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Matric Operational Calculus and Its Applications

by

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In analysing the linear physical systems with many variables, a good method is to use the matrix functions and the operational calculus.

This paper describes the fundamental properties of the operational calculus for the matrix functions based on the Mikusinsky's method, and then, presents the method to analyse the periodically excited linear systems (periodically interrupted electric circuits of second genus).

1. Introduction

One of the purposes of modern engineering is to investigate the physical systems with many variables. These systems are composed of the physical elements having constant or variable parameters with respect to time and space. Electrical networks, electro-mechanical systems and electro-acoustical systems are typical examples of them.

In analysing such a system, we must find a mathematical model of the system, and then, get the state equations having necessary and sufficient variables (state variables). In the case where these equations are linear and time-invariant, the operational calculus provides an extremely powerful tool for solving them. The operational calculus bases its exposition on the Laplace transform. Mikusinsky introduced operators algebraically as a kind of fraction. This method is based on Titchmarsh's convolution theorem and is simpler and more general than the Laplace transform method[1].

As stated above, we shall treat many variables when the systems to be analysed are very large. In analysing such a very large system, matrix algebra provides a systematic method for the manipulation and solution of system equations. Hence in the present paper, we shall try to apply the operational calculus to the matrix functions. Following to the Mikusinsky's method, we shall present the basic definition and the fundamental properties of the operational calculus in the matrix

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functions, and then, on the basis of these results, consider the linear systems excited periodically.

2. Matrix Functions and Matric Operators

Here, Mikusinsky's operational calculus is extended and applied to the matrix functions, and then, their fundamental properties are given. The functions (elements of matrix functions) to be considered are defined and continuous in the interval $0 \leq t < \infty$. Furthermore, function and value of the functions are distinguished and expressed schematically in the following form.

$\{\boldsymbol{F}(t)\}$: matrix function $\boldsymbol{F}(t)$

$\boldsymbol{F}(t)$: value of the matrix function $\boldsymbol{F}(t)$ at the point t

2.1 Sum and convolution of matrix functions. For the matrix function stated above, sum and convolution are defined as follows.

Sum: As in the case of the ordinary matrix algebra, when matrix functions $\{\boldsymbol{F}(t)\}$, $\{\boldsymbol{G}(t)\}$ and $\{\boldsymbol{H}(t)\}$ satisfy the following condition, $\{H_{ij}(t)\} = \{F_{ij}(t)\} + \{G_{ij}(t)\}$, we shall express them in the following form

$$\{\boldsymbol{H}(t)\} = \{\boldsymbol{F}(t)\} + \{\boldsymbol{G}(t)\}.$$

$\{\boldsymbol{H}(t)\}$ is called the sum of $\{\boldsymbol{F}(t)\}$ and $\{\boldsymbol{G}(t)\}$.

Convolution: When matrices $\boldsymbol{F}(t)$ and $\boldsymbol{G}(t)$ are conformable and not nilfactors (See Appendix), and a matrix function $\{\boldsymbol{H}(t)\}$ satisfies the following condition, $\{H_{ij}(t)\} = \left\{\sum_k \int_0^t F_{ik}(t-\tau)\, G_{kj}(\tau)\, d\tau\right\}$, we shall express them in the following form

$$\{\boldsymbol{H}(t)\} = \{\boldsymbol{F}(t)\}\,\{\boldsymbol{G}(t)\}.$$

$\{\boldsymbol{H}(t)\}$ is called the convolution of $\{\boldsymbol{F}(t)\}$ and $\{\boldsymbol{G}(t)\}$.

For the sum and the convolution of the matrix functions, we have the same fundamental properties (associativity, distributivity and so on) as for the sum and product of the ordinary matrix algebra.

2.2 Matric integral operator. According to the definition of the convolution, we have

$$\{\mathbf{1}\}\,\{\boldsymbol{F}(t)\} = \left\{\int_0^t \boldsymbol{F}(\tau)d\tau\right\}. \tag{2.1}$$

What distinguishes the unit matrix function $\{\mathbf{1}\}$ is the formation of its convolution with an arbitrary matrix function $\{\boldsymbol{F}(t)\}$ causes the integration of the latter in the interval from 0 to t. Consequently, the unit matrix function $\{\mathbf{1}\}$ will be termed the matrix integral operator and denoted by the letter $\boldsymbol{l}$: $\boldsymbol{l} = \{\mathbf{1}\}$.

From the definition of the convolution, we have

$$\boldsymbol{l}^n=\left\{\frac{t^{n-1}}{(n-1)!}\boldsymbol{1}\right\} \qquad n=1,\ 2,\ \ldots\ldots. \tag{2.2}$$

Using the associativity of the convolution, we have

$$\boldsymbol{l}^n\{\boldsymbol{F}(t)\}=\Big\{\underbrace{\int_0^t dt \ldots\ldots \int_0^t}_{n} \boldsymbol{F}(t)dt\Big\}=\Big\{\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\boldsymbol{F}(\tau)d\tau\Big\}. \tag{2.3}$$

2.3 Matric operators. When matrix function $\{\boldsymbol{A}\}$ and $\{\boldsymbol{B}\}$ are given, we shall consider matrix functions $\{\boldsymbol{X}\}$ and $\{\boldsymbol{Y}\}$ satisfying the following equations.

$$\{\boldsymbol{A}\}\{\boldsymbol{X}\}=\{\boldsymbol{B}\} \qquad \{\boldsymbol{Y}\}\{\boldsymbol{A}\}=\{\boldsymbol{B}\}$$

Here $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are square matrices and $\boldsymbol{A}$ is assumed to be non-singluar. Since these equations are not ordinary matrix equations, matrix function $\{\boldsymbol{X}\}$ and $\{\boldsymbol{Y}\}$ satisfing them do not exist always. When the operation inverse to the convolution cannot be performed, a new mathematical concept, matrix operators are introduced. For the multication of matrices, the commutative law does not always hold, therefore, $\{\boldsymbol{X}\}$ and $\{\boldsymbol{Y}\}$ are not expressed as a fraction. But when $\boldsymbol{A}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are commutative, $\{\boldsymbol{X}\}$ and $\{\boldsymbol{Y}\}$ are denoted as $\{\boldsymbol{B}\}/\{\boldsymbol{A}\}$. This fraction represents a matric operator (which is no longer a matrix function).

2.4 Matric numerical operators. Now we shall take for instance $\{\boldsymbol{A}\}=\{\boldsymbol{1}\}$ and $\{\boldsymbol{B}\}=\{\boldsymbol{\alpha}\}$, that is

$$\{\boldsymbol{1}\}\{\boldsymbol{X}\}=\{\boldsymbol{X}\}\{\boldsymbol{1}\}=\{\boldsymbol{\alpha}\}$$

where $\{\boldsymbol{1}\}$ is the unit matrix function and $\{\boldsymbol{\alpha}\}$ is an arbitrary constant numerical matrix function. As stated above, a matric operator $\{\boldsymbol{X}\}$ is expressed as a fraction. We shall denote them by $[\boldsymbol{\alpha}]=\{\boldsymbol{\alpha}\}/\{\boldsymbol{1}\}$, and call it a matric numerical operator. For matric numerical operator $[\boldsymbol{\alpha}]$ and $[\boldsymbol{\beta}]$, we have

$$[\boldsymbol{\alpha}]+[\boldsymbol{\beta}]=[\boldsymbol{\alpha}+\boldsymbol{\beta}] \qquad [\boldsymbol{\alpha}][\boldsymbol{\beta}]=[\boldsymbol{\alpha}\boldsymbol{\beta}]. \tag{2.4}$$

Owing to these formulas the brackets [ ] can be omitted in the operational calculus. For a matric numerical operator $\boldsymbol{\alpha}$ and a matrix function $\{\boldsymbol{\beta}\}$, we have

$$\boldsymbol{\alpha}\{\boldsymbol{\beta}\}=[\boldsymbol{\alpha}]\{\boldsymbol{\beta}\}=\{\boldsymbol{\alpha}\boldsymbol{\beta}\}. \tag{2.5}$$

specially when $\{\boldsymbol{\beta}\}=\{\boldsymbol{1}\}$, we have

$$\boldsymbol{\alpha}\{\boldsymbol{1}\}=\{\boldsymbol{\alpha}\}. \tag{2.6}$$

2.5 Matric differential operator. In the operational calculus we shall define the matric differential operator as the inverse of the matric integral operator, which will be denoted by

$$\boldsymbol{s}=\frac{\boldsymbol{1}}{\boldsymbol{l}}=\frac{\boldsymbol{1}}{\{\boldsymbol{1}\}}. \tag{2.7}$$

By definition we have $\boldsymbol{ls}=\boldsymbol{sl}=\boldsymbol{1}$, and $\boldsymbol{s}$ is a diagonal matrix. For the matric differential operator $\boldsymbol{s}$, we have the following theorem.

Theorem : If a matrix function $\{\boldsymbol{X}(t)\}$ (not always a square matrix) has a derivative matrix $\{\boldsymbol{X}'(t)\}$ continuous for $0 \leq t < \infty$, then we have the formula

$$\boldsymbol{s}\{\boldsymbol{X}(t)\} = \{\boldsymbol{X}'(t)\} + \boldsymbol{X}(0) \tag{2.8}$$

where $\boldsymbol{X}(0)$ is the value of $\{\boldsymbol{X}(t)\}$ at the point $t=0$.

Appling this theorem for a matrix function $\{\boldsymbol{X}(t)\}$ having a n-th derivative matrix $\{\boldsymbol{X}^{(n)}(t)\}$ continuous for $0 \leq t < \infty$, we have the follwing general form.

$$\boldsymbol{s}^n\{\boldsymbol{X}(t)\} = \{\boldsymbol{X}^{(n)}(t)\} + \boldsymbol{X}^{(n-1)}(0) + \boldsymbol{s}\boldsymbol{X}^{(n-2)}(0) + \ldots\ldots + \boldsymbol{s}^{n-1}\boldsymbol{X}(0) \tag{2.9}$$

2.6 Operational form for certain matrix functions. Appling eq. (2.8) we shall have the operational forms for certain matrix functions. First we shall consider $\{e^{\boldsymbol{A}t}\}$, where $\boldsymbol{A}$ is assumed to be a constant numerical square matrix. From eq. (2.8) we obtain the equality

$$\boldsymbol{s}\{e^{\boldsymbol{A}t}\} = \boldsymbol{A}\{e^{\boldsymbol{A}t}\} + \boldsymbol{1}$$

from which we easily arrive at

$$\{e^{\boldsymbol{A}t}\} = \frac{\boldsymbol{1}}{\boldsymbol{s}-\boldsymbol{A}}. \tag{2.10}$$

By the definition of the convolution we have the following general form.

$$\frac{\boldsymbol{1}}{(\boldsymbol{s}-\boldsymbol{A})^n} = \left\{\frac{t^{n-1}}{(n-1)!}e^{\boldsymbol{A}t}\right\} \qquad n=1,\ 2,\ \ldots\ldots . \tag{2.11}$$

Next we shall consider $\{\cos \boldsymbol{B}t\}$ and $\{\sin \boldsymbol{B}t\}$, where $\boldsymbol{B}$ is assumed to be a constant numerical square matrix. Using eq. (2.10) we have

$$\{\cos \boldsymbol{B}t\} = \frac{1}{2}\{e^{i\boldsymbol{B}t} + e^{-i\boldsymbol{B}t}\} = \frac{1}{2}\left(\frac{\boldsymbol{1}}{\boldsymbol{s}-i\boldsymbol{B}} + \frac{\boldsymbol{1}}{\boldsymbol{s}+i\boldsymbol{B}}\right)$$

$$\{\sin \boldsymbol{B}t\} = \frac{1}{2i}\{e^{i\boldsymbol{B}t} - e^{-i\boldsymbol{B}t}\} = \frac{1}{2i}\left(\frac{\boldsymbol{1}}{\boldsymbol{s}-i\boldsymbol{B}} - \frac{\boldsymbol{1}}{\boldsymbol{s}+i\boldsymbol{B}}\right).$$

The matrix $\boldsymbol{s}$ and $\boldsymbol{B}$ are commutative, therefore, we have

$$\{\cos \boldsymbol{B}t\} = \frac{\boldsymbol{s}}{\boldsymbol{s}^2+\boldsymbol{B}^2} \tag{2.12}$$

$$\{\sin \boldsymbol{B}t\} = \frac{\boldsymbol{B}}{\boldsymbol{s}^2+\boldsymbol{B}^2}. \tag{2.13}$$

In the same way, we have

$$\{\cosh \boldsymbol{B}t\} = \frac{\boldsymbol{s}}{\boldsymbol{s}^2-\boldsymbol{B}^2} \tag{2.14}$$

$$\{\sinh \boldsymbol{B}t\} = \frac{\boldsymbol{B}}{\boldsymbol{s}^2-\boldsymbol{B}^2}. \tag{2.15}$$

In finding the operational forms for an arbitrary matrix function, attention must be called to the fact that the commutative law does not always hold for multiplication.

When matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are commutative, we have

$$e^{A}e^{B}=e^{B}e^{A}$$

therefore, we have the following equations.

$$\{e^{At}\cos \boldsymbol{B}t\}=\frac{s-\boldsymbol{A}}{(s-\boldsymbol{A})^2+\boldsymbol{B}^2} \tag{2.16}$$

$$\{e^{At}\sin \boldsymbol{B}t\}=\frac{\boldsymbol{B}}{(s-\boldsymbol{A})^2+\boldsymbol{B}^2} \tag{2.17}$$

$$\{e^{At}\cosh \boldsymbol{B}t\}=\frac{s-\boldsymbol{A}}{(s-\boldsymbol{A})^2-\boldsymbol{B}^2} \tag{2.18}$$

$$\{e^{At}\sinh \boldsymbol{B}t\}=\frac{\boldsymbol{B}}{(s-\boldsymbol{A})^2-\boldsymbol{B}^2} \tag{2.19}$$

2.7 First order simultaneous ordinary differential equations. According to the results given above, we shall consider the following differential equation,

$$\{\dot{\boldsymbol{x}}-\boldsymbol{A}\boldsymbol{x}\}=\{\boldsymbol{B}\boldsymbol{u}\} \tag{2.20}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant numerical matrices. Using eq. (2.8) we have

$$s\{\boldsymbol{x}\}-\boldsymbol{x}(0)=\boldsymbol{A}\{\boldsymbol{x}\}+\{\boldsymbol{B}\boldsymbol{u}\}. \tag{2.21}$$

For the operational solution, we have

$$\{\boldsymbol{x}\}=\frac{1}{s-\boldsymbol{A}}\boldsymbol{x}(0)+\frac{1}{s-\boldsymbol{A}}\boldsymbol{B}\{\boldsymbol{u}\}. \tag{2.22}$$

By the operational form given above and the definition of the convolution, we have the following equality satisfing eq. (2.20).

$$\{\boldsymbol{x}(t)\}=\left\{e^{At}\boldsymbol{x}(0)+\int_0^t e^{A(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau\right\} \tag{2.23}$$

The properties of linear time-invariant physical systems can be expressed by the differential equation the same as (2.20) when state variables are used. Therefore, their properties can be given exactly by eq. (2.23).

2.8 Second order simultaneous ordinary differential equations. Here we shall consider the following differential equations,

$$\{\ddot{\boldsymbol{x}}+\boldsymbol{\alpha}_1\dot{\boldsymbol{x}}+\boldsymbol{\alpha}_0\boldsymbol{x}\}=\{\boldsymbol{\beta}\boldsymbol{u}\} \tag{2.24}$$

where $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}$ are constant numerical matrices. Using eq. (2.9), we have

$$(s^2+\boldsymbol{\alpha}_1 s+\boldsymbol{\alpha}_0)\{\boldsymbol{x}\}=\boldsymbol{x}(0)+(s+\boldsymbol{\alpha}_1)\boldsymbol{x}(0)+\boldsymbol{\beta}(\boldsymbol{u}). \tag{2.25}$$

In solving the second order differential equation for matrix functions, attention must be called to the fact that the commutative law does not always hold for the multiplication of the matrices. With regard to this, we shall consider eq. (2.24). The characteristic equation of this equation is assumed to be solved into the factors below

$$s^2+\boldsymbol{\alpha}_1 s+\boldsymbol{\alpha}_0=(s-\boldsymbol{A})(s-\boldsymbol{B}) \tag{2.26}$$

where A and B are generally complex matrices.

Then we shall formally express the solution of eq. (2.25) in the following form.

$$\{x\} = \frac{1}{(s-A)(s-B)}\dot{x}(0) + \frac{s}{(s-A)(s-B)}x(0) + \frac{a_1}{(s-A)(s-B)}x(0) + \frac{1}{(s-A)(s-B)}\beta\{u\} \tag{2.27}$$

When matrices A and B are commutative and A-B is non-singular, we have

$$\frac{1}{(s-A)(s-B)} = (A-B)^{-1}\{e^{At}-e^{Bt}\}$$

$$\frac{s}{(s-A)(s-B)} = (A-B)^{-1}\{Ae^{At}-Be^{Bt}\}.$$

By the relations between roots and coefficients, we have

$$a_1 = -(A+B) \qquad a_0 = AB$$

therefore, a_1, a_0, A and B are commutative, and we have

$$\frac{a_1}{(s-A)(s-B)} = a_1(A-B)^{-1}\{e^{At}-e^{Bt}\}$$

and

$$\{x\} = \Big\{(A-B)^{-1}[(e^{At}-e^{Bt})\dot{x}(0) + (Ae^{At}-Be^{Bt})x(0) + a_1(e^{At}-e^{Bt}) \cdot x(0) + \int_0^t (e^{A(t-\tau)} - e^{B(t-\tau)})\beta u(\tau)d\tau]\Big\}. \tag{2.28}$$

As stated above, in order to solve the second order differential equation of (2.24), it is necessary that matrices a_1 and a_0 are commutative.

For a simple example, we shall consider the following equation,

$$\{A\ddot{x} + Bx\} = 0 \tag{2.29}$$

where A and B are symmetric and positive definite. Using eq. (2.9), we have

$$(s^2 + C^2)\{x\} = \dot{x}(0) + sx(0) \tag{2.30}$$

where $C = (A^{-1}B)^{1/2}$ and in this case C is expressed by the polynominals of $A^{-1}B$[2]. As the operational solution, we have

$$\{x\} = \frac{1}{s^2+C^2}\dot{x}(0) + \frac{s}{s^2+C^2}x(0) \tag{2.31}$$

and then

$$\{x(t)\} = C^{-1}\sin Ct\dot{x}(0) + \cos Ctx(0). \tag{2.32}$$

3. Linear Systems Excited Periodically

Here we shall consider a linear time-invariant system excited periodically

(periodically interrupted electric circuit of second genus[3]) using matric operational calculus.

3.1 Matric translation operator and periodic functions. Now we shall introduce the matric translation operator. First denote by $\{\boldsymbol{H}_\lambda(t)\}$ a matrix function of the following form

$$\{\boldsymbol{H}_\lambda(t)\} = \left\{ \begin{array}{l} \boldsymbol{0}: \ 0 \leq t < \lambda \\ \boldsymbol{1}: \ 0 \leq \lambda < t \end{array} \right\} \tag{3.1}$$

and is called the Heviside's function. It is not so much the Heviside's function as the matric operator

$$\boldsymbol{h}^\lambda = \boldsymbol{s}\,\{\boldsymbol{H}_\lambda(t)\} \tag{3.2}$$

concerned with it and termed the matric translation operator that plays an important role in the operational calculus. If $\{\boldsymbol{F}(t)\}$ is an arbitrary matrix function, then

$$\boldsymbol{h}^\lambda\{\boldsymbol{F}(t)\} = \left\{ \begin{array}{l} \quad \boldsymbol{0}: \quad 0 \leq t < \lambda \\ \boldsymbol{F}(t-\lambda): \ 0 \leq \lambda < t \end{array} \right\}. \tag{3.3}$$

If we multiply by the matric operator $\boldsymbol{1}/(\boldsymbol{1}-\boldsymbol{h}^{t_0})$ a matrix function $\{\boldsymbol{F}(t)\}$ which outside the interval $0 \leq t < t_0$ is equal to $\boldsymbol{0}$, we obtain a preiodic matrix function whose period is t_0, for the matric operator $\boldsymbol{1}/(\boldsymbol{1}-\boldsymbol{h}^{t_0})$ is expressed in the following infinite series

$$\frac{\boldsymbol{1}}{\boldsymbol{1}-\boldsymbol{h}^{t_0}} = \boldsymbol{1} + \boldsymbol{h}^{t_0} + \boldsymbol{h}^{2t_0} + \ldots\ldots$$

and then, we have

$$\frac{\boldsymbol{1}}{\boldsymbol{1}-\boldsymbol{h}^{t_0}}\{\boldsymbol{F}(t)\} = \sum_{n=0}^{\infty} \boldsymbol{F}(t-nt_0) = \boldsymbol{G}(t) \tag{3.4}$$

where

$$\boldsymbol{G}(t) = \boldsymbol{G}(t-nt_0) \qquad nt_0 < t < (n+1)t_0 \qquad n=1,\ 2,\ \ldots\ldots. \tag{3.5}$$

3.2 Linear systems excited periodically. The state equation of these systems can be reduced to the following form

$$\{\dot{\boldsymbol{x}}\} = \{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}\} \tag{3.6}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant matrices, $\{\boldsymbol{x}\}$ is the state vector and $\{\boldsymbol{u}\}$ is a periodic function whose period is t_0, that is

$$\{\boldsymbol{u}(t)\} = \{\boldsymbol{u}(t+t_0)\}.$$

By the matric operational calculus, we have

$$\boldsymbol{s}\,\{\boldsymbol{x}\} = \boldsymbol{A}\,\{\boldsymbol{x}\} + \boldsymbol{x}(0) + \boldsymbol{B}\,\{\boldsymbol{u}\}. \tag{3.7}$$

It follows from the periodicity of the input function that it is of the form

$$\{\boldsymbol{u}\}=\frac{\{\boldsymbol{u}_0\}}{1-\boldsymbol{h}^{t_0}}, \qquad \{\boldsymbol{u}_0(t)\}=\left\{\begin{array}{ll}\boldsymbol{u}(t): & 0\leq t<t_0\\ \boldsymbol{0}: & t_0<t\end{array}\right\}$$

and then we have

$$\{\boldsymbol{x}\}=\frac{1}{s-\boldsymbol{A}}\boldsymbol{x}(0)+\frac{1}{s-\boldsymbol{A}}\boldsymbol{B}\frac{\{\boldsymbol{u}_0\}}{1-\boldsymbol{h}^{t_0}}. \tag{3.8}$$

Here we shall consider the solution of this equation by the following three terms. The first term $\{\boldsymbol{x}_0\}$ is the transient term caused by the initial values, the second term $\{\boldsymbol{x}_t\}$ is the transient term when initial value is $\boldsymbol{0}$ and the third term $\{\boldsymbol{x}_s\}$ is the steady-state term. For the first term $\{\boldsymbol{x}_0\}$, we have

$$\{\boldsymbol{x}_0(t)\}=\frac{1}{s-\boldsymbol{A}}\boldsymbol{x}(0)=\{e^{\boldsymbol{A}t}\boldsymbol{x}(0)\}=\{\boldsymbol{\chi}(t)\boldsymbol{x}(0)\}. \tag{3.9}$$

For the remaining two terms, we suppose that

$$\frac{1}{s-\boldsymbol{A}}\boldsymbol{B}\frac{\{\boldsymbol{u}_0\}}{1-\boldsymbol{h}^{t_0}}=\{\boldsymbol{X}_s\}\frac{1}{1-\boldsymbol{h}^{t_0}}+\{\boldsymbol{x}_t\}=\{\boldsymbol{x}_s+\boldsymbol{x}_t\} \tag{3.10}$$

where

$$\{\boldsymbol{X}_s\}=\frac{1}{s-\boldsymbol{A}}\boldsymbol{B}\{\boldsymbol{u}_0\}-\{\boldsymbol{x}_t\}(1-\boldsymbol{h}^{t_0}) \qquad t_0<t<\infty. \tag{3.11}$$

Because $\{\boldsymbol{x}_s\}$ is a stead-state solution and $\{\boldsymbol{X}_s\}$ is the periodic part, $\{\boldsymbol{X}_s\}$ is equal to $\boldsymbol{0}$ in the interval $t_0<t<\infty$, therefore

$$\frac{1}{s-\boldsymbol{A}}\boldsymbol{B}\{\boldsymbol{u}_0\}=\{\boldsymbol{x}_t\}(1-\boldsymbol{h}^{t_0}). \tag{3.12}$$

This function $\{\boldsymbol{x}_t\}$ satisfies the following differential equation in the interval $t_0<t<\infty$.

$$\{\dot{\boldsymbol{x}}\}=\{\boldsymbol{A}\boldsymbol{x}\}$$

Suppose that $\boldsymbol{C}$ is an arbitrary constant matrix, then we have

$$\{\boldsymbol{x}_t\}=\frac{1}{s-\boldsymbol{A}}\boldsymbol{C}=\{e^{\boldsymbol{A}t}\boldsymbol{C}\}=\{\boldsymbol{\chi}(t)\boldsymbol{C}\}. \tag{3.13}$$

In this case, the right side of eq. (3.11) is

$$\{\boldsymbol{x}_t\}(1-\boldsymbol{h}^{t_0})=\{\boldsymbol{\chi}(t)\boldsymbol{C}\}-\{\boldsymbol{\chi}(t-t_0)\boldsymbol{C}\}=\{-\boldsymbol{\chi}_0^{-1}(1-\boldsymbol{\chi}_0)\boldsymbol{\chi}\boldsymbol{C}\} \tag{3.14}$$

where $\boldsymbol{\chi}_0=\boldsymbol{\chi}(t_0)=e^{\boldsymbol{A}t_0}$, $\boldsymbol{\chi}=\boldsymbol{\chi}(t)=e^{\boldsymbol{A}t}$ and these matrices are commutative, and the left side is

$$\begin{aligned}\frac{1}{s-\boldsymbol{A}}\boldsymbol{B}\{\boldsymbol{u}_0\}&=\left\{\int_0^t\boldsymbol{\chi}(t-\tau)\boldsymbol{B}\boldsymbol{u}_0(\tau)d\tau\right\}\\&=\left\{\boldsymbol{\chi}(t-t_0)\int_0^t\boldsymbol{\chi}(t_0-\tau)\,\boldsymbol{B}\boldsymbol{u}_0(\tau)d\tau\right\}=\{\boldsymbol{\chi}\boldsymbol{\chi}_0^{-1}\boldsymbol{\varphi}_0\}\end{aligned} \tag{3.15}$$

where

$$\boldsymbol{\varphi}_0=\int_0^t\boldsymbol{\chi}(t_0-\tau)\boldsymbol{B}\boldsymbol{u}_0(\tau)d\tau,$$

We can get the function $\{\boldsymbol{x}_t\}$ deciding the matrix $\boldsymbol{C}$ by eqs. (3.11), (3.14) and (3.15), that is

$$\{\boldsymbol{x}_t(t)\} = \{-\boldsymbol{\chi}(\boldsymbol{1}-\boldsymbol{\chi}_0)^{-1}\boldsymbol{\varphi}_0\}. \tag{3.16}$$

For the periodic part $\{\boldsymbol{X}_s\}$ in the interval $0 \leq t < t_0$, using eq. (3.10), we have

$$\{\boldsymbol{X}_s\} = \frac{1}{s-\boldsymbol{A}}\boldsymbol{B}\{\boldsymbol{u}_1\} - \{\boldsymbol{x}_t\} \tag{3.17}$$

where

$$\{\boldsymbol{u}_1(t)\} = \left\{ \begin{array}{l} \boldsymbol{u}(t): \ 0 \leq t < t_0 \\ \text{arbitrary} \end{array} \right\}$$

therefore, we have

$$\begin{aligned} \{\boldsymbol{X}_s\} &= \left\{ \int_0^t \boldsymbol{\chi}(t-\tau)\boldsymbol{B}\boldsymbol{u}_1(\tau)d\tau \right\} + \{\boldsymbol{\chi}(\boldsymbol{1}-\boldsymbol{\chi}_0)^{-1}\boldsymbol{\varphi}_0\} \\ &= \{\boldsymbol{\varphi}+\boldsymbol{\chi}(\boldsymbol{1}-\boldsymbol{\chi}_0)^{-1}\boldsymbol{\varphi}_0\}. \end{aligned} \tag{3.18}$$

This equality can be used only in the interval $0 \leq t < t_0$. The steady-state term $\{\boldsymbol{x}_s\}$ is expressed by $\{\boldsymbol{X}_s\}$ in the following form.

$$\{\boldsymbol{x}_s(t)\} = \{\boldsymbol{X}_s(t-nt_0)\} \qquad nt_0 < t < (n+1)t_0 \qquad n=0,\ 1,\ 2,\ \ldots\ldots \tag{3.19}$$

We can get the perfect solution of eq. (3.6) in the following form.

$$\{\boldsymbol{x}(t)\} = \{\boldsymbol{x}_0(t) + \boldsymbol{x}_t(t) + \boldsymbol{x}_s(t)\} \tag{3.20}$$

4. Conclusion

As mentioned above, a method has been presented to apply the Mikusinsky's operational calculus to the matrix functions, and then, using these results to analyse the linear time-invariant systems excited periodically. In case of the matrix functions, what differs from the caes of the schalar functions is that the sphere in applying this method is restricted by the non-commutativity of matrices in multiplication.

Further work is proceeding on analysis of the systems whose properties are expressed by the simultaneous partial differential equations.

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Appendix

The operational calculus stated in this paper is based on the following Titchmarsh's convolution theorem for the matrix functions. First we shall prove a few preparatory propositions, and then, Titchmarsh's theorem.

1. Theorem of Phragmén.

If $\boldsymbol{G}$ is a matrix function whose elements are continuous in the interval $0 \leq t \leq T$, then

$$\lim_{x\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^T e^{kx(t-\tau)} \boldsymbol{G}(\tau)d\tau = \int_0^t \boldsymbol{G}(\tau)d\tau \qquad 0 \leq t \leq T. \tag{1.1}$$

Proof: If we assume

$$\lim_{x\to\infty} \sum_{k=1}^{\infty} \quad \text{and} \quad \int_0^T \quad \text{can change place with each other,} \tag{1.2}$$

the left side of eq. (1.1) is

$$\int_0^T \boldsymbol{G}(\tau)d\tau \lim_{x\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kx(t-\tau)} d\tau = \int_0^T \boldsymbol{G}(\tau) \lim_{x\to\infty} [1 - \exp(e^{-x(t-\tau)})] d\tau \tag{1.3}$$

and then

$$\lim_{x\to\infty} \exp(e^{-x(t-\tau)}) = \begin{cases} 0: & \tau < t \\ 1: & \tau > t \end{cases}$$

therefore we have the formula (1.1). The formula (1.2) can be proved by Lebesgue's theorem, q. e. d.

2. Theorem on moments.

I. If $\boldsymbol{F}$ is a matrix function whose elements are continuous in the interval $0 \leq t \leq T$, and there exists such a number N that

$$\left| \int_0^T e^{nT} \boldsymbol{F}(t)dt \right| \leq N\boldsymbol{H} \qquad n = 1, 2, \ldots\ldots \tag{2.1}$$

then $\boldsymbol{F}(t) = 0$ in the interval $0 \leq t \leq T$, where $\boldsymbol{H}$ is a matrix whose elements are all unity.

Proof: Phragmén's formula (1.1) can be written in the form

$$\lim_{x\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{-kx(T-t)} \int_0^T e^{kx(T-\tau)} \boldsymbol{G}(\tau)d\tau = \int_0^t \boldsymbol{G}(\tau)d\tau \qquad 0 \leq t \leq T \tag{2.2}$$

If k and x are natural numbers and

$$\boldsymbol{G}(\tau) = \boldsymbol{F}(T-\tau) \tag{2.3}$$

then by the assumption (2.1), we have

$$\left| \int_0^T e^{kx(T-\tau)} \boldsymbol{G}(\tau)d\tau \right| = \left| \int_0^T e^{kx(T-\tau)} \boldsymbol{F}(T-\tau)d\tau \right| = \left| \int_0^T e^{kxt} \boldsymbol{F}(t)dt \right| \leq N\boldsymbol{H}.$$

Consequently the expression preceeded by the sign "lim" in formula (2.2) is not greater than

$$\boldsymbol{NH}\sum_{k=1}^{\infty}\frac{1}{k!}e^{-kx(T-t)}=\boldsymbol{NH}\{\exp(e^{-x(T-t)}-1)\}$$

and therefore thens to $\mathbf{0}$ as x increases infinitely running over natural values. The existence of a limit of this expression for $x\to\infty$ is ensured by Phragmén's theorem for x running over arbitrary positive value: this limit must always equal $\mathbf{0}$, since it is equal to $\mathbf{0}$ if x runs over natural values, therefore

$$\int_0^T \boldsymbol{G}(\tau)d\tau=\mathbf{0} \qquad 0\leq t<T.$$

Differentiating this equality, we have $\boldsymbol{G}(t)=\mathbf{0}$ for $0<t<T$ and by (2.3) $\boldsymbol{F}(t)=\mathbf{0}$ for $0<t<T$. Because $\boldsymbol{F}(t)$ is continuous, we must have $\boldsymbol{F}(t)=\mathbf{0}$ for $0\leq t\leq T$, q.e.d.

II. If a matrix function $\boldsymbol{G}$ is continuous in the interval $1\leq x\leq X$ and there exists a number N such that

$$\left|\int_1^X x^n\boldsymbol{G}(x)dx\right|\leq \boldsymbol{NH} \qquad n=1, 2, \ldots\ldots \tag{2.4}$$

then $\boldsymbol{G}(x)=\mathbf{0}$ in the interval $1\leq x\leq X$.

Proof: By the substitution $x=e^t$, $X=e^T$ and $x\boldsymbol{G}(x)=\boldsymbol{F}(t)$ inequalities (2.4) change into (2.1). It follows that $\boldsymbol{F}(t)=\mathbf{0}$ for $0\leq t\leq T$ i.e., that $x\boldsymbol{G}(x)=\mathbf{0}$ for $1\leq x\leq X$, q.e.d.

III. If a matrix function $\boldsymbol{F}$ is continuous in the interval $0\leq t\leq T$ and

$$\int_0^T t^n\boldsymbol{F}(t)dt=\mathbf{0} \qquad n=1, 2, \ldots\ldots \tag{2.5}$$

then $\boldsymbol{F}(t)=\mathbf{0}$ in the interval $0\leq t\leq T$.

Proof: Let θ be an arbitrary fixed number from the interval $0\leq t\leq T$. By the substitution

$$t=\theta x, \quad T=\theta X \quad \text{and} \quad \boldsymbol{F}(t)=\boldsymbol{G}(x)$$

equality (2.5) yields

$$\theta^{n+1}\int_0^x x^n\boldsymbol{G}(x)dx=\mathbf{0} \qquad n=1, 2, \ldots\ldots$$

and

$$\left|\int_1^X x^n\boldsymbol{G}(x)dx\right|=\left|\int_0^1 x^n\boldsymbol{G}(x)dx\right|\leq\int_0^1\left|\boldsymbol{G}(x)\right|dx=\boldsymbol{NH} \qquad n=1, 2, \ldots\ldots.$$

By theorem II $\boldsymbol{G}(x)=\mathbf{0}$ for $1\leq x\leq X$, i.e., $\boldsymbol{F}(t)=\mathbf{0}$ for $\theta\leq t\leq T$. Since θ can be fixed arbitrary small and $\boldsymbol{F}$ is continuous, $\boldsymbol{F}$ must be equal to $\mathbf{0}$ at $t=0$, q.e.d.

3. Titchmarsh's theorem in the case $\boldsymbol{F}=\boldsymbol{G}$.

If $\boldsymbol{F}$ is a matrix function continuous in the interval $0\leq t\leq 2T$ and that

$$\int_0^T \boldsymbol{F}(t-\tau)[\boldsymbol{F}(\tau)]^t d\tau = \boldsymbol{0} \qquad 0 \leq t \leq 2T \tag{3.1}$$

where $[\boldsymbol{F}]^t$ is the transposed matrix of $\boldsymbol{F}$, then $\boldsymbol{F}(t)=\boldsymbol{0}$ in the interval $0 \leq t \leq T$.
Proof: From (3.1)

$$\boldsymbol{I}_n = \int_0^{2T} e^{n(2T-t)} dt \int_0^t \boldsymbol{F}(t-\tau)[\boldsymbol{F}(\tau)]^t d\tau = \boldsymbol{0}. \tag{3.2}$$

The iterated integral (3.2) can be represented as follows

$$\boldsymbol{I}_n = \iint_A e^{n(2T-t)} \boldsymbol{F}(t-\tau)[\boldsymbol{F}(\tau)]^t d\tau$$

where A is a triangle defined bey the inequalities

$$0 \leq \tau \leq t \leq 2T.$$

After the substitution

$$t = 2T-u-v, \qquad \tau = T-v$$

we have

$$\boldsymbol{I}_n = \iint_B e^{n(u+v)} \boldsymbol{F}(T-u)[\boldsymbol{F}(T-v)]^t dudv$$

where B is a triangle defined by the inequalities

$$0 \leq u+v, \qquad u \leq T, \qquad v \leq T.$$

We can write

$$\iint_{B+C} = \iint_B + \iint_C$$

where C is a triangle, defined by the inequalities

$$-T \leq u, \qquad -T \leq v, \qquad u+v \leq 0.$$

Since $\boldsymbol{I}_n = \boldsymbol{0}$, we have

$$\iint_{B+C} e^{nu} \boldsymbol{F}(T-u) e^{nv} [\boldsymbol{F}(T-v)]^t dudv$$
$$= \iint_C e^{n(u+v)} \boldsymbol{F}(T-u)[\boldsymbol{F}(T-v)]^t dudv\,.$$

If $h>0$ and M denotes the maximum absolute value of the elements of $\boldsymbol{F}$, we have

$$\left| \int_{-T}^T e^{nu} \boldsymbol{F}(T-u) du \int_{-T}^T e^{nv} [\boldsymbol{F}(T-v)]^t dv \right| = \iint_C M^2 \boldsymbol{H} dudv = 2T^2 M^2 \boldsymbol{H}$$

and consequently

$$\left| \int_{-T}^T e^{nu} \boldsymbol{F}(T-u) du \right| \leq \sqrt{2}\, TM\boldsymbol{H}\,.$$

Therefore

$$\left| \int_0^T e^{nu} \boldsymbol{F}(T-u) du \right| = \left| \int_{-T}^T e^{nu} \boldsymbol{F}(T-u) du - \int_{-T}^0 e^{nu} \boldsymbol{F}(T-u) du \right|$$
$$\leq \sqrt{2}\, TM\boldsymbol{H} + \left| \int_{-T}^0 e^{nu} \boldsymbol{F}(T-u) du \right|.$$

But in the last interval e^{nu} is less than 1, and thus

$$\left|\int_0^T e^{nu}F(T-u)du\right| \leq \sqrt{2}\ TMH + \int_{-T}^0 MHdu = (\sqrt{2}+1)TMH.$$

By the first theorem on moments, $F(T\text{-}u)=0$ for $0 \leq u \leq T$, i.e., $F(t)=0$ for $0 \leq t \leq T$, e.q.d.

Now if the matrix function F is continuous in every infinite interval $0 \leq t < \infty$ and the equality

$$\int_0^t F(t-\tau)[F(\tau)]^t d\tau = 0 \tag{3.3}$$

always hold in that interval, then it holds in every interval $0 \leq t \leq 2T$. It follows that $F(t)=0$ in every interval $0 \leq t \leq T$ and consequently in the interval $0 \leq t < \infty$.

4. Titchmarsh's theorem of general form.

If matrix functions F and G are conformable and not nilfactors, and are not identically equal to 0, then neigher is their convolution identically equal to 0.

Proof: In the interval $0 \leq t < \infty$, we have

$$\int_0^t (t-\tau)F(t-\tau)G(\tau)d\tau + \int_0^t F(t-\tau)\tau G(\tau)d\tau = t\int_0^t F(t-\tau)G(\tau)d\tau. \tag{4.1}$$

Introducing the notation

$$F_1(t) = tF(t) \quad \text{and} \quad G_1(t) = tG(t) \qquad 0 \leq t < \infty$$

we can express (4.1) in the operational symbols as

$$\{F_1\}\{G\} + \{F\}\{G_1\} = 0.$$

Multiplying, $\{G^t{}_1\}\{F^t\}$, we have

$$\{G_1{}^t\}\{F^t\}\{F_1\}\{G\} + \{G^t{}_1\}\{F^t\}\{F\}\{G_1\} = 0.$$

Since by hypothesis, we have

$$\{G_1{}^t\}\{F^t\}\{F_1\}\{G\} = \{G_1{}^t\}\{F_1{}^t\}\{F\}\{G\} = 0,$$

therefore

$$\{G_1{}^t\}\{F^t\}\{F\}\{G\} = [\{G_1{}^t\}\{F^t\}][\{G_1{}^t\}\{F^t\}]^t = 0.$$

By (3.3) we have $\{F\}\{G_1\}=0$ i. e.,

$$\int_0^t F(t-\tau)\tau G(\tau)d\tau = 0 \qquad 0 \leq t < \infty. \tag{4.2}$$

In the same way we have

$$\int_0^t F(t-\tau)\tau^n G(\tau)d\tau = 0.$$

Hence by the third theorem on moments

$$F(t-\tau)G(\tau) = 0 \qquad 0 \leq \tau \leq t < \infty.$$

If $G(\tau_0) \neq 0$ for a certain $\tau_0 \geq 0$, then $F=0$ in the interval $0 \leq t < \infty$. If no such τ_0 exists, then $G=0$ in the interval $0 \leq t < \infty$, q. e. d.