A Network-Topological Study on Statical Analysis of Rigid Framed Structure

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By

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After the studies of G.H. Kron, who applied the network theory to structural analysis, many investigations in this field were done. They are the analysis by analogy of framed structures to electrical circuits, and the purpose of these studies is to establish the effective formulations of framed structural analysis, with respect to the Network-Topological properties of structures which are suited to the numerical analysis by use of the digital computer which requires systematic notation- and calculation methods. In the network-topological analysis of electrical circuits, there are two basic different methods, i.e. "Node Method" and "Mesh Method". As a variation of "Node Method", the "Tree Method" exists. In the structural analysis, the two basic methods are already formulated as "Displacement Method" and "Force Method" respectively, but the last one is not used. When these methods are used for the analysis of large structures by use of computer, there arise many problems, which are the problems of the capacity of the computer and also that of calculation-time. To decrease the difficulties of these problems, the method of tearing and interconnecting of the structure is used.

In this paper, a tearing and interconnecting method by an information, which is got from Incidence Matrix, is presented. And the application of Tree Method in the problems of electric circuits to the structural analysis and a tearing and interconnecting method for it by successive application of the Housholder's formula is also considered.

1. Introduction

Two basic network-topological formulations for the structural analysis were already established and one of them is called "Displacement Method". In this method a matrix called Incidence Matrix is used. Incidence Matrix expresses the topological properties of framed structures, in other words, the relation between nodes and branches. It means that the matrix prescribes the relationship between nodes and cobundaries, therefore it is possible to think of it as a kind of cut-set matrices, and it gives some information about tearing of structures. Then a structure can be thought of as follows; a structure is a gathering of many substructures which are made of a node and all branches which are connected to the

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node at one end and are fixed at the other end.

In this paper, the tearing and interconnecting method according to this information is presented, and the merits of this method are discussed.

As the variation of "Node Method" which is already formulated as Displacement Method in structural analysis, there is Tree Method in the problems of electrical circuits. This method is also capable of being applied to the analysis of rigid framed structures, but till today this application has not been done. The system in the problems of electric circuits, whose branches are divided into tree and link branches, can be transformed to some tree systems, whose quantities are equal to those of primitive system, by the transformations of all quantities of link branches to the equivalent quantities of tree branches.

In this paper, the application of Tree Method for the analysis of rigid framed structures is considered, and an effective method of it for computer is also presented by the application of Housholder's formula.

2. Tearing and Interconnecting Method with respect to the Incidence Matrix

To solve a complex structure with high redundancy by use of digital computer, the system must be torn into many substructures in accordance with the capacity of the digital computer being used.

The Incidence Matrix, which can be got from a framed structure, implies some informations about tearing of the system, because the matrix prescribes the relationship between nodes and branches (coboundaries of nodes). According to this information, a node and all members connected to it are considered as a basic substructure. Then a given structure is divided into as many as the number of nodes.

The given structure is shown in Fig. 1–a.

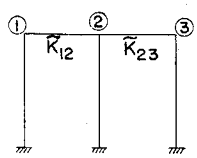

Fig. 1-a Given Structure

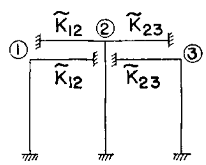

Fig. 1-b Fundamental Set of Substructures

By the Displacement Method in a global coordinate system, the displacements at every node are calculated by the following equation,

$$\tilde{D} = (A^t \tilde{K} A)^{-1} \tilde{P} \tag{1}$$

in which

$\tilde{D}$; the displacements at joints expressed in the global coordinate system
$\tilde{P}$; the external loads at joints expressed in the global coordinate system
$\tilde{K}$; the stiffness matrix of all members expressed in the global coordinate system
A ; incidence matrix

The structure shown in Fig. 1–a is divided into finite substructures shown in Fig. 1–b. Then the incidence matrix is given as follows,

$$A = [A_1, A_2, \cdots, A_i, \cdots, A_n] , \tag{2}$$

where A_i is the i-th column matrix of the incidence matrix A and the subscript n is the number of the joints.

Since the given structure has high redundancy, it is often impossible to solve structures as a whole by using computers with finite capacity. At this stage, the tearing and interconnecting method with respect to the information, which is given from the incidence matrix, is used. The given structure (shown in Fig. 1–a) is torn into many substructures, which consist of all members connecting to a node and it is shown in Fig. 1–b.

Using eq. (2), let us define stiffness matrix $\tilde{K}^{(0)}$ of the 0-th order which indicates the totality of the joint stiffness matrices at every nodal point of unconnected substructures, given in Fig. 1–b.

$$\tilde{K}^{(0)} = \begin{bmatrix} A_1^t\tilde{K}A_1 & & & & \\ & A_2^t\tilde{K}A_2 & & 0 & \\ & & A_3^t\tilde{K}A_3 & & \\ & 0 & & \ddots & \\ & & & & A_n^t\tilde{K}A_n \end{bmatrix} \tag{3}$$

where $A_i^t\tilde{K}A_i$; the joint stiffness matrix of i-th substructure which consists of all members connecting to the i-th node.

A permutation matrix Ω_i is defined as follows,

$$\Omega_i = [0, 0, \cdots, \boldsymbol{E}_i, 0, \cdots, 0] \tag{4}$$

where 0 ; (6×6) zero matrix
$\boldsymbol{E}_i$; (6×6) unit matrix, which locates at i-th column in expression for the permutation matrix

Using eq's (3) and (4), the joint stiffness matrix of the given structure yields to

$$A^t\tilde{K}A = \begin{bmatrix} A_1^t\tilde{K}A_1 & & 0 \\ & A_2^t\tilde{K}A_2 & \\ & & \ddots & \\ 0 & & & A_n^t\tilde{K}A_n \end{bmatrix} - \sum_{ij}^{n} \Omega_i^t\tilde{K}_{ij}\Omega_j \tag{5}$$

where $\bar{K}_{ij}$; the primitive stiffness matrix of a member which is connected to i-th and j-th joints, namely $\bar{K}_{ii}=0$

Equation (5) shows that the joint stiffness matrix of the given structure, left side of eq. (5), can be deformed to the right side of eq. (5), and at the same time this equation indicates the interconnecting method of the unconnected substructures. The stiffness matrix of a given structure consists of the stiffness matrix of the divided substructures and that of all members except those members which connect nodes to the support joints.

For the first step to calculate the flexibility matrix $\bar{F}$ of the given structure from eq. (5), the flexibility matrix $\bar{F}^{(0)}$ of the gathering of the divided structures is defined as follows

$$\bar{F}^{(0)}=\begin{bmatrix} (A_1^t\bar{K}A_1)^{-1} & & 0 \\ & (A_2^t\bar{K}A_2)^{-1} & \\ & & \ddots \\ 0 & & (A_n^t\bar{K}A_n)^{-1}\end{bmatrix}=[\bar{K}^{(0)}]^{-1} \qquad (6)$$

The Housholder's formula is given as

$$(Z+U^tKV)^{-1}=Z^{-1}-Z^{-1}U^t(K^{-1}+VZ^{-1}U^t)^{-1}VZ^{-1} \qquad (7)$$

where Z and K are nonsingular square matrices.

Using this formula, the inverse matrix $\bar{F}^{(1)}$ of the sum of the first term and one symmetric element, $-(\Omega_i^t\bar{K}_{ij}\Omega_j+\Omega_j^t\bar{K}_{ij}\Omega_i)$, of the second term of the left side of eq. (5) can be calculated and the kl-th element of $\bar{F}^{(1)}$ is as follows,

$$\begin{aligned}\bar{F}_{kl}^{(1)} &= \bar{F}_{kl}^{(0)}+\bar{F}_{ki}^{(0)}\bar{K}_{ij}\bar{F}_{jl}^{(0)}+(\bar{F}_{kj}^{(0)}+\bar{F}_{ki}^{(0)}\bar{K}_{ij}\bar{F}_{jj}^{(0)})\\ &\quad\times(\bar{K}_{ij}^{-1}-\bar{F}_{ii}^{(0)}\bar{K}_{ij}\bar{F}_{jj}^{(0)})(\bar{F}_{il}^{(0)}+\bar{F}_{ii}^{(0)}\bar{K}_{ij}\bar{F}_{jl}^{(0)})\end{aligned} \qquad (8)$$

where $\bar{F}_{kl}^{(0)}$ is the kl-th element of matrix $\bar{F}^{(0)}$.

By successive application of this formula for all i and j, the flexibility matrix $\bar{F}_{kl}$ is found as

$$\begin{aligned}\bar{F}_{kl} = \bar{F}_{kl}^{(m)} &= \bar{F}_{kl}^{(m-1)}+\bar{F}_{ki}^{(m-1)}\bar{Z}_{ij}^{(m-1)}\bar{F}_{jl}^{(m-1)}\\ &\quad+(\bar{F}_{kj}^{(m-1)}+\bar{F}_{ki}^{(m-1)}\bar{Z}_{ij}^{(m-1)}\bar{F}_{jj}^{(m-1)})(\bar{Z}_{ij}^{-1}-\bar{F}_{ii}^{(m-1)}\bar{Z}_{ij}^{(m-1)}\bar{F}_{jj}^{(m-1)})\\ &\quad\times(\bar{F}_{il}^{(m-1)}+\bar{F}_{ii}^{(m-1)}\bar{Z}_{ij}^{(m-1)}\bar{F}_{jl}^{(m-1)})\end{aligned} \qquad (9)$$

in which

$\bar{F}_{ij}^{(m-1)}$; the ij-th element of matrix $\bar{F}^{(m-1)}$ which is calculated by the successive application of $(m-1)$ times of Housholder's formula

m ; the number of members which are not connected with the support joints, and $\bar{Z}_{ij}^{(m-1)}=(\bar{K}_{ij}^{-1}-\bar{F}_{jl}^{(m-1)})^{-1}$

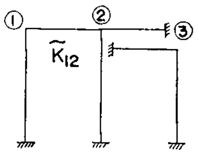

Fig. 1-c $\tilde{F}^{(1)}$

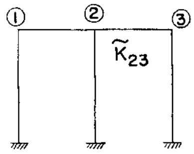

Fig. 1-d $\tilde{F}^{(2)}=\tilde{F}$

In Fig. 1–c and Fig. 1–d, the divided substructures are connected one after another by the stiffness matrices $\tilde{K}_{12}$ and $\tilde{K}_{23}$, respectively. In this example, there are two members which are not connected to the support joints, so $m=2$. Therefore eq. (8) means the step of calculation of $\tilde{F}^{(1)}$ for Fig. 1–c, and eq. (9) means that for Fig. 1–d.

Then at every intermediate step for the calculation of $\tilde{F}$ of given structure, we need not calculate the flexibility matrix of intermediate step with respect to whole k and l of the structure but may calculate only some elements of the flexibility matrix with respect to k and l which are included in the substurcture being connected at every step and other elements are invariant at the step.

3. Application of Tree Method to Structural Analysis

"Tree" in "Graph Theory" is one of the simplest types of graph and it is defined as any connected graph that contains no closed paths. When the figure of a framed structure is shown by a graph, tree systems including all nodal points can be easily gotten. These systems are basic determinate systems in structural analysis, and link members connect these tree systems to each other and make many closed paths.

In Displacement Method the nodal displacements are used as auxiliary variables, but in Tree Method the deformations of tree members are used as auxiliary variables and the latter method can be derived by some transformations from the former, therefore Tree Method is a variation of Displacement Method.

In this section, at first, some formulas of Tree Method in the global coordinate system are presented, and after it by some transformation matrices, from the global coordinate system to the local coordinate system, Tree Method in the local coordinate system is derived. And the application of Housholder's formula to this method is also considered.

3.1 Coordinate System and Transformation Matrix

The coordinate systems for a structure is shown in Fig. 2–a.

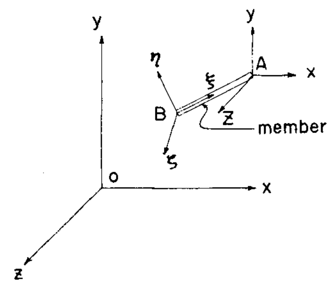

Fig. 2-a Coordinate System

1. 0-xyz ; A global coordinate system with origin at 0 which is arbitrarily chosen in space
2. A-xyz ; A global coordinate system with origin at A which includes all nodes before deformation
3. B-$\xi\eta\zeta$; A local coordinate system with origin at B which is an initial point of all members and ξ axis has the same orientation on the directed line containing the initial and final node of the member considered.
4. C-$\xi\eta\zeta$; A local coordinate system with origin at C which is an initial point of all tree members and ξ axis has the same orientation on the directed line containing the initial and final node of the member considered

A force P_A and a displacement u_A at point A represented in the local coordinate system are specified as six-dimensional vectors $(P_\xi^A, P_\eta^A, P_\zeta^A, m_\xi^A, m_\eta^A, m_\zeta^A)$ and $(u_\xi^A, u_\eta^A, u_\zeta^A, \theta_\xi^A, \theta_\eta^A, \theta_\zeta^A)$ respectively. And similarly, $\tilde{P}_A$ and $\tilde{u}_A$ denote the force vector and the displacement vector at point A represented in the global coordinate system, respectively.

Between these vectors, we have following relations

$$P_A = T_{AO}\tilde{P}_A \tag{10}$$

$$u_A = [T_{AO}^{-1}]^t \tilde{u}_A \tag{11}$$

where T_{AO} is a (6×6) transformation matrix from the global coordinate system 0-xyz to the local coordinate system A-$\xi\eta\zeta$, and matrix T_{AO} is given as

$$T_{AO} = \begin{bmatrix} \Lambda & 0 \\ \Lambda X & \Lambda \end{bmatrix} \tag{12}$$

in which

$$A = \begin{bmatrix} \lambda_\xi & \mu_\xi & \nu_\xi \\ \lambda_\eta & \mu_\eta & \nu_\eta \\ \lambda_\zeta & \mu_\zeta & \nu_\zeta \end{bmatrix} \tag{13}$$

and

$$X = \begin{bmatrix} 0 & z_A & -y_A \\ -z_A & 0 & x_A \\ y_A & -x_A & 0 \end{bmatrix} \tag{14}$$

$(\lambda_\xi, \mu_\xi, \pi_\xi)$; the direction cosine of ξ axis relative to $x, y,$and z
(x_A, y_A, z_A); the global coordinate of point A

3.2 Tree Method in the Global Coordinate System

Applied free-end member distortions $\tilde{V}$ represented in the global coordinate system cause the deformations of meshes and cause the mesh (redundant) forces, of cause. These redundant forces induce member forces $\tilde{r}$.

Similarly the sum of applied fixed member forces $\tilde{R}$ at each joint are equal to the applied joint loads and they cause the joint displacements. The differences of the displacements of neighbouring joints are equal to the member distortions $\tilde{v}$.

A structure is given as shown in Fig. 2–b.

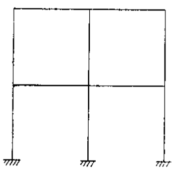

Fig. 2-b Given Structure

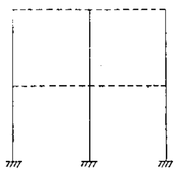

Fig. 2-c

Since all the members are divided into two types which consist of tree and link members (Fig. 2–c), the total member distorsions $\tilde{U}$ are given as follows

$$\begin{bmatrix} \tilde{U}_T \\ \tilde{U}_L \end{bmatrix} = \begin{bmatrix} \tilde{V}_T \\ \tilde{V}_L \end{bmatrix} + \begin{bmatrix} \tilde{v}_T \\ \tilde{v}_L \end{bmatrix} \tag{15}$$

The subscript T indicates tree members and subscript L link members.

By continuity, the member distorsions $\tilde{v}$ are given as

$$\begin{bmatrix} \tilde{v}_T \\ \tilde{v}_L \end{bmatrix} = \begin{bmatrix} A_T \\ A_L \end{bmatrix} \tilde{u} \quad \text{or} \quad \tilde{v} = A\tilde{u} \tag{16}$$

where A ; incidence matrix
$\tilde{u}$; displacements of joints

From eq. (16), following equation is given

$$\tilde{u} = (B_T)^t \tilde{v}_T \tag{17}$$

where B_T ; node-to-datum path matrix.

The load-deflection relationship is shown as

$$\tilde{S} = \tilde{R} + \tilde{r} = \tilde{K}(\tilde{V} + \tilde{v}) \tag{18}$$

where $\tilde{S}$; total member forces
$\tilde{R}$; applied fixed-end member forces
$\tilde{r}$; member forces induced by redundants
$\tilde{K}$; the member stiffness matrix of members

by the arrangement of terms of eq. (18), it becomes as follows

$$(\tilde{R} - \tilde{K}\tilde{V}) + \tilde{r} = \tilde{K}\tilde{v} \tag{19}$$

Multiplying A^t by eq. (19), the second term of eq. (19) vanishes according to the property of Graph Theory. Then the equation becomes as follows

$$A^t(\tilde{R} - \tilde{K}\tilde{V}) = A^t\tilde{K}\tilde{v} \tag{20}$$

Also we have the relation from Graph Theory

$$B_T A^t = D^t \quad \text{or} \quad A B_T^t = D \tag{21}$$

Combining eq's (16), (17), and (21) with eq. (20), we find

$$D^t(\tilde{R} - \tilde{K}\tilde{V}) = D^t\tilde{K}D\tilde{v}_T \tag{22}$$

where $D^t\tilde{K}D$ is the tree stiffness matrix.

The solution of this equation is

$$\tilde{v}_T = (D^t\tilde{K}D)^{-1}D^t(\tilde{R} - \tilde{K}\tilde{V}) \tag{23}$$

in which $(D^t\tilde{K}D)^{-1}$ is the tree flexibility matrix.

Here we use the relation $\tilde{v} = D\tilde{v}_T$, which is arrived at by combining eq's (16), (17), and also $\tilde{r} = \tilde{K}(\tilde{V} + \tilde{v}) - \tilde{R}$, which is derived from eq. (18), to compute the member distorsions $\tilde{v}$ and the induced member forces $\tilde{r}$. Then the solutions are

$$\tilde{v} = D(D^t\tilde{K}D)^{-1}D^t(\tilde{R} - \tilde{K}\tilde{V}) \tag{24}$$

$$\tilde{r} = [\tilde{K} - \tilde{K}D(D^t\tilde{K}D)^{-1}D^t\tilde{K}](\tilde{V} - \tilde{F}\tilde{R}) \tag{25}$$

in which $D(D^t\tilde{K}D)^{-1}D^t$ and $[\tilde{K} - \tilde{K}D(D^t\tilde{K}D)^{-1}D^t\tilde{K}]$ are member flexibility and member stiffness matrices in another form respectively, and $\tilde{F}$ is the primitive flexibility matrix of members.

Combining these results with eq's (15) and (18), the total member distorsions $\tilde{U}$ and the total member forces $\tilde{S}$ are

$$\tilde{U} = D(D^t\tilde{K}D)^{-1}D^t(\tilde{R}-\tilde{K}\tilde{V})+\tilde{V} \tag{26}$$

$$\tilde{S} = \tilde{K}[D(D^t\tilde{K}D)^{-1}D^t(\tilde{R}-\tilde{K}\tilde{V})+\tilde{V}] \tag{27}$$

These results are represented in the global coordinate system.

3.3 Tree Method in the local Coordinate System

In the foregoing section, Tree Method in the global coordinate system is shown. Then the values which are calculated by the foregoing method are the ones in the global coordinate system and must be transformed to the values in the local coordinate system when the values are actually used in designs.

In this section, the formulas of Tree Method in the local coordinate system are derived by transformations of formulas in foregoing section to vanish this demerit.

Here two transformation matrices T_{BO} and T_{CO}, which transform the quantities of members from the global coordinate system to the local coordinate system, are defined and they are as follows

$T_{BO}=[T^t_{BO}]$; Transformation matrix for all members
$T_{CO}=[T^t_{CO}]$; Transformation matrix for tree members only

The quantities which are used in previous section are transformed as follows

$$\begin{aligned}
R &= T_{BO}\tilde{R} \\
r &= T_{BO}\tilde{r} \\
V &= (T_{BO}^{-1})^t\tilde{V} \\
v &= (T_{BO}^{-1})^t\tilde{v} \\
V_T &= (T_{CO}^{-1})^t\tilde{V}_T \\
v_T &= (T_{CO}^{-1})^t\tilde{v}_T \\
K &= T_{BO}\tilde{K}T^t_{BO} \\
F &= (T_{BO}^{-1})^t\tilde{F}(T_{BO})^{-1}
\end{aligned} \tag{28}$$

Substituting these results into eq. (23) yield to

$$D^t(T_{BO})^{-1}K(T_{BO}^{-1})^tDT^t_{CO}v_T = D^t(T_{BO})^{-1}[R-KV]$$

By multiplying a matrix T_{CO} to both sides of this equation, it yields to

$$T_{CO}D^t(T_{BO})^{-1}K(T_{BO}^{-1})^tDT^t_{CO}v_T = T_{CO}D^t(T_{BO})^{-1}[R-KV] \tag{29}$$

The term, $(T_{BO}^{-1})^tDT^t_{CO}$, in eq. (29) is written by the symbol $\boldsymbol{D}$.

Then the equation is written as follows

$$\boldsymbol{D}^t K \boldsymbol{D} v_T = \boldsymbol{D}^t[R-KV] \tag{30}$$

The solution is, of course,

$$v_T = (\boldsymbol{D}^t K \boldsymbol{D})^{-1} \boldsymbol{D}^t [R-KV] \tag{31}$$

Using the same procedure, the eq's (24), (25), (26) and eq. (27) are transformed and the results are as follows

$$v = \boldsymbol{D}(\boldsymbol{D}^t K \boldsymbol{D})^{-1} \boldsymbol{D}^t [R-KV] \tag{32}$$

$$r = [K - K\boldsymbol{D}(\boldsymbol{D}^t K \boldsymbol{D})^{-1} \boldsymbol{D} K](V-FR) \tag{33}$$

and

$$U = \boldsymbol{D}(\boldsymbol{D}^t K \boldsymbol{D})^{-1} \boldsymbol{D}^t [R-KV] + V \tag{34}$$

$$S = K[\boldsymbol{D}(\boldsymbol{D}^t K \boldsymbol{D})^{-1} \boldsymbol{D}^t (R-KV) + V] \tag{35}$$

The matrix $\boldsymbol{D}$, which is used in this section, is called "Modified Basic Cut-Set Matrix".

Since v_T, v, U, r, and S, which are calculated in this section, are the values in the local coordinate system, we can use these values at once when we design.

3.4 Application of the Housholder's Formula to Tree Method

If a structure is given as shown in Fig. 2-b, all members are divided into tree and link members, as shown in Fig. 2-c.

By Tree Method in a global coordinate system, the induced member distorsions $\tilde{v}_T$, are calculated by the equation (23),

$$\tilde{v}_T = (D^t \tilde{K} D)^{-1} D^t (\tilde{R} - \tilde{K}\tilde{V}) \tag{23}$$

The basic cut-set matrix D and the stiffness matrix $\tilde{K}$ can be divided into two parts, tree and link parts. Therefore it is possible to treat the tree stiffness matrix $D^t \tilde{K} D$ as follows,

$$\begin{aligned} D^t \tilde{K} D &= [\boldsymbol{I}_T \ D_L^t] \begin{bmatrix} \tilde{K}_T & 0 \\ 0 & \tilde{K}_L \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_T \\ D_L \end{bmatrix} \\ &= \tilde{K}_T + D_L^t \tilde{K}_L D_L \end{aligned} \tag{36}$$

where $\boldsymbol{I}_T$; a unit matrix

$\tilde{K}_T$; a stiffness matrix of tree members only

$\tilde{K}_L$; a stiffness matrix of link members only

By using eq. (36), the tree stiffness matrix $D^t \tilde{K} D$ of the given structure can be represented as the sum of the stiffness matrix of tree members only and that of link members only.

Application of the Housholder's formula to eq. (36) yields to

$$\begin{aligned}(D^t\tilde{K}D)^{-1} &= (\tilde{K}_T+D_L^t\tilde{K}_LD_L)^{-1} \\ &= \tilde{F}_T-\tilde{F}_TD_L^t(\tilde{F}_L+D_L\tilde{F}_TD_L^t)^{-1}D_L\tilde{F}_T \end{aligned} \tag{37}$$

where $\tilde{F}_T$; the flexibility matrix of tree members only

$\tilde{F}_L$; the flexibility matrix of link members only

If it is necessary to calculate $D^t\tilde{K}D$, it is evident from eq. (36) that we need not calculate $D^t\tilde{K}D$ itself but may calculate $D_L^t\tilde{K}_LD_L$ only, therefore the number of calculations proportionally decreases by the square of the number of tree members. And eq. (37) shows that the dimension of $(D^t\tilde{K}D)^{-1}$ decreases to that of link members. By using this equation the calculation-time can be shortened.

If a given structure has high redundancy, "the tearing and interconnecting method" must be used for the analysis of the structure when a computer with finite capacity is used.

Eq. (36) gives us an information for tearing and interconnecting of a structure. The first term of eq. (36) is the stiffness of tree members only and these systems (with tree members only) are statically determinate systems as shown in Fig. 2–d. To add the second term $(D_L^t\tilde{K}_LD_L)$ to the first term $\tilde{K}_T$ meams that these systems are connected by link members to each other.

According to this information, eq. (36) yields to

$$D^t\tilde{K}D = \tilde{K}_T+\sum_{n=1}^{n}(D_L^n)^t\tilde{K}_L^nD_L^n \tag{37'}$$

where $\tilde{K}_L^n$; a stiffness matrix of link members which connects two tree systems

D_L^n ; row matrix with respect to the link members of the basic cut-set matrix D

For the first step to calculate the tree flexibility matrix $(D^t\tilde{K}D)^{-1}$ of the given structure from eq. (37), the flexibility matrix $\tilde{F}^{(0)}$ of the gathering of the divided structure is defined as follows

$$\tilde{F}^{(0)} = \begin{bmatrix} \tilde{F}_T^1 & & 0 \\ & \tilde{F}_T^2 & \\ 0 & & \ddots \ \tilde{F}_T^n \end{bmatrix} = [\tilde{K}_T^{(0)}]^{-1} \tag{38}$$

Using the Housholder's formula, the inverse matrix $\tilde{F}^{(1)}$ of $[\tilde{K}_T+(D_L^1)^t\tilde{K}_L^1D_L^1]$ can be calculated as follows,

$$\begin{aligned}\tilde{F}^{(1)} &= [\tilde{K}_T^{(0)}+(D_L^1)^t\tilde{K}_L^1D_L^1]^{-1} = [\tilde{K}^{(1)}]^{-1} \\ &= \tilde{F}^{(0)}-\tilde{F}^{(0)}(D_L^1)^t[\tilde{F}_L^1+D_L^1\tilde{F}^{(0)}(D_L^1)^t]^{-1}D_L^1\tilde{F}^{(0)}\end{aligned} \tag{39}$$

where $\tilde{F}_L^1$ is the inverse matrix of $\tilde{K}_L^1$.

By successive application of this formula to eq. (37), the tree flexibility matrix is found as

$$\begin{aligned}(D^t\tilde{K}D)^{-1} &= \tilde{F}^{(n)} = [\tilde{K}^{(n)}]^{-1} = [\tilde{K}^{(n-1)}+(D_L^n)^t\tilde{K}_L^n D_L^n]^{-1} \\ &= \tilde{F}^{(n-1)}-\tilde{F}^{(n-1)}(D_L^n)^t[\tilde{F}_L^n+D_L^n\tilde{F}^{(n-1)}(D_L^n)^t]^{-1}D_L^n\tilde{F}^{(n-1)} \qquad (40)\end{aligned}$$

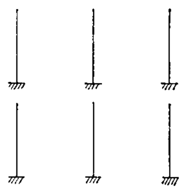

Fig. 2-d $[\tilde{K}^{(0)}]^{-1}=\tilde{F}^{(0)}$

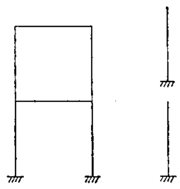

Fig. 2-e $[\tilde{K}^{(1)}]^{-1}=\tilde{F}^{(1)}$

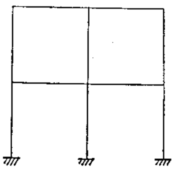

Fig. 2-f $[\tilde{K}^{(2)}]^{-1}=\tilde{F}^{(2)}=(D^t\tilde{K}D)^{-1}$

In Fig. 2–e and Fig. 2–f, the divided substurctures are connected one after another by link members. These procedures are equal to eq's (39) and (40), respectively.

From the property of the basic cut-set matrix, this method is useful for the analysis of these structures which have a few link members or a few stories.

Conclusion

In this paper, a new method for tearing and interconnecting of a structure is shown by using the information implied in the incidence matrix, which is gotten only from the figure of given framed structure. And Tree Method in the global coordinate system for the analysis of rigid framed structure is shown by using Tree Method in the electrical circuit problems and also by using these results Tree Method in the local coordinate system is obtained through the transformations from the global coordinate system to the local coordinate system. A tearing and interconnecting method in case of Tree Method is also presented by the successive application of the Housholder's formula.

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Appendix

(1) Graph Theory

To express Network Theory which is the basic concept of the network-topological analysis for structures, Graph Theory is used. And the following matrices are defined by this theory.

1. Branch-node incidence matrix; A

   The ij-th element of A is defined as follows

$$a_{ij} = (1, -1, 0)$$

   if the i-th branch is positively, negatively, and not incident on the j-th node respectively. By the proper rearrangement of rows of A, A may be written as follows

$$A = \begin{bmatrix} A_T \\ A_L \end{bmatrix}$$

   in which the subscript T indicates tree branches and the subscripts L link branches.

2. Node-to-datum path matrix; B_T

   This matrix is defined for tree branches only and the ij-th element of B_T is defined as follows

$$b_{ij} = (1, -1, 0)$$

   if the i-th branch is positively, negatively, and not included in the j-th node-to-datum path.

   The relation between A_T and B_T is given as follows

$$A_T^{-1} = B_T^t$$

3. Branch-mesh matrix; C

   The ij-th element of C is defined as follows

$$C_{ij} = (1, -1, 0)$$

   if the i-th branch is positively, negatively and not included in the j-th basic mesh. By the proper rearrangement of C, matrix C may be written as follows

$$C = \begin{bmatrix} C_T \\ C_L \end{bmatrix} = \begin{bmatrix} C_T \\ \boldsymbol{I}_L \end{bmatrix}$$

   in which $\boldsymbol{I}_L$ is a unit matrix. And we have following relations.

$$A^t C = C^t A = 0$$

$$C_T = -B_T A_L^t$$

4. Basic cut set matrix; D

The ij-th element of D is defined as follows

$$d_{ij} = (1, -1, 0)$$

if the i-th branch is positively, negatively, and not included in the j-th basic cut set.

If the graph is divided into the tree and the link parts and rows of D are rearranged properly, the matrix D may written as follows

$$D = \begin{bmatrix} D_T \\ D_L \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_T \\ D_L \end{bmatrix}$$

And we have following relations between A, B, C and D.

$$D_L = -C_T^t$$

$$D = \begin{bmatrix} \boldsymbol{I}_T \\ -C_T^t \end{bmatrix} = \begin{bmatrix} A_T B_T^t \\ A_L B_T^t \end{bmatrix} = A B_T^t$$

$$D^t C = C^t D = 0$$

(2) Modifications of A, B_T, C and D Matrices

A, B_T, C and D are modified by the transformation matrices, and they are defined as follows.

1. Modified branch-node incidence matrix; $\boldsymbol{A}$

The ij-th element a_{ij} of A is modified as follows

$$\boldsymbol{a}_{ij} = (T_i, -T_i^*, 0)$$

if $a_{ij}=(1, -1, 0)$ respectively,

in which

$$T_i = \begin{bmatrix} \Lambda_i & 0 \\ 0 & \Lambda_i \end{bmatrix}, \quad T_i^* = \begin{bmatrix} \Lambda_i & M_i \\ 0 & \Lambda_i \end{bmatrix}$$

$$M_i = \begin{bmatrix} 0 & 0 & 0 \\ l_i\lambda_\zeta^t & l_i\mu_\zeta^t & l_i\nu_\zeta^t \\ -l_i\lambda_\eta^t & -l_i\mu_\eta^t & -l_i\nu_\eta^t \end{bmatrix}$$

And the matrix $\boldsymbol{A}$ is divided into tree branches and link branches

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_T \\ \boldsymbol{A}_L \end{bmatrix}$$

2. Modified node-to-datum path matrix; $\boldsymbol{B}_T$

The ij-th element b_{ij} is modified as follows

$$\boldsymbol{b}_{ij} = (T_i, \bar{T}_i, 0)$$

if $b_{ij}=(1, -1, 0)$ respectively,
in which

$$\bar{T}_i = \begin{bmatrix} A_i & 0 \\ M_i & A_i \end{bmatrix}$$

And we have the following relation.

$$\boldsymbol{A}_T^{-1} = \boldsymbol{B}_T^t$$

3. Modified branch-mesh matrix; $\boldsymbol{C}$
Matrix C is defined as follows,

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_T \\ \boldsymbol{C}_L \end{bmatrix} = \begin{bmatrix} -\boldsymbol{B}_T\boldsymbol{A}_L^t \\ \boldsymbol{I}_L \end{bmatrix}$$

4. Modified basic cut set matrix; $\boldsymbol{D}$
The ij-th element d_{ij} is modified as follows

$$\boldsymbol{d}_{ij} = (\Omega, -\Omega, 0)$$

if $d_{ij}=(1, -1, 0)$ respectively,
in which

$$\Omega = \begin{bmatrix} A_iA_k^t & A_iXA_k^t \\ 0 & A_iA_k^t \end{bmatrix}$$

and X is the transformation matrix from the initial point of the k-th branch to that of the i-th branch and it is written as

$$X = \begin{bmatrix} 0 & z_i - z_k & -y_i + y_k \\ -z_i + z_k & 0 & x_i - x_k \\ y_i - y_k & -x_i + x_k & 0 \end{bmatrix}$$

where (x_i, y_i, z_i) is the values of the initial point of the i-th branch in the global coordinate system.
And we have following relation between $\boldsymbol{A}$, $\boldsymbol{B}_T$, and $\boldsymbol{D}$.

$$\boldsymbol{D} = \boldsymbol{A}\boldsymbol{B}_T^t$$

(3) Housholder's formula
The formula is given as

$$(Z+U^tKV)^{-1} = Z^{-1} - Z^{-1}U^t(K^{-1}+VZ^{-1}U^t)^{-1}VZ^{-1}$$

in which Z and K are nonsingular square matrices.