$ is the coefficient analogous to $a_{k_1\cdots k_n l_1\cdots l_n}$ in the original H_4. Here we assume* an incommensurable condition of λ_j (n frequencies of linear system (2) with $\alpha=0$), that is,

$$\sum_{j=1}^{n} \lambda_j(l_j - k_j) \neq 0 \quad \text{for } l_j,\, k_j = 0,\, 1, \cdots, 4,\ \sum_{j=1}^{n}(k_j + l_j) = 4 \quad \text{and} \sum_{j=1}^{n}|l_j - k_j| \neq 0.$$

Therefore if we take the coefficients $a_{k_1\cdots k_n l_1\cdots l_n}$ appropriately, we can eliminate the terms of $\bar{H}_4$ except those of the forms

$$(7) \quad \lambda_{kk}(\bar{p}_k\bar{q}_k)^2, \quad \lambda_{kl}\bar{p}_k\bar{q}_k\bar{p}_l\bar{q}_l \qquad (k, l = 1, \cdots, n).$$

Then it is sufficient for the explicit expression of $\bar{H}_4(\bar{p}, \bar{q})$ to calculate in (6) the coefficients of $\xi_k\eta_k\xi_l\eta_l$ that are the same as those of the terms (7) in $\bar{H}_4$. Thus we consider the four terms in (6) separately, each of which contains the summation with respect to indexes k, l, m, j in (4). Since out of the product $(\xi_k - \eta_k)$ $(\xi_l - \eta_l)$ $(\xi_m - \eta_m)$ $(\xi_j - \eta_j)$, at most, the following terms are possible not to vanish in $\bar{H}_4$: $\xi_k\xi_l\eta_m\eta_j + \xi_k\eta_l\xi_m\eta_j + \xi_k\eta_l\eta_m\xi_j + \eta_k\xi_l\xi_m\eta_j + \eta_k\xi_l\eta_m\xi_j + \eta_k\eta_l\xi_m\xi_j \equiv J(\xi, \eta)$, it is the following term that is possible not to vanish in $\bar{H}_4$ out of the first term in (6):

$$\frac{-\alpha}{16\mathcal{N}} \sum_{k+l+m+j=2(n+1)} \frac{\beta_k\beta_l\beta_m\beta_j - 6\alpha_k\alpha_l\beta_m\beta_j + \alpha_k\alpha_l\alpha_m\alpha_j}{\sqrt{\lambda_k\lambda_l\lambda_m\lambda_j}} J(\xi, \eta) =$$

$$= \frac{-\alpha}{16\mathcal{N}} \sum \frac{\beta_k\beta_l\beta_m\beta_j - 6\alpha_k\alpha_l\beta_m\beta_j + \alpha_k\alpha_l\alpha_m\alpha_j}{\sqrt{\lambda_k\lambda_l\lambda_m\lambda_j}} \cdot$$

$$\cdot(\xi_k\xi_l\eta_m\eta_j + 4\xi_k\eta_l\xi_m\eta_j + \eta_k\eta_l\xi_m\xi_j).$$

If we choose the terms of the forms $C\xi_k\eta_k\xi_l\eta_l$ from this, we arrive at the following:

* it is mentioned in the last part of the paper

$$(8)\quad \frac{-\alpha}{16N}\{12\sum_{k+l=n+1}\xi_k\eta_k\xi_l\eta_l\,(\beta_k^2\beta_l^2-4\alpha_k\alpha_l\beta_k\beta_l-\alpha_k^2\beta_l^2-\alpha_l^2\beta_k^2+$$

$$+\alpha_k^2\alpha_l^2)/\lambda_k\lambda_l-6\sum_{k=1}^{n}\delta_{n+1-k}^{k}\,(\xi_k\eta_k)^2(\beta_k^4-6\alpha_k^2\beta_k^2+\alpha_k^4)/\lambda_k^2\},$$

where δ_l^k is Kronecker's delta. Since in the second and third terms in (6) the summation is such that $k+l+m=j$, $k+l+m=j+2(n+1)$ and $k, l, m, j=1,2,\cdots, n$, they do not contain the terms of the forms $C\xi_k\eta_k\xi_l\eta_l$. From the fourth term we may choose the following:

$$(9)\quad \frac{-\alpha}{16N}\{24\sum_{k,l=1}^{n}\xi_k\eta_k\xi_l\eta_l\,(\beta_k^2\beta_l^2+\beta_k^2\alpha_l^2+\alpha_k^2\beta_l^2+\alpha_k^2\alpha_l^2)/\lambda_k\lambda_l-$$

$$-6\sum_{k=1}^{n}(\xi_k\eta_k)^2(\beta_k^4+2\beta_k^2\alpha_k^2+\alpha_k^4)/\lambda_k^2\}.$$

Then by (8) and (9) we obtain the explicit expression of $\bar{H}_4$.

$$\bar{H}_4=\frac{-3\alpha}{8N}\{2\sum_{k+l=n+1}\bar{p}_k\bar{q}_k\bar{p}_l\bar{q}_l\,(\beta_k^2\beta_l^2-4\alpha_k\alpha_l\beta_k\beta_l-\alpha_k^2\beta_l^2-$$

$$-\alpha_l^2\beta_k^2+\alpha_k^2\alpha_l^2)/\lambda_k\lambda_l+4\sum_{k,l=1}^{n}\bar{p}_k\bar{q}_k\bar{p}_l\bar{q}_l\,(\beta_k^2\beta_l^2+\beta_k^2\alpha_l^2+$$

$$+\alpha_k^2\beta_l^2+\alpha_k^2\alpha_l^2)/\lambda_k\lambda_l-\sum_{k=1}^{n}(\bar{p}_k\bar{q}_k)^2(\beta_k^4+2\beta_k^2\alpha_k^2+\alpha_k^4+$$

$$+\delta_{n+1-k}^{k}(\beta_k^4-6\alpha_k^2\beta_k^2+\alpha_k^4))/\lambda_k^2\}=$$

$$=-\frac{3\alpha}{8N}\{-2\sum_{k+l=n+1}\lambda_k\lambda_l\bar{p}_k\bar{q}_k\bar{p}_l\bar{q}_l+4\sum_{k,l=1}^{n}\lambda_k\lambda_l\cdot$$

$$\cdot\bar{p}_k\bar{q}_k\bar{p}_l\bar{q}_l-\sum_{k=1}^{n}(1-\delta_{n+1-k}^{k})\lambda_k^2(\bar{p}_k\bar{q}_k)^2\}$$

$$\equiv-\frac{3\alpha}{8N}\sum_{k,l=1}^{n}a_{kl}\lambda_k\lambda_l\bar{p}_k\bar{q}_k\bar{p}_l\bar{q}_l,$$

where we use $\beta_k=\sin kh$, $\alpha_k=\cos kh-1$, $\lambda_k=\sqrt{-2\alpha_k}$.
By the transformation $\bar{p}_k=\frac{1-i}{2}(\bar{\bar{p}}_k-i\bar{\bar{q}}_k)$, $\bar{q}_k=\frac{1-i}{2}(\bar{\bar{p}}_k+i\bar{\bar{q}}_k)$ we get

$$H(\bar{\bar{p}},\bar{\bar{q}})=H(\bar{p},\bar{q})\,\Big|_{\bar{p},\bar{q}\to\bar{\bar{p}},\bar{\bar{q}}}=\frac{1}{2}\sum_{k=1}^{n}\lambda_k(\bar{\bar{p}}_k^2+\bar{\bar{q}}_k^2)+$$

$$+\frac{3\alpha}{32N}\sum_{k,l=1}^{n}a_{kl}\lambda_k\lambda_l\,(\bar{\bar{p}}_k^2+\bar{\bar{q}}_k^2)(\bar{\bar{p}}_l^2+\bar{\bar{q}}_l^2)+\tilde{H}(\bar{\bar{p}},\bar{\bar{q}}).$$

Furthermore by the transormation using action-angle variables
$\bar{\bar{p}}_k=\sqrt{2\tau_k}\cos Q_k$, $\bar{\bar{q}}_k=\sqrt{2\tau_k}\sin Q_k$, we get the following:

$$(10)\quad H(\tau,Q) = \sum_{k=1}^{n} \lambda_k\tau_k + \frac{3\alpha}{8N}\sum_{k,l=1}^{n} a_{kl}\lambda_k\lambda_l\tau_k\tau_l + \tilde{H}(\tau, Q)$$
$$\equiv H_0(\tau) + \tilde{H}(\tau, Q),$$

where by virture of the reduction by Birkhoff's transformation to this normal form there exists $\varepsilon_0 > 0$ such that

(11) $\tilde{H}(\tau, Q)$ is analytic in the region $G = \{|\tau_j - \varepsilon_0| < \varepsilon_0, |I_m Q| < 1\}$ and $\tilde{H}$ begins with at least the sixth degree term in $\bar{\bar{p}}$, $\bar{\bar{q}}$, that is,

(12) $|\tilde{H}(\tau, Q)| \leq C|\tau|^3$ in G.

We note that the motion by the canonical equation with the Hamiltonian $H_0(\tau)$ is the following:

$$\dot{\tau}_k = -\partial H_0/\partial Q_k = 0, \quad \dot{Q}_k = \partial H_0/\partial \tau_k = \lambda_k + \frac{3\alpha}{4N}\sum_{l=1}^{n} a_{kl}\lambda_k\lambda_l\tau_l \equiv \omega_k,$$

that is, $\tau = (\tau_1(t), \cdots, \tau_n(t)) = (\tau_1(0), \cdots, \tau_n(0))$ is an invariant torus. When $\omega = (\omega_1, \cdots, \omega_n)$ is rationally independent, the motion is conditionally periodic and $\tau(t)$, $Q(t)$ fills the torus $\tau = \tau(0)$ everywhere densely.

4. Results by virtue of Kolomgorov-Arnol'd-Moser's theorem

We remember a theorem of Kolmogorov-Arnol'd-Moser in dynamical systems, which concerns the motion near the equilibrium state in the theory of oscillations. It may be described as follows;
The motion is described by the canonical equation

(13) $\dot{p}_j = -\partial H/\partial q_j, \quad \dot{q}_j = \partial H/\partial p_j, \quad j = 1, 2, \cdots, n,$

where the Hamiltonian is assumed to be

$$H = H_0(p, q) + H_1(p, q),$$

where

(14) $H_0 = \sum_{j=1}^{n} \lambda_j\tau_j + \sum_{j,k=1}^{n} \lambda_{jk}\tau_j\tau_k, \quad 2\tau_j = p_j^2 + q_j^2,$

λ_j, $\lambda_{kj} = \lambda_{jk}$ are constants, $H_1(p, q)$ is analytic with respect to p, q in the domain $G = \{|\tau_j - \varepsilon_0| < \varepsilon_0, j = 1, 2, \cdots, n\}$, and it satisfies

(15) $|H_1| \leq C|\tau|^{5/2}$ in G.

If the condition

(16) $\det(2\lambda_{jk}) = \det(\partial^2 H_0/\partial\tau_j\partial\tau_k) \neq 0$ in G

is valid, then for any $\kappa>0$ it is possible to find $\varepsilon>0$ $(\varepsilon_0>\varepsilon)$ such that: I. The domain $G_\varepsilon=\{|\tau_j-\varepsilon|<\varepsilon\}$ consists of two sets F_ε and f_ε, one of which F_ε is invariant with respect to the motions of (13) and the other f_ε is small: mes $f_\varepsilon<\kappa$ mes F_ε, where mes denotes the ordinary Lebesgue measure. II. F_ε consists of invariant n-dimensional analytic tori T_ω given by the parametrically represented equations

$$p_j=\sqrt{2(\tau_j{}^\omega+f_j{}^\omega(Q))}\ \cos\ (Q_j+g_j{}^\omega(Q))$$
$$q_j=\sqrt{2(\tau_j{}^\omega+f_j{}^\omega(Q))}\ \sin\ (Q_j+g_j{}^\omega(Q)),$$

where $f_j{}^\omega(Q_k+2\pi)=f_j{}^\omega(Q_k),\quad g_j{}^\omega(Q_k+2\pi)=g_j{}^\omega(Q_k),\quad Q=(Q_1,\cdots,Q_n)$

is angular parameter and $\tau^\omega=(\tau_1{}^\omega\cdots\cdots,\ \tau_n{}^\omega)$ is constant depending on the number of the torus ω. III. The invariant tori T_ω differ little from the tori $\tau=\tau^\omega=$constant, i.e., $|f_j{}^\omega(Q)|,\ |g_j{}^\omega(Q)|<\kappa\varepsilon$. IV. The motion determined by (13) on the torus T_ω is conditionally periodic with n-frequencies ω; $\dot{Q}=\omega=\partial H_0/\partial\tau^\omega$, $\dot{\tau}=0$.

In order to apply the above theorem to our case it remains only to verify that the condition (16) holds. From (10) and (14) we get the following:

$$\lambda_{jk}=\frac{3\alpha}{8N}a_{jk}\lambda_j\lambda_k,\quad j,k=1,\cdots,n,$$

where $a_{jk}=4-2\delta^j_{n+1-k}-\delta_k{}^j+\delta_k{}^j\delta^j_{n+1-k}$,

$\lambda_j=\sqrt{2(1-\cos jh)},\quad h=\pi/N.$

Thus $\det(2\lambda_{jk})=\det\left(\frac{3\alpha}{4N}\lambda_j\lambda_k a_{jk}\right)=\left(\frac{3\alpha}{4N}\right)^n\prod_{j=1}^n\lambda_j{}^2\det\,(a_{jk}).$

After calculation we obtain the value $f(n)$ of the determinant of (a_{jk}):

If $n=2m$ $(m=1,2,\cdots)$, then $f(n)=(-3)^{m-1}(8m-3)$.

If $n=2m-1$ $(m=2,3,\cdots)$, then $f(n)=\frac{1}{2}(f(m)-f(m-1))=$

$$=-2(-3)^{m-2}(8m-5).$$

It follows from this that the condition (16) is satisfied and that the conclusion of the theorem is valid, that is,

If n-frequencies of the linear system λ_j are incommensurable in the above-mentioned sense, then we may conclude that the large part of the oscillations starting from initial data sufficiently near the origin of the phase space of the system (2) *is a conditionally periodic motion, or it is a "recurrence" phenomenon.*

At last we note a result on the incommensurability of $\lambda_j=2\sin\ (j\pi/2N)$ $(j=1,2,\cdots,n=N-1)$, which was announced by Izumi (10). If $N=2^m$ $(m=1,2,$

$\cdots\cdots$) or $\mathcal{N}$ is a prime number, then λ_j $(j=1,\cdots,n)$ is rationally independent, i.e., $\sum_{j=1}^{n} k_j \lambda_j \neq 0$ for integers k_j and $\sum_{j=1}^{n} |k_j| \neq 0$. This contains the case of Fermi, Pasta and Ulam's experiments. If $\mathcal{N}$ is the other natural number, then $\lambda_j(j=1,\cdots,n)$ is rationally dependent. Therefore in the latter case a further consideration is needed for the justification of the incommensurability of λ_j assumed in the above proposition.

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