On a Generalized Problem of Disc Electrodes II

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By

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A formal solution of a set of quadruple integral equations is given. The integral equations are reduced to a pair of simultaneous integral equations of Fredholm type which may be solved by a method similar to that used for a single equation. As an example, the problem of a circular plate condenser with guard rings is considered. It is shown that all the well-known disc electrode problems may be treated as special cases.

1. Introduction

Mixed boundary value problems often arise in electrostatics and other mathematical physics. A certain class of mixed boundary value problems may be reduced to dual or triple integral equations. General and systematic investigation of multiple integral equations, originated by Titchmarsh[1], has been promoted extensively in recent years and numerous problems have been solved in electrostatics, elastostatics and heat conduction problems. To extend the applicability of the theory, it is desirable that more complicated systems may be solved.

In this paper we consider a circular plate condenser with guard rings. The problem may be described as quadruple integral equations with Hankel kernel of 0-th order. However in section 1, we discuss quadruple integral equations in a generalized form. Although the closed form solution cannot be obtained, the integral equations are reduced to a pair of Fredholm integral equations. Since a pair of Fredholm integral equations can be solved by a method similar to that used for a single equation, one may consider that the problem is "solved" by the present paper, in the sense of Cooke[3], who says: "In the old days a problem was considered to be solved if it could be 'reduced to a quadrature'. Nowadays one can surely consider that a problem is solved if it can be reduced to an integral equation which can be solved by well-tried procedures."

After the general discussion of integral equations, the specific problem of

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electrostatics, that is, the problem of a circular plate condenser with guard rings is discussed.

Our analysis is purely formal and we don't justify the change of the order of integrations and other limiting processes.

2. Integral Equations

In this paper we consider the set

$$\int_0^\infty u^{-1}\{1+h_1(u)\}A(u)J_\nu(\rho u)du = f_1(\rho) \qquad 0 \leqq \rho < a \tag{1}$$

$$\int_0^\infty A(u)J_\nu(\rho u)du = 0 \qquad a<\rho<b \tag{2}$$

$$\int_0^\infty u^{-1}\{1+h_3(u)\}A(u)J_\nu(\rho u)du = f_3(\rho) \qquad b<\rho<c \tag{3}$$

$$\int_0^\infty A(u)J_\nu(\rho u)du = 0 \qquad c<\rho \tag{4}$$

$$\nu > -\frac{1}{2}.$$

Functions $f_i(\rho)$ and $h_i(u)$ $(i=1, 3)$ are known functions and the problem is to determine the unknown function $A(u)$. We assume that $\rho^\nu f_1(\rho)$ and $\rho^\nu f_3(\rho)$ are continuously differentiable in $[0, a]$ and $[b, c]$, respectively and that $h_i(u)$ have properties necessary for integrations. We shall use the method due to Noble[2] and Cooke[3]. The following lemma is due to Noble[2].

Lemma 1.

$$\begin{aligned} L(\rho, y) &= \int_0^\infty J_\nu(\rho u)J_\nu(yu)du \\ &= \frac{2(\rho y)^{-\nu}}{\pi}\int_0^{\min(\rho, y)} \frac{s^{2\nu}ds}{\sqrt{(\rho^2-s^2)(y^2-s^2)}}, \end{aligned} \tag{5}$$

where $\nu > -\frac{1}{2}$.

We define the quantity $I(t, u, \alpha)$ by

$$I(t, u, \alpha) = \frac{d}{dt}\int_\alpha^t \frac{y^{\nu+1}J_\nu(yu)}{\sqrt{t^2-y^2}}dy. \tag{6}$$

It should be noted that $I(t, u, \alpha)$ satisfies the integral equation

$$\frac{2}{\pi}y^{-\nu}\int_\alpha^y \frac{I(t, u, \alpha)}{\sqrt{y^2-t^2}}dt = J_\nu(yu). \tag{7}$$

From Eqs. (2) and (4), we may write

$$A(u) = u\int_0^a yg_1(y)J_\nu(yu)dy + u\int_b^c yg_3(y)J_\nu(yu)dy\,. \tag{8}$$

From Eqs. (5) and (8), we have

$$\int_0^\infty u^{-1}A(u)J_\nu(\rho u)\,du = \int_0^a yg_1(y)L(\rho, y)\,dy + \int_b^c yg_3(y)\;L(\rho, y)dy\,. \tag{9}$$

We define $G_1(s)$ and $G_3(s)$ by

$$G_1(s) = \int_s^a \frac{y^{-\nu+1}g_1(y)}{\sqrt{y^2-s^2}}\,dy\,, \tag{10}$$

$$G_3(s) = \int_s^c \frac{y^{-\nu+1}g_3(y)}{\sqrt{y^2-s^2}}\,dy\,, \tag{11}$$

which are equivalent to

$$y^{-\nu+1}g_1(y) = -\frac{2}{\pi}\frac{d}{dy}\int_y^a \frac{sG_1(s)}{\sqrt{s^2-y^2}}ds \tag{12}$$

and

$$y^{-\nu+1}g_3(y) = -\frac{2}{\pi}\frac{d}{dy}\int_y^c \frac{sG_3(s)}{\sqrt{s^2-y^2}}ds\,, \tag{13}$$

respectively.

At first, we consider Eq. (1).

If we substitute Eq. (5) into the right hand side of Eq. (9) and invert the order of integration, we have

$$\begin{aligned}\int_0^a yg_1(y)L(\rho, y)dy &= \frac{2}{\pi}\int_0^a yg_1(y)dy\int_0^{\min(\rho, y)} \frac{(\rho y)^{-\nu}s^{2\nu}}{\sqrt{(\rho^2-s^2)(y^2-s^2)}}ds \\ &= \frac{2}{\pi}\rho^{-\nu}\int_0^\rho \frac{s^{2\nu}}{\sqrt{\rho^2-s^2}}\int_s^a \frac{y^{-\nu+1}g_1(y)}{\sqrt{y^2-s^2}}dy \\ &= \frac{2}{\pi}\rho^{-\nu}\int_0^\rho \frac{s^{2\nu}G_1(s)}{\sqrt{\rho^2-s^2}}ds \end{aligned} \tag{14}$$

and

$$\begin{aligned}\int_b^c yg_3(y)L(\rho, y)dy &= \frac{2}{\pi}\int_b^c yg_3(y)dy\int_0^{\min(\rho, y)} \frac{(\rho y)^{-\nu}s^{2\nu}}{\sqrt{(\rho^2-s^2)(y^2-s^2)}}ds \\ &= \frac{2}{\pi}\rho^{-\nu}\int_0^\rho \frac{s^{2\nu}ds}{\sqrt{\rho^2-s^2}}\int_b^c \frac{y^{-\nu+1}g_3(y)}{\sqrt{y^2-s^2}}dy \\ &= \frac{2}{\pi}\rho^{-\nu}\int_0^\rho \frac{s^{2\nu}ds}{\sqrt{\rho^2-s^2}}\int_b^c \frac{1}{\sqrt{y^2-s^2}}\left[-\frac{2}{\pi}\frac{d}{dy}\int_y^c \frac{tG_3(t)}{\sqrt{t^2-y^2}}dt\right]dy \\ &= \left(\frac{2}{\pi}\right)^2\rho^{-\nu}\int_0^\rho \frac{s^{2\nu}ds}{\sqrt{\rho^2-s^2}}\int_b^c \frac{t\sqrt{b^2-s^2}}{(t^2-s^2)\sqrt{t^2-b^2}}G_3(t)dt, \end{aligned} \tag{15}$$

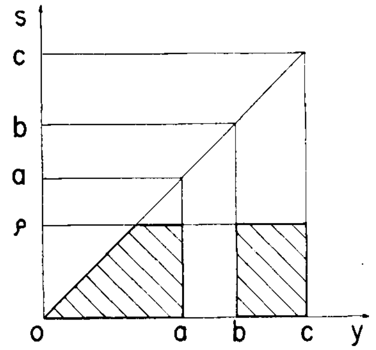

Fig. 1. The field of integration in Eqs. (14) and (15).

where the field of integration is shown in Fig. 1.
Substitution of Eqs. (9), (14) and (15) into Eq. (1) gives

$$\frac{2}{\pi}\int_0^\rho \frac{ds}{\sqrt{\rho^2-s^2}}\left\{s^{2\nu}G_1(s)+\frac{2}{\pi}\int_b^c \frac{s^{2\nu}t\sqrt{b^2-s^2}}{(t^2-s^2)\sqrt{t^2-b^2}}G_3(t)dt\right\} = M(\rho)\,, \tag{16}$$

where

$$M(\rho) = \rho^\nu f_1(\rho)-\int_0^\infty h_1(u)\rho^\nu J_\nu(\rho u)du\int_0^a yg_1(y)J_\nu(yu)dy$$
$$-\int_0^\infty h_1(u)\rho^\nu J_\nu(\rho u)du\int_b^c yg_3(y)J_\nu(yu)dy\,. \tag{17}$$

For the reduction of Eq. (17), we give the following lemma.

Lemma 2.

$$\int_0^a yg_1(y)J_\nu(yu)dy = \frac{2}{\pi}\int_0^a G_1(t)I(t,\ u,\ 0)dt \tag{18}$$

$$\int_b^c yg_3(y)J_\nu(yu)dy = \frac{2}{\pi}\int_b^c G_3(t)\,I(t,\ u,\ b)dt\,. \tag{19}$$

We now give the proof. From Eqs. (7) and (10), we have

$$\int_0^a yg_1(y)J_\nu(yu)dy = \int_0^a yg_1(y)dy\int_0^y \frac{2}{\pi}\frac{y^{-\nu}I(t,\ u,\ 0)}{\sqrt{y^2-t^2}}dt$$
$$= \frac{2}{\pi}\int_0^a I(t,\ u,\ 0)dt\int_t^a \frac{y^{-\nu+1}g_1(y)}{\sqrt{y^2-t^2}}dy$$
$$= \frac{2}{\pi}\int_0^a G_1(t)I(t,\ u,\ 0)dt\,.$$

Eq. (19) may be verified in a similar manner.

Now substitution of Eqs. (18), (19) into Eq. (17) gives

$$M(\rho) = \rho^{\nu} f_1(\rho) - \frac{2}{\pi} \int_0^\infty h_1(u) \rho^{\nu} J_\nu(\rho u) du \int_0^a G_1(t) I(t, u, 0) dt$$
$$- \frac{2}{\pi} \int_0^\infty h_1(u) \rho^{\nu} J_\nu(\rho u) du \int_b^c G_3(t) I(t, u, b) dt . \tag{20}$$

Considering Eq. (16) as an integral equation and solving it, we have

$$\frac{2}{\pi} s^{2\nu} G_1(s) + \left(\frac{2}{\pi}\right)^2 \int_b^c \frac{s^{2\nu} t \sqrt{b^2 - s^2}}{(t^2 - s^2)\sqrt{t^2 - b^2}} G_3(t) dt = \frac{2}{\pi} \frac{d}{ds} \int_0^s \frac{\rho M(\rho)}{\sqrt{s^2 - \rho^2}} d\rho . \tag{21}$$

From Eqs. (20), (21) and (6), we arrive at the integral equation

$$s^{2\nu} G_1(s) + \int_0^a K_{11}(s, t) G_1(t) dt + \int_b^c K_{13}(s, t) G_3(t) dt = \frac{d}{ds} \int_0^s \frac{\rho^{\nu+1} f_1(\rho)}{\sqrt{s^2 - \rho^2}} d\rho , \tag{22}$$

where we have written

$$K_{11}(s, t) = \frac{2}{\pi} \int_0^\infty h_1(u) I(s, u, 0) I(t, u, 0) du , \tag{23}$$

$$K_{13}(s, t) = \frac{2}{\pi} \frac{s^{2\nu} t \sqrt{b^2 - s^2}}{(t^2 - s^2)\sqrt{t^2 - b^2}} + \frac{2}{\pi} \int_0^\infty h_1(u) I(s, u, 0) I(t, u, b) du . \tag{24}$$

We now consider Eq. (3). It may be reduced to an integral equation in a similar way to that used in reduction of Eq. (1). From Eq. (5), we have

$$\int_0^a y g_1(y) L(\rho, y) dy = \frac{2}{\pi} \int_0^a y g_1(y) dy \int_0^{\min(\rho, y)} \frac{(\rho y)^{-\nu} s^{2\nu} ds}{\sqrt{(\rho^2 - s^2)(y^2 - s^2)}}$$
$$= \frac{2}{\pi} \rho^{-\nu} \int_0^a \frac{s^{2\nu}}{\sqrt{\rho^2 - s^2}} ds \int_s^a \frac{y^{-\nu+1} g_1(y)}{\sqrt{y^2 - s^2}} dy$$
$$= \frac{2}{\pi} \rho^{-\nu} \int_0^a \frac{s^{2\nu} G_1(s)}{\sqrt{\rho^2 - s^2}} ds \tag{25}$$

and

$$\int_b^c y g_3(y) L(\rho, y) dy = \frac{2}{\pi} \rho^{-\nu} \int_b^\rho \frac{s^{2\nu} ds}{\sqrt{\rho^2 - s^2}} \int_s^c \frac{y^{-\nu+1} g_3(y)}{\sqrt{y^2 - s^2}} dy$$
$$+ \frac{2}{\pi} \rho^{-\nu} \int_0^b \frac{s^{2\nu} ds}{\sqrt{\rho^2 - s^2}} \int_b^c \frac{y^{-\nu+1} g_3(y)}{\sqrt{y^2 - s^2}} dy$$
$$= \frac{2}{\pi} \rho^{-\nu} \int_b^\rho \frac{s^{2\nu} G_3(s)}{\sqrt{\rho^2 - s^2}} ds + \frac{2}{\pi} \rho^{-\nu} \int_0^b \frac{s^{2\nu} ds}{\sqrt{\rho^2 - s^2}} \int_b^c \frac{y^{-\nu+1} g_3(y)}{\sqrt{y^2 - s^2}} dy , \tag{26}$$

where the field of integration is shown in Fig. 2.
Substitution of Eq. (9) into Eq. (3) with Eqs. (25), (26) gives

$$\frac{2}{\pi} \int_b^\rho \frac{s^{2\nu} G_3(s)}{\sqrt{\rho^2 - s^2}} ds = \rho^{\nu} f_3(\rho) - \frac{2}{\pi} \int_0^a \frac{t^{2\nu} G_1(t)}{\sqrt{\rho^2 - t^2}} dt - \frac{2}{\pi} \int_0^b \frac{t^{2\nu} dt}{\sqrt{\rho^2 - t^2}} \int_b^c \frac{y^{-\nu+1} g_3(y)}{\sqrt{y^2 - t^2}} dy$$

$$-\frac{2}{\pi}\int_0^a G_1(t)dt\int_0^\infty h_3(u)I(t,\ u,\ 0)\rho^\nu J_\nu(\rho u)du$$
$$-\frac{2}{\pi}\int_b^c G_3(t)dt\int_0^\infty h_3(u)I(t,\ u,\ b)\rho^\nu J_\nu(\rho u)du\ , \tag{27}$$

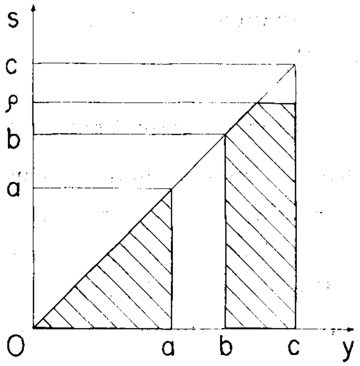

Fig. 2. The field of integration in Eqs. (25) and (26).

where lemma 2 is used. If we denote the right hand side of Eq. (27) by $N(\rho)$, we have

$$\frac{2}{\pi}s^{2\nu}G_3(s)=\frac{2}{\pi}\frac{d}{ds}\int_b^s\frac{\rho N(\rho)}{\sqrt{s^2-\rho^2}}d\rho\ . \tag{28}$$

An elementary computation shows that

$$\frac{d}{ds}\int_b^s\frac{t^{2\nu}\rho}{\sqrt{(s^2-\rho^2)(\rho^2-t^2)}}d\rho=\frac{t^{2\nu}s\sqrt{b^2-t^2}}{(s^2-t^2)\sqrt{s^2-b^2}}\ . \tag{29}$$

In exactly the same way that Cooke[3] has used in his reduction of triple integral equations into a Fredholm integral equation, one may show

$$\frac{2}{\pi}\frac{d}{ds}\int_b^s\frac{\rho d\rho}{\sqrt{s^2-\rho^2}}\int_0^b\frac{t^{2\nu}dt}{\sqrt{\rho^2-t^2}}\int_b^c\frac{y^{-\nu+1}g_3(y)}{\sqrt{y^2-t^2}}dy$$
$$=\left(\frac{2}{\pi}\right)^2\int_b^c G_3(t)dt\cdot\frac{ts}{\sqrt{t^2-b^2}\sqrt{s^2-b^2}}\int_0^b\frac{x^{2\nu}(b^2-x^2)}{(t^2-x^2)(s^2-x^2)}dx\ . \tag{30}$$

Substitution of Eqs. (29), (30) and (6) into Eq. (28) gives the final result

$$s^{2\nu}G_3(s)+\int_0^a K_{31}(s,\ t)G_1(t)dt+\int_b^c K_{33}(s,\ t)G_3(t)dt=\frac{d}{ds}\int_b^s\frac{\rho^{\nu+1}f_3(\rho)}{\sqrt{s^2-\rho^2}}d\rho\ , \tag{31}$$

where we have written

$$K_{31}(s,\ t)=\frac{2}{\pi}\frac{t^{2\nu}s\sqrt{b^2-t^2}}{(s^2-t^2)\sqrt{s^2-b^2}}+\frac{2}{\pi}\int_0^\infty h_3(u)I(s,\ u,\ b)\ I(t,\ u,\ 0)du \tag{32}$$

$$K_{33}(s,\,t)=\left(\frac{2}{\pi}\right)^2\frac{ts}{\sqrt{t^2-b^2}\sqrt{s^2-b^2}}\int_0^b\frac{x^{2\nu}(b^2-x^2)}{(t^2-x^2)(s^2-x^2)}dx$$
$$+\frac{2}{\pi}\int_0^\infty h_3(u)I(s,\,u,\,b)I(t,\,u,\,b)du\,. \qquad (33)$$

Thus the quadruple integral equations (1)-(4) are reduced to a pair of Fredholm integral equations (22) and (31).

3. Electrostatic Potential

We consider the problem of a circular plate condenser with guard rings. This problem is important in electrostatics in its own right but it is all the more important because all the well-known problems involving disc electrodes may be treated as special cases. As shown in Fig. 3, we consider a uniform plate of thickness 2τ and of dielectric constant ε_1. At $z=\pm\tau$, this plate is bounded by uniform medium of directric constant ε_2. We use the cylindrical coordinates of Fig. 3. On this

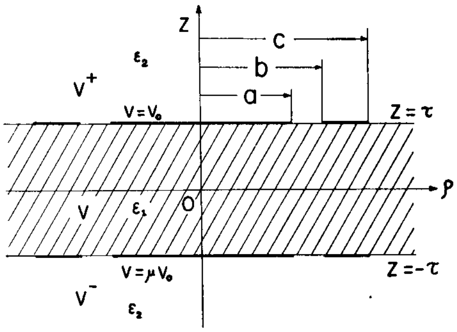

Fig. 3. A plate condenser with guard electrodes.

plate, we arrange two identical conducting discs with flat guard rings. Suppose that the radius of the disc is a and that inner and outer radii of the flat ring are b and c, respectively. We maintain discs and flat rings at $z=\pm\tau$ at prescribed potentials V_0 and μV_0, respectively, where $\mu=1$ or -1. We attempt to find the distribution of electric potential in the whole space. We denote potential functions for $z>\tau$, $-\tau<z<\tau$ and $z<-\tau$ by V^+, V and V^-, respectively. Then potentials must satisfy Laplace's equation in appropriate regions and boundary conditions to be satisfied at $z=\pm\tau$ are as follows:

$$\left.\begin{array}{lll} V^+=V=V_0 & z=\tau & 0\leqq\rho\leqq a,\quad b\leqq\rho\leqq c \\ V^-=V=\mu V_0 & z=-\tau & 0\leqq\rho\leqq a,\quad b\leqq\rho\leqq c \\ \varepsilon_1\dfrac{\partial V}{\partial z}=\varepsilon_2\dfrac{\partial V^\pm}{\partial z} & z=\pm\tau & a<\rho<b,\quad c<\rho\,. \end{array}\right\} \qquad (34)$$

In view of symmetric setup of the problem, we may assume

$$\left.\begin{aligned}
V^{+} &= V_0\int_0^\infty u^{-1}\{e^{-(z-\tau)u}+\mu e^{-(z+\tau)u}\}B(u)J_0(\rho u)\,du \\
V &= V_0\int_0^\infty u^{-1}\{e^{(z-\tau)u}+\mu e^{-(z+\tau)u}\}B(u)J_0(\rho u)\,du \\
V^{-} &= V_0\int_0^\infty u^{-1}\{e^{(z-\tau)u}+\mu e^{(z+\tau)u}\}B(u)J_0(\rho u)\,du \\
\mu &= \pm 1\,.
\end{aligned}\right\} \tag{35}$$

These functions have the property at $z=\pm\tau$

$$\left.\begin{aligned}
V^{+} = V &= V_0\int_0^\infty u^{-1}(1+\mu e^{-2\tau u})B(u)J_0(\rho u)\,du \qquad z=\tau \\
V^{-} = V &= \mu V_0\int_0^\infty u^{-1}(1+\mu e^{-2\tau u})B(u)J_0(\rho u)\,du \qquad z=-\tau
\end{aligned}\right\} \tag{36}$$

and

$$\left.\begin{aligned}
\frac{\partial V^{+}}{\partial z}\bigg|_{z=\tau} &= -V_0\int_0^\infty (1+\mu e^{-2\tau u})B(u)J_0(\rho u)\,du \\
\frac{\partial V}{\partial z}\bigg|_{z=\tau} &= V_0\int_0^\infty (1-\mu e^{-2\tau u})B(u)J_0(\rho u)\,du \\
\frac{\partial V}{\partial z}\bigg|_{z=-\tau} &= -\mu V_0\int_0^\infty (1-\mu e^{-2\tau u})B(u)J_0(\rho u)\,du \\
\frac{\partial V^{-}}{\partial z}\bigg|_{z=-\tau} &= \mu V_0\int_0^\infty (1+\mu e^{-2\tau u})B(u)J_0(\rho u)\,du\,.
\end{aligned}\right\} \tag{37}$$

We define following quantities by

$$\kappa = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\,, \tag{38}$$

$$A(u) = (1-\mu\kappa e^{-2\tau u})B(u) \tag{39}$$

and

$$h(u) = \frac{\mu(\kappa+1)e^{-2\tau u}}{1-\mu\kappa e^{-2\tau u}}\,. \tag{40}$$

Substitution of Eqs. (36)–(40) into Eqs. (34) gives the set of integral equations

$$\left.\begin{aligned}
&\int_0^\infty u^{-1}\{1+h(u)\}A(u)J_0(\rho u)\,du = 1 && 0\leqq\rho<a \\
&\int_0^\infty A(u)J_0(\rho u)\,du = 0 && a<\rho<b \\
&\int_0^\infty u^{-1}\{1+h(u)\}A(u)J_0(\rho u)\,du = 1 && b<\rho<c \\
&\int_0^\infty A(u)J_0(\rho u)\,du = 0 && c<\rho\,.
\end{aligned}\right\} \tag{41}$$

By the use of the result of section 2, the set of integral equations (41) may be reduced to a pair of integral equations

$$G_1(s)+\int_0^a K_{11}(s, t)G_1(t)dt+\int_b^c K_{13}(s, t)G_3(t)dt = 1\,, \qquad 0<s<a\,, \tag{42}$$

$$G_3(s)+\int_0^a K_{31}(s, t)G_1(t)dt+\int_b^c K_{33}(s, t)G_3(t)dt = \frac{s}{\sqrt{s^2-b^2}}\,, \qquad b<s<c\,. \tag{43}$$

Since $\nu=0$, we have from Eq. (6)

$$\begin{aligned} I(t, u, 0) &= \frac{d}{dt}\int_0^t \frac{yJ_0(yu)}{\sqrt{t^2-y^2}}dy \\ &= \cos tu. \end{aligned} \tag{44}$$

Kernels of the integral equations (42) and (43) are given by

$$\begin{aligned} K_{11}(s, t) &= \frac{2}{\pi}\int_0^\infty h(u)\cos su \cos tudu \\ &= \frac{1}{\sqrt{2\pi}}\{F_c(s-t)+F_c(s+t)\} \end{aligned} \tag{45}$$

$$F_c(\xi) = \sqrt{\frac{2}{\pi}}\int_0^\infty h(u)\cos \xi\, udu \tag{46}$$

$$K_{13}(s, t) = \frac{2}{\pi}\frac{t\sqrt{b^2-s^2}}{(t^2-s^2)\sqrt{t^2-b^2}}+\frac{2}{\pi}\int_0^\infty h(u)I(t, u, b)\cos sudu \tag{47}$$

$$K_{31}(s, t) = \frac{2}{\pi}\frac{s\sqrt{b^2-t^2}}{(s^2-t^2)\sqrt{s^2-b^2}}+\frac{2}{\pi}\int_0^\infty h(u)I(s, u, b)\cos tudu \tag{48}$$

$$\begin{aligned} K_{33}(s, t) &= \frac{2}{\pi^2}\frac{1}{(s^2-t^2)}\left\{t\sqrt{\frac{s^2-b^2}{t^2-b^2}}\log\frac{s+b}{s-b}-s\sqrt{\frac{t^2-b^2}{s^2-b^2}}\log\frac{t+b}{t-b}\right\} \\ &\quad +\frac{2}{\pi}\int_0^\infty h(u)I(s, u, b)I(t, u, b)du\,, \end{aligned} \tag{49}$$

where $F_c(\xi)$ denotes Fourier cosine transform of $h(u)$.
It should be noted that $K_{13}(s, t)=K_{31}(t, s)$.

We denote the charge density at $z=\tau$, the charge on the inner disc at $z=\tau$ and the charge on the guard ring at $z=\tau$ by $\sigma(\rho)$, Q_1 and Q_3, respectively. From Eqs. (37)–(40), we have

$$\begin{aligned} \sigma(\rho) &= -\varepsilon_2\frac{\partial V^+}{\partial z}\bigg|_{z=\tau}+\varepsilon_1\frac{\partial V}{\partial z}\bigg|_{z=\tau} \\ &= (\varepsilon_1+\varepsilon_2)V_0\int_0^\infty A(u)J_0(\rho u)du\,. \end{aligned} \tag{50}$$

From Eqs. (8), (10), (11) and (50), we have

$$
\begin{aligned}
Q_1 &= \int_0^a 2\pi\rho\sigma(\rho)d\rho \\
&= 2\pi(\varepsilon_1+\varepsilon_2)aV_0\int_0^\infty u^{-1}A(u)J_1(au)du \\
&= 4(\varepsilon_1+\varepsilon_2)V_0\int_0^a G_1(s)ds
\end{aligned} \tag{51}
$$

and

$$
\begin{aligned}
Q_3 &= \int_b^c 2\pi\rho\sigma(\rho)d\rho \\
&= 2\pi(\varepsilon_1+\varepsilon_2)V_0\int_0^\infty u^{-1}A(u)\{cJ_1(cu)-bJ_1(bu)\}du \\
&= 4(\varepsilon_1+\varepsilon_2)V_0\int_b^c \frac{sG_3(s)}{\sqrt{s^2-b^2}}ds\,,
\end{aligned} \tag{52}
$$

where use is made of formulas such as

$$
\left.
\begin{aligned}
&\int \rho J_0(\rho u)d\rho = \frac{\rho}{u}J_1(\rho u) \\
&\int_0^\infty J_0(\alpha u)J_1(\beta u)du = \begin{cases} 0 & \alpha>\beta \\ \dfrac{1}{2\beta} & \alpha=\beta \\ \dfrac{1}{\beta} & \alpha<\beta\,. \end{cases}
\end{aligned}
\right\} \tag{53}
$$

It is remarkable that Q_1 and Q_3 may be determined only by $G_1(s)$ and $G_3(s)$, respectively. If we denote capacity between disc electrodes by C_1, we have from Eq. (51)

$$
\begin{aligned}
C_1 &= Q_1/2V_0 \\
&= \frac{4\varepsilon_2}{1-\kappa}\int_0^a G_1(s)ds\,.
\end{aligned} \tag{54}
$$

If electric field between the two disc electrodes is uniform and charge on the upper side of disc electrode at $z=\tau$ is negligible, then the capacity is given by

$$
C_0 = \varepsilon_1\frac{\pi a^2}{2\tau}\,. \tag{55}
$$

The ratio of C_1 to C_0 is given by

$$
C_1/C_0 = \frac{8\tau}{\pi a^2(1+\kappa)}\int_0^a G_1(s)ds\,. \tag{56}
$$

4. Special Cases

We now show all the well known problems of disc electrodes as special cases only by changing parameters of $h(u)$. This is important not only from theoretical point of view but also from practical point of view, because a single computer program is necessary.

a. A charged disc

This is a classical problem solved by Weber. We may treat this problem by putting $\kappa=-1$*, $\tau=0$, $\mu=1$ and $b=c$. Substitution of $\kappa=-1$ into Eq. (40) gives $h(u)=0$. Therefore from Eq. (42), we have $G_1(s)=1$. One may easily verify that $g_1(y)=\frac{2}{\pi}\frac{1}{\sqrt{a^2-y^2}}$ and $A(u)=\frac{2}{\pi}\sin au$ which are identical with Weber's result. Of course we can treat a charged disc of radius c by putting $\kappa=-1$*, $\tau=0$ $\mu=1$ and $a=b$. In this case one may verify that $G_1(s)=\frac{2}{\pi}\tan^{-1}\sqrt{\frac{a^2-s^2}{c^2-a^2}}$ $(0\leqq s<a)$ and $G_3(s)=1$ $(a<s<c)$ satisfy integral equations (42) and (43).

b. A charged annular disc

Cooke[3] has solved this problem by reducing triple integral equations to a Fredholm integral equation. We may treat this problem by putting $\kappa=-1$*, $\tau=0$, $\mu=1$ and $a=0$. Eq. (43) may be reduced to a Fredholm integral equation identical with that of Cooke. It should be noticed that a problem of two parallel flat rings may be treated as a special case.

c. A circular plate condenser

This problem has been studied by Nicholson[4], Love[5], Nomura[6], Cooke[7], Sneddon[8] and others. We can treat this problem by putting $\kappa=0$ and $b=c$. The integral equation (42) becomes identical with that derived by Love. The case $\mu=1$ corresponds to a problem of two equally charged discs and the case $\mu=-1$ corresponds to that of two oppositely charged discs. We also treat a circular plate condenser of radius c by putting $\kappa=0$ and $a=b$ but we must treat a pair of integral equations (42), (43). Substitution of $\kappa=0$ into Eq. (38) gives $\varepsilon_1=\varepsilon_2$. Thus $\kappa=0$ corresponds to an air condenser. From the practical point of view, the case $\kappa\neq 0$ is quite important. Of course this may be treated by Eq. (42).

d. The problem of two disc electrodes applied to an infinite plate conductor

We now consider a problem of steady current. We consider an infinite plate of thickness 2τ and of finite electrical conductivity σ. This plate is bounded by vacuum at $z=\pm\tau$ and to these planes two identical perfectly conducting discs of radius a are applied as electrodes. Between these electrodes, we force a constant

* $\kappa=-1$ corresponds to $\varepsilon_1=0$, but in the special case where $\tau=0$ and $\mu=1$, there is no differrence $\kappa=-1$ and $\kappa=0$ $(\varepsilon_1=\varepsilon_2)$.

current I to flow. It is required to find the potential in the plate and to compute the resistance between these electrodes. This problem was studied by Riemann. Kiyono et al[9)10)] showed that the Riemann's solution quoted in the well-known book of Gray and Mathews[11)] gave only a first approximation and that the exact solution could be obtained by solving dual integral equations. Numerical result of this problem is given in our former report[12)]. We can treat this problem by putting $\kappa=1$ (replacing ε_1 and ε_2 by σ_1 and σ_2, respectively and putting $\sigma_2=0$), $\mu=-1$ and $b=c$. The constant V_0 may be determined from the condition that the total current flowing from the disc I is equal to Q_1 in Eq. (51).

5. Numerical Solutions

Integral equations (42) and (43) may be solved by a similar method which is used for a single equation. Utilizing the Legendre-Gauss quadrature formula, the integral equations (42) and (43) may be reduced to a simultaneous linear equations. In order to diminish rounding errors, we adopt the similar variable transformation used by Cooke[3)]. If we write for $b\leqq s\leqq c$,

$$s = b\cdot\sec\theta \qquad 0\leqq\theta\leqq\sec^{-1}(c/b) \tag{57}$$

$$H(\theta) = G_3(b\sec\theta)\sec^2\theta \tag{58}$$

$$F(\theta, u) = \sin\theta I(b\sec\theta, u, b)\,, \tag{59}$$

then integral equations (42) and (43) may be reduced to

$$G_1(s)+\int_0^a K_{11}(s, t)G_1(t)dt+b\int_0^{\sec^{-1}(c/b)} T_{13}(s, \phi)H(\phi)d\phi = 1\,, \qquad 0<s<a \tag{60}$$

$$H(\theta)\sin\theta\cos^2\theta+\int_0^a T_{31}(\theta, t)G_1(t)dt+\int_0^{\sec^{-1}(c/b)} T_{33}(\theta, \phi)H(\phi)d\phi = 1$$

$$0\leqq\theta<\sec^{-1}(c/b), \tag{61}$$

where

$$T_{13}(s, \theta) = T_{31}(\theta, s) = \frac{2}{\pi}\frac{\sqrt{b^2-s^2}}{b^2\sec^2\theta-s^2}+\frac{2}{\pi}\int_0^\infty h(u)F(\theta, u)\cos su\,du \tag{62}$$

$$T_{33}(\theta, \phi) = \frac{4}{\pi^2}\frac{\sec\phi\sin^2\phi\log\left(\tan\frac{\phi}{2}\right)-\sec\theta\sin^2\theta\log\left(\tan\frac{\theta}{2}\right)}{\sec^2\theta-\sec^2\phi}$$

$$+\frac{2}{\pi}b\int_0^\infty h(u)F(\theta, u)F(\phi, u)du\,. \tag{63}$$

When $\theta=\phi$ in Eq. (63), we have by L'Hopital's theorem

$$T_{33}(\theta, \theta) = -\frac{4}{\pi^2}\left(1-\frac{1}{2}\sin^2\theta\right)\cos\theta\log\left(\tan\frac{\theta}{2}\right)-\frac{2}{\pi^2}\cos^2\theta$$
$$+\frac{2}{\pi}b\int_0^\infty h(u)F(\theta, u)F(\theta, u)du\,. \tag{64}$$

Substitution of Eq. (58) into Eq. (52) gives

$$Q_3 = 4(\varepsilon_1+\varepsilon_2)V_0 b\int_0^{\sec^{-1}(c/b)} H(\theta)d\theta\,. \tag{65}$$

We now refer to the computation of kernels of integral equations (60) and (61). In the previous section, we show that the flexibility of $h(u)$ plays an important role from the practical point of view. On the other hand the flexibility of $h(u)$ itself prevents infinite integrals involving $h(u)$ from being simplified. In his paper which treats triple integral equations, Cooke[3] says that though most important, "it seems scarcely practicable" to solve triple integral equations in any specified case of $h(u)\neq 0$. However, since $h(u)$ in our cases decrease exponentially as seen from Eq. (40), infinite integrals involving $h(u)$ may be computed quite efficiently by the Laguerre-Gauss quadrature formula. As to the evaluation of $F(\theta, u)$, substitution of Eq. (6) into Eq. (59) and some elementary computation show that

$$F(\theta, u) = J_0(bu)-bu\tan\theta\int_0^\theta J_1(bu\sec\theta\cos\varphi)d\varphi\,. \tag{66}$$

Thus $F(\theta, u)$ may be estimated by the Legendre-Gauss quadrature formula. Attention should be paid to symmetry of kernels, that is $K_{11}(s, t)=K_{11}(t, s)$, $T_{13}(s, \theta)=T_{31}(\theta, s)$ and $T_{33}(\theta, \phi)=T_{33}(\phi, \theta)$. Symmetry of kernels and the use of Gaussian quadrature formulas diminish the amount of computation of kernels of Eq. (60) and (61) to a great extent. Thus it is quite practicable to solve Eqs. (60) and (61) in an appropriate computer time, though they may appear to be very complicated at first sight.

Several special problems were solved and results were in good agreement with those obtained by other methods. We give some results in Appendix. It should be noted that those results show the correctness of our quadruple integral equations approach. Finally a problem of the plate condenser with guard rings was solved to investigate the effect of the guard electrode. Eqs. (60) and (61) are solved for $\kappa=0.5$, $\mu=-1$, $a=1.00$ and $b=1.01$, changing c from 1.01 to 1.70. Integrals over $[0, a]$ and $[0, \sec^{-1}(c/b)]$ are approximated by the Legendre- Gauss 20- and 15-point formula, respectively, so that Eqs. (60) and (61) are reduced to 35 linear equations. C_1/C_0 *vs* c for $\tau=0.3$ and $\tau=0.5$ are given in Fig. 4. Computations were carried out using computer system FACOM 230-60 at the

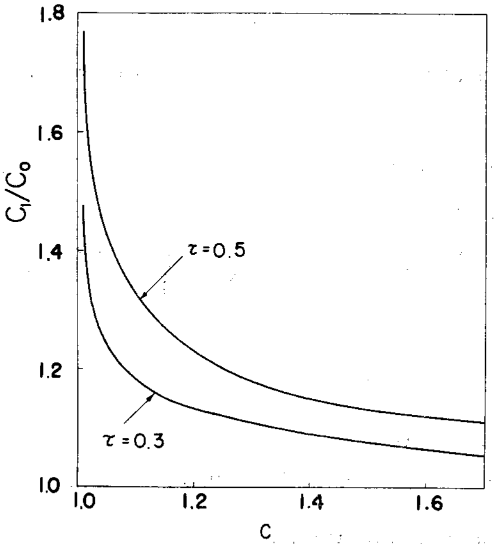

Fig. 4. Capacity between disc electrodes for $\kappa=0.5$, $\mu=-1$, $a=1.0$ and $b=1.01$.

Data Processing Center, Kyoto University. The computer time used to solve Eqs. (60) and (61) for a set of parameters was 40 seconds.

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Appendix

We give numerical results of several special problems referred to in section 4.

a. A charged disc

Parameters: $\kappa=-1$, $\tau=0$, $\mu=1$, $a=b=1$, $c=2$.

The total charge on the disc is given by Q_1+Q_3.

$$\frac{Q_1+Q_3}{4(\varepsilon_1+\varepsilon_2)V_0}=\int_0^a G_1(s)ds+b\int_0^{\sec^{-1}(c/b)}H(\theta)d\theta$$

Computed 2.00000

Exact 2.

b. A charged annular disc

Parameters: $\kappa=-1$, $\tau=0$, $\mu=1$, $a=0$, $b=1$, $c=2$.

$$\frac{\text{capacity of the annular disc}}{\text{capacity of a disc of radius } c}=\frac{b}{c}\int_0^{\sec^{-1}(c/b)}H(\theta)d\theta$$

Computed 0.981007

Cooke's value[3] 0.9810

c. A circular plate condenser

1. The problem of equally charged discs

1.1 Parameters: $\kappa=0$, $\tau=0.5$, $\mu=1$, $a=b=c=1$.

The total charge on the upper disc is given by Q_1.

$$\frac{Q_1}{4(\varepsilon_1+\varepsilon_2)V_0}=\int_0^a G_1(s)ds$$

Computed 0.691207

1.2 Parameters: $\kappa=0$, $\tau=0.5$, $\mu=1$, $a=b=0.5$, $c=1$.

The total charge on the upper disc is given by Q_1+Q_3.

$$\frac{Q_1+Q_3}{4(\varepsilon_1+\varepsilon_2)V_0}=\int_0^a G_1(s)ds+b\int_0^{\sec^{-1}(c/b)}H(\theta)d\theta\ .$$

Computed 0.691205

1.3 Nomura-Cooke's value[8] 0.6912.

2. The problem of equally charged discs

2.1 Parameters: $\kappa=0$, $\tau=0.5$, $\mu=-1$, $a=b=c=1$.

The total charge on the upper disc is given by Q_1.

$$\frac{Q_1}{4(\varepsilon_1+\varepsilon_2)V_0}=\int_0^a G_1(s)ds$$

Computed 1.820787

2.2 Parameters: $\kappa=0$, $\tau=0.5$, $\mu=-1$, $a=b=0.5$, $c=1$.

The total charge on the upper disc is given by Q_1+Q_3.

$$\frac{Q_1+Q_3}{4(\varepsilon_1+\varepsilon_2)V_0}=\int_0^a G_1(s)ds+b\int_0^{\sec^{-1}(c/b)}H(\theta)d\theta\ .$$

Computed 1.820786

2.3 Nomura-Cooke's value[8] 1.8208.

d. The problem of two disc electrodes applied to an infinite plate conductor

We denote the resistance between the two disc electrodes by R. As a normalization constant, we use R_0 given by $R_0=\dfrac{1}{2\sigma c}$.

1. Parameters: $\kappa=1$, $\tau=0.5$, $\mu=-1$, $a=b=c=1$.

$$\frac{R}{R_0}=\frac{c}{\int_0^a G_1(s)ds}$$

Computed 0.429055

2. Parameters: $\kappa=1$, $\tau=0.5$, $\mu=-1$, $a=b=0.5$, $c=1$.

$$\frac{R}{R_0}=\frac{c}{\int_0^a G_1(s)ds+b\int_0^{\sec^{-1}(c/b)}H(\theta)d\theta}$$

Computed 0.429054

3. Kiyono-Shimasaki's value[12] 0.429057.