# A New Rounding Algorithm for Integer Linear Programming



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# A New Rounding Algorithm for Integer Linear Programming

By

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An algorithm is given for optimizing a linear function subject to integer linear constraints by a rounding method which is an extension of Gomory's.

The range of computation to obtain the optimal solution to the integer linear problem by this algorithm is less than that of Gomory's.

In particular, this algorithm is very effective in the case where the value of the product of the pivots is much larger than the number of nonbasic variables.

## 1. Introduction

In the area of the integer linear programming, many different approaches have been devised. These are developed with a view to modifying the linear programming problem. Typical approaches are the Cutting plane method and the Rounding method by R. E. Gomory, the Branch and Bound method by A. H. Land and A. G. Doig, the Partition method by J.F. Benders, and so on.

Some approaches for mixed integer linear programming are derived by E.M. L. Beale, R. E. Gomory, J. F. Benders, et al and some approaches for special case of the (0, 1) variable problem are dealt with by M.E.Balas, J. F. Benders, and R. Bellman.

This paper is an extension of Gomory's rounding method for solving the integer linear programming problem. This new approach is quite effective for the case where the value of the product of pivots is much larger than the number of nonbasic variables, and effectively the steps to obtain the solution of the problem are less than those of Gomorys' method.

Section 2 presents the general description of the rounding method. Section 3 presents the new rounding algorithm. Section 4 illustrates the new algorithm with an example and finally section 5 remarks on the results.

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## 2. General Description

Generally, the integer linear programming problem is to find integer valued $X$ with the following constraints,

(P1) maximize $DX$,
when $AX = A_0,\ X \geqq 0$,

where $A$, $A_0$, $D$ and $X$ are the following vectors and matrices,

$$A = \begin{bmatrix} a_{1l} & \cdots & a_{1j} & \cdots & a_{1n} 1000 & \cdots & 0 \\ \vdots & & & & & & \\ a_{il} & \cdots & a_{ij} & \cdots & a_{in} 0001 & \cdots & 0 \\ \vdots & & & & & & \\ a_{ml} & \cdots & a_{mj} & \cdots & a_{mn} 0000 & \cdots & 1 \end{bmatrix}$$

($a_{ij}$ ; integer)

$A_0^T = (a_{0l}, \cdots, a_{0m}), D = (d_1, \cdots, d_{m+n}), X^T = (x_1, \cdots, x_{m+n})$.

Let $A_1$ be a basis, i. e. a set of $m$ linearly independent columns of $A$ and divide the vectors into two parts as follows;

$$A = (A_1, A_2), D = (D_1, D_2), X^T = (X_1, X_2).$$

Then the problem of (P1) can be rewritten as;

(P2) maximize $D_1X_1 + D_2X_2$,
when $A_1X_1 + A_2X_2 = A_0,\ X_1, X_2 \geqq 0$.

Solving for $X_1$ in terms of $X_2$, (P2) becomes the same problem as,

(P3) maximize $D_1A_1^{-1}A_0 + (D_2 - D_1A_1^{-1}A_2)X_2$,
when $X_1 = A_1^{-1}A_0 - A_1^{-1}A_2X_2,\ X_1, X_2 \geqq 0$.

For the (P3) problem, if the following conditions are satisfied,

(4) $A_1^{-1}A_0 \geqq 0$,
$D_2 - D_1A_1^{-1}A_2 \leqq 0$,

the optimal solution to (P3) is $X_2 = 0$ and $X_1 = A_1^{-1}A_0$, because taking the independent variables $X_2$ to be any number larger than zero can only decrease the value of the objective function of (P3), as G. Danzig has pointed out in the simplex method.

To simlify the notation, let

$$A_1^{-1}A_0 = \alpha_0,\ A_1^{-1}A_2 = (\alpha_2, \cdots, \alpha_n),\ D_1A_1^{-1}A_0 = c_0,\ D_2 - D_1A_1^{-1}A_2 = (c_1, \cdots, c_n),$$

then (P3) is equivalent to finding $X_1$ and $[x_j]$ in the problem, (P5) maximize $(c_0+\sum_{j=1}^{n} c_j x_j), \quad c_j \leqq 0,$

when $X_1 = \alpha_0 - \sum_{j=1}^{n} \alpha_j x_j, \quad X_1, x_j \geqq 0.$

Let us consider the problem (P5). If all components of $\alpha_0$ are integers, then $X_2=0$, $X_1=\alpha_0$ are an optimal solution. Otherwise, the independent variables $X_2$ of (P5) must take on some non negative integer values ($x_2 \neq 0$) in such a way that $\alpha_0 - \sum_{j=1}^{n} \alpha_j x_j$ is a non negative integer.

As we mentioned before, the above two requirements must be met for solving the integer linear programming problem. For the moment, we consider only the requirement for integer constraints, and consider later the non negative requirements for the solution which is found on the previous stage. At first, we gave only the integer requirements when finding the solution of the (P5) problem. Then (P5) is equivalent to finding $x$ in the problem;

(P6) maximize $\sum_{j=1}^{n} c_j x_j,$

when $\sum_{j=1}^{n} \alpha_j x_j = \alpha_0 \pmod 1, \quad x_j \geqq 0.$

Clearly, any column $\alpha_j (j = 0, 1, \cdots, n)$ may be replaced by a column $\bar{\alpha}_j$ so long as $\alpha_j = \bar{\alpha}_j (\bmod 1)$; in particular $\bar{\alpha}_j$ can be taken to be the column of the fractional part of $\alpha_j$ and the problem is defined as follows;

(P7) maximize $\sum_{j=1}^{n} c_j x_j,$

when $\sum_{j=1}^{n} \bar{\alpha}_j x_j = \bar{\alpha}_0 \pmod 1, \quad x_j \geqq 0.$

If there exists an integer feasible solution of (P6), then there must exist a solution of (P7), since every $c_j \geqq 0$.

It is shown [4] by Gomory that the $\bar{\alpha}_j$ generate a group $H$ with respect to the modulo 1 operation.

Therefore, if the absolute value of determinant of $A_1$ is $d$, then the order of the group $H$ is $d$ and $\bar{\alpha}_j$ can be expressed as the form of $h_j/d$, where $h_j$ and

$d$ are integer and $h_j$ is less than or equal to $d-1$. Moreover, $H$ is the direct sum of the cyclic subgroups $H_j$ each of which is generated by $\bar{\alpha}_j x_j$.

It is clear that an optimal solution $x_j$ of (P7) has

$$x_j \leqq d-1,$$

where $j=1,2,\cdots,n$, since $d\alpha_j=0$ (mod 1).

Here, we apply these facts to obtain the solution of (P7) and if,

$$\alpha_0 - \sum_{j=1}^{n} \alpha_j x_j \geqq 0,$$

then this $\{x_j\}$ is the solution of (P1).

## 3. Revised Rounding Algorithm

In section 2, we have observed that $\bar{\alpha}_j x_j$ generates subgroup $H_j$ of $H$ which is the set of $\bar{\alpha}$ and that $H$ is the dircet sum of $H_j$. Therefore, to find the solution of the integer problem (P7) is eaquivlent to choosing one element from each subgroup so that the sum of $x_j$ corresponding to the above elements must satisfy the optimal condition of (P7)

In the previous section, we have given the range

$$0 \leqq x_j \leqq d-1.$$

However, we may restrict $x_j$ to a more narrow range as shown later. This enables us to improve on Gomory's algorithm. Here, we restrict the range of $x_j$ to find the feasible soltuion of the problem (P7); we may consider the following problem

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{when} \quad \sum_{j=1}^{n} \bar{\alpha}_j x_j = \bar{\alpha}_0, \quad 0 \leqq x_j \leqq m_j,$$

$$m_j \bar{\alpha}_j = \bar{\alpha}_0 \pmod 1, \quad j=1,2,\cdots,n.$$

The reason why we can include the above constraints is as follows. Assume $x_j$ to be greater than $m_j$ and the feasible solution satisfies the following realtion.

$$\sum_{j=1}^{n} \bar{\alpha}_j x_j = \bar{\alpha}_0 \pmod 1, \quad \text{maximize} \quad \sum_{j=1}^{n} c_j x_j = x_0'.$$

Then, the above equations may be rewritten as,

(8) $$x_0' = \sum_{j=1}^{n} c_j x_j = c_j x_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i x_i < c_j x_j < c_j m_j,$$

where $c_j \leqq 0, \quad x_j \geqq 0, \quad j = 1, 2, \cdots, n.$

This result contradicts the fact that $x_j$ is the feasible solution of (P7). Hence the assumption that $x_j$ is greater than $m_j$ is incorrect and consequently the range of $x_j$ is $x_j \leqq m_j$.

Moreover, the condition of $x_j \leqq d-1$ must be satisfied, since $\bar{\alpha}_j d = 1 \pmod 1$. Then, $m_j \leqq d-1$.

As the consequence of these facts, it is sufficient to choose one element from each column of Table 1 which corresponds to the subgroup $H_j$ of $H$ for solving the problem (P7).

Table 1

| $x$ \ $\bar{\alpha}$ | $\bar{\alpha}_1$ | $\bar{\alpha}_2$ | $\cdots$ | $\bar{\alpha}_j$ | $\cdots$ | $\bar{\alpha}_n$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $\cdots$ | 0 | $\cdots$ | 0 |
| 1 | $\bar{\alpha}_1$ | $\bar{\alpha}_2$ | $\cdots$ | $\bar{\alpha}_j$ | $\cdots$ | $\bar{\alpha}_n$ |
| 2 | $2\bar{\alpha}_1$ | $2\bar{\alpha}_2$ | $\cdots$ | $2\bar{\alpha}_j$ | $\cdots$ | $2\bar{\alpha}_n$ |
| $\vdots$ | | | | | | |
| $m$ | $m\bar{\alpha}_1$ | $m\bar{\alpha}_2$ | $\cdots$ | $m\bar{\alpha}_j$ | $\cdots$ | $m\bar{\alpha}_n$ |
| $\vdots$ | | | | | | |
| $r$ | $r\bar{\alpha}_1$ | $r\bar{\alpha}_2$ | $\cdots$ | $r\bar{\alpha}_j$ | $\cdots$ | $r\bar{\alpha}_n$ |

Note that, $0 \leqq m_j, r_j \leqq d-1, m_j \bar{\alpha}_j = \bar{\alpha}_0 \pmod 1$ and $d$ is the absolute value of the determinant of the matrix $A_1$.

In order to compute the solution of the problem (P7), we apply dynamic programming as follows; we define $\Psi_s(\bar{\alpha})$ which maximizes $\sum_{j=1}^{s} c_j x_j$ for $s$ stages and satisfies the condition of $\sum_{j=1}^{s} \bar{\alpha}_j x_j = \bar{\alpha} \pmod 1$, $\bar{\alpha} \in H$, i.e.,

(9) $$\Psi_s(\bar{\alpha}) = \max\left[\sum_{j=1}^{s} c_j x_j \middle| \sum_{j=1}^{s} \bar{\alpha}_j x_j = \bar{\alpha} \pmod 1, \bar{\alpha} \in H\right].$$

Since $x_j \geqq 1$, or $x_j = 0$, we obtain the following relation by dynamic programming;

(10) $$\Psi_s(\bar{\alpha}) = \max\left[\Psi_s(\bar{\alpha} - \bar{\alpha}_s) + c_s, \Psi_{s-1}(\alpha)\right].$$

The above equation is reduced to;

(11) $$\Psi_s(\bar{\alpha}_s) = \max\,[\Psi_s(0)+c_s,\ \Psi_{s-1}(\bar{\alpha}_s)]$$
$$\Psi_s(r\bar{\alpha}_s) = \max\,[\Psi_s(r\bar{\alpha}_s-\bar{\alpha}_s)+c_s,\ \Psi_{s-1}(r\bar{\alpha}_s)],$$

where $r=2, 3, \cdots, m_j$ or $d-1$.

Next we offer the following three hypotheses.

Hyp. 1 $\Psi_s(0)=0$

Hyp. 2 $\Psi_s(\bar{\alpha}) = -\infty$, for $\bar{\alpha} \in \left\{ \overset{s-1}{\underset{j=1}{U}} H_j \right\}^c$

Hyp. 3 $\Psi_{s-1}(\bar{\alpha})$ is known for all $\bar{\alpha}$, $\bar{\alpha} \in \left\{ \overset{s-1}{\underset{j=1}{U}} H_j \right\}^c$

where $\{\ \}^c$ in Hyp. 2 means the complement of the set of $\left\{ \overset{s-1}{\underset{j=1}{U}} H_j \right\}$ and $U$ denotes the logical sum of some sets.

The validity of the above hypotheses can be shown as follows.

Hyp. 1 Since $\bar{\alpha}=0$ in this case, $\alpha$ is integer value. Therefore the optimal solution is $x_j=0$, $j=1, 2, \cdots, n$, hence $\Psi_s(0)=0$.

Hyp. 2 In case $\bar{\alpha} \in \left\{ \overset{s-1}{\underset{j=1}{U}} H_j \right\}^c$, there does not exist $x_j$ which satisfies the constraints $\sum_{j=1}^{s-1} \bar{\alpha}_j x_j = \bar{\alpha}_0 \pmod 1$, i.e. there are no optimal solutions at the $s-1$ stage computation. Therefore, it is no contradiction to make the value of $\Psi_s(\bar{\alpha})$ equal to $-\infty$.

Hyp. 3 As we compute the value of $\Psi_{s-1}(\bar{\alpha})$ from $\Psi_1(\bar{\alpha})$ successively, it is possible to assume the hypotheses.

Under the above three hypotheses (in fact, these hypotheses are true) we may calculate the value of $\Psi_s(r\bar{\alpha}_s)$ by the equation (11).

In course of the computation of (11), we may determine the value of $\Psi_s(\bar{\alpha})$ in which $\bar{\alpha}$ is generated by $\bar{\alpha}_s r$.

However, there may be another element which is not generated by integer multiplication of $r\bar{\alpha}_s$ but is generated in the preceding stage computation. For this element, we must know the value $\Psi_s(\bar{\alpha})$. Because of this, we have the following algorithm for computing the value $\Psi_s(\bar{\alpha})$ in which $\bar{\alpha}$ is not generated by $r\bar{\alpha}_s$.

If there exists any $\bar{\alpha}$ which belongs to the set $K$,

$$K = \left\{ \overset{s-1}{\underset{j=1}{U}} H_j \right\} \cap \{r\bar{\alpha}_s\}^c$$

we must find $\bar{\alpha}'$ such that the following equation holds.

(12) $$\max_{\alpha\in K} \Psi_{s-1}(\bar{\alpha}) = \Psi_{s-1}(\bar{\alpha}').$$

We put

(13) $$\Psi_s(\bar{\alpha}') = \Psi_{s-1}(\bar{\alpha}').$$

Then, we may compute the following equation.

(14) $$\Psi_s(\bar{\alpha}+r'\bar{\alpha}_s) = \max[\Psi_s(\bar{\alpha}+r'\bar{\alpha}_s-\bar{\alpha}_s)+c_s, \Psi_{s-1}(\bar{\alpha}+r'\bar{\alpha}_s)],$$

where the range of $r'$ is

$$r' = 1, 2, \cdots, h'', \quad \text{where} \quad \bar{\alpha}+h''\bar{\alpha}_s = r\bar{\alpha}_s.$$

Note that only if the equation (13) holds, we can know the value $\Psi_s(\bar{\alpha}+r'\bar{\alpha}_s)$ by equation (14).

The reason why we have (13) can be easily shown as follows. Let $\Psi_s(\bar{\alpha}')$ be

$$\Psi_s(\bar{\alpha}') \neq \Psi_{s-1}(\alpha').$$

Then, from equation (10),

(15) $$\Psi_s(\alpha') = \Psi_s(\bar{\alpha}'-\bar{\alpha}_s)+c_s > \Psi_{s-1}(\bar{\alpha}').$$

It is noted that $\Psi_s(\bar{\alpha}'-\bar{\alpha}_s)$ may be expressed as the following relation from the equation (10).

(16) $$\Psi_s(\bar{\alpha}'-\bar{\alpha}_s) = \max[\Psi_s(\alpha'-2\bar{\alpha}_s)+c_s, \Psi_{s-1}(\bar{\alpha}'-\bar{\alpha}_s)]$$

If the element $(\bar{\alpha}'-\bar{\alpha}_s)$ does not belong to the set $K$, we have the following relation by Hyp. 2.

$$\Psi_{s-1}(\bar{\alpha}'-\bar{\alpha}_s) = -\infty$$

If the element $(\bar{\alpha}'-\bar{\alpha}_s)$ belongs to the set $K$, we have the following relation by the equation (12).

$$\Psi_{s-1}(\bar{\alpha}'-\bar{\alpha}_s) \leqq \Psi_{s-1}(\bar{\alpha}').$$

Therefore, by equation (15)

$$\Psi_s(\bar{\alpha}'-\bar{\alpha}_s) > c_s+\Psi_s(\bar{\alpha}'-\bar{\alpha}_s) > \Psi_{s-1}(\bar{\alpha}') \geqq \Psi_{s-1}(\bar{\alpha}'-\bar{\alpha}_s).$$

So, the value of $\Psi_s(\bar{\alpha}'-\bar{\alpha}_s)$ should be equal to

$$\Psi_s(\bar{\alpha}'-2\bar{\alpha}_s)+c_s.$$

Then we obtain the following equation.

$$\Psi_s(\bar{\alpha}') = \Psi_s(\bar{\alpha}'-2\bar{\alpha}_s)+2c_s.$$

Moreover,

$$\Psi_s(\bar{\alpha}'-2\bar{\alpha}_s) = \max[\Psi_s(\bar{\alpha}'-3\bar{\alpha}_s)+c_s, \Psi_{s-1}(\bar{\alpha}'-2\bar{\alpha}_s)].$$

When we use the same procedure for the above equation, we have the following relation.

$$(17) \qquad \Psi_s(\bar{\alpha}') = \Psi_s(\bar{\alpha}'-\bar{\alpha}_s)+c_s = \Psi_s(\bar{\alpha}'+d\bar{\alpha}_s)+c_s = \cdots\cdots = \Psi_s(\bar{\alpha}'-\bar{\alpha}_s)+dc_s.$$

This result contradicts the fact. If $c_s=0$, from the equation (11) and $\Psi_s(0)=0$, this implies that the eaqution (13) itself holds. Hence the validity of the equation (13) is proved.

We can obtain the value of $\Psi_s(\bar{\alpha})$ in the way just mentioned above. It is noted that, if there exists any element which is not computed in the $s$ stage but belongs to the set of the previous stage elements, we must repeat the same procedure. If the following relation is satisfied, we may stop the $s$ stage computation.

$$\overset{s-1}{\underset{j=1}{U}} H_j \geqq H_s$$

In this algorithm, we continue the above computation to the $n$ stage and finally find the following value.

$$(18) \qquad \max[\sum_{j=1}^{n} c_j x_j | \sum_{j=1}^{n} \bar{\alpha}_j x_j = \bar{\alpha}_0 (\text{mod } 1)].$$

If $x_j$ which is obtained form the equation (18) satisfies the following condition,

$$(19) \qquad \alpha_0 - \sum_{j=1}^{n} \alpha_j x_j \geqq 0.$$

then this $\{x_j\}$ is the optimal solution of (P1).

In order to retrieve $\{x_j\}$ from $\Psi_n(\bar{\alpha}_0)$ which satisfies the equation (18), we consider the index $i$ written as;

$$(20) \qquad i(s, \bar{\alpha}) \begin{cases} = i(s-1, \bar{\alpha}) & \text{if } \Psi_s(\bar{\alpha}) = \Psi_{s-1}(\bar{\alpha}) \\ = s & \text{otherwise,} \end{cases}$$

in other words, this index $i$ corresponds to $x_j$ which maximizes ${}_s(\bar{\alpha})$. Then if the pairs of $(\Psi_n(\bar{\alpha}), i(n, \bar{\alpha}))$ are given letting $x_{i(n,\bar{\alpha})}=1$, find $\Psi_n(\bar{\alpha}-\bar{\alpha}_{i(n,\bar{\alpha})})$ and increase $x_{i(n,\bar{\alpha})}$ by 1 and cnotinue this procedure until $\Psi_n(0)$ can be reached and stop the computation to retrieve the set $\{x_j\}$.

Revised Algorithm:

[1] Compute the element of the subgroup which is generated by $\bar{\alpha}_j x_j$ until

$x_j$ reaches to $m_j$ or $d'$, where $m_j\bar{\alpha}_j=\bar{\alpha}_0$ (mod 1) and $d'\alpha_j=0$ (mod 1). As a matter of course $d'$, $m_j \leqq d-1$ for all $\bar{\alpha}_j$, $j=1,2,\cdots,n$. Rearrange the column so as to $m_j \leqq m_{j+1}$ and then the subscript of $\bar{\alpha}_j$ is denoted according to this rearranged sequence.

[2] Let $\Psi_0(\bar{\alpha})=b$ where $b$ is very small value (e.g., $b=\min(dc_j)$, for all $\bar{\alpha}\in H$).

[3] Compute $\Psi_s(\bar{\alpha})$ of a stage by the equation (11). The range of computation is

$$r=1,2,\cdots,m_s \quad \text{or} \quad d', \quad \text{where} \quad m_s\bar{\alpha}_0=\bar{\alpha}_0\,(\text{mod } 1),$$
$$d'\bar{\alpha}_s=0\,(\text{mod } 1).$$

[4] If there exists some $\bar{\alpha}$ which satisfies

$$\bar{\alpha}\in\left\{\bigcup_{j=1}^{s-1}H_j\right\}\cap\{H_s'\}^c,$$

where $H_s'$ denotes the set of the elements which is generated for $s$ stage computation, jump to [5], otherwise to [6].

[5] Compute $\bar{\alpha}$ by the equation (14). The range of computation is

$$r=1,2,\cdots,h', \quad \text{where} \quad \bar{\alpha}+h'\bar{\alpha}_s=r\bar{\alpha}_s.$$

Jump to [4].

[6] If $s=n$, jump to [7], otherwise increase $s$ by 1 and jump to [3].

[7] Retrieval of $x$ which constitutes the optimal solution.

(i) Determine the index $i(s,\bar{\alpha})$ by the equation (20).

(ii) Put $x_{i(n,\bar{\alpha})}=1$

(iii) Find $\Psi_n(\bar{\alpha}-\bar{\alpha}_{i(n,\bar{\alpha})})$ and increase $x_{i(n,\bar{\alpha})}$ by 1.

(iv) Stop the computation when $\Psi_n(0)$ is derived.

[8] If there exists $\{x_i\}$ such that

$$\bar{\alpha}_0-\sum_{j=1}^{n}\bar{\alpha}_jx_j\geqq 0,$$

this $\{x_j\}$ is the optimal solution of (P1).

## 4. The Illustrative Example

Problem: Find $x_j$, $j=1,2,\cdots,5$, such that,

$$\begin{aligned} &\text{maximize} && x^0=-3x_1-4x_2,\\ &\text{when} && -5x_1-3x_2+x_3=-10\\ &&& -6x_1-2x_2+x_4=-8\\ &&& -x_1-3x_2+x_5=-6. \end{aligned}$$

Solution: Solve the problem as a linear programming problem applying the dual simplex method. Then we obtain the following table; where, the star denotes pivot.

Table 2

| | 1 | $-x_1$ | $-x_2$ |
|---|---|---|---|
| $x_3$ | $-10$ | $-5^*$ | $-3$ |
| $x_4$ | $-8$ | $-6$ | $-2$ |
| $x_5$ | $-6$ | $-1$ | $-3$ |
| $x_1$ | 0 | $-1$ | 0 |
| $x_2$ | 0 | 0 | $-1$ |
| $x^0$ | 0 | 3 | 4 |

Table 3

| | 0 | $-x_3$ | $-x_2$ |
|---|---|---|---|
| $x_3$ | 4 | $-1$ | 0 |
| $x_4$ | $-4$ | $-6/5$ | $8/5$ |
| $x_5$ | 2 | $-1/5$ | $-12/5^*$ |
| $x_1$ | 0 | $-1/5$ | $3/5$ |
| $x_2$ | $-6$ | 0 | $-1$ |
| $x^0$ | | $3/5$ | $11/5$ |

Table 4

| | 1 | $-x_3$ | $-x_5$ |
|---|---|---|---|
| $x_3$ | 0 | $-1$ | 0 |
| $x_4$ | $16/12$ | $-8/6$ | $8/12$ |
| $x_5$ | 0 | 0 | $-1$ |
| $x_1$ | 1 | $-1/4$ | $3/12$ |
| $x_2$ | $20/12$ | $1/12$ | $-5/12$ |
| $x^0$ | $-116/12$ | $5/12$ | $11/12$ |

From the result of pivot operation, we obtain Table 4 and find the following optimal solution by linear programming;

$$x^0 = -116/12 - 5/12x_3 - 11/12x_5$$
$$x_1 = 1 + 1/4x_3 - 3/12x_5$$
$$x_2 = 20/12 - 1/12x_3 + 5/12x_5$$
$$x_4 = 16/12 + 8/6x_3 - 8/12x_5.$$

The solution of the above problem by linear programming is

$$x_1 = 1, x_2 = 20/12, \quad x^0 = -116/12.$$

However, this solution is not an all integer solution. Thus we write this problem in the form of (P7). From Table 4 we obtain the following equation,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 20/12 \\ 16/12 \end{bmatrix} - \begin{bmatrix} -1/4 \\ 1/12 \\ -8/6 \end{bmatrix} x_3 - \begin{bmatrix} 3/12 \\ -5/12 \\ 8/12 \end{bmatrix} x_5$$

Therefore, the problem in the form of (P7) is to find $x_j$, $j = 3, 5$, such that, maximize $x^{0\prime} = -5/12x_3 - 11/12x_5$

when

$$\begin{bmatrix} 9/12 \\ 1/12 \\ 8/12 \end{bmatrix} x_3 + \begin{bmatrix} 3/12 \\ 7/12 \\ 8/12 \end{bmatrix} x_5 = \begin{bmatrix} 0 \\ 8/12 \\ 4/12 \end{bmatrix} (\text{mod } 1).$$

According to the notation of (P7), we have

$$\bar{\alpha}_1 = \begin{bmatrix} 9/12 \\ 1/12 \\ 8/12 \end{bmatrix}, \quad \bar{\alpha}_2 = \begin{bmatrix} 3/12 \\ 7/12 \\ 8/12 \end{bmatrix}, \quad \bar{\alpha}_0 = \begin{bmatrix} 0 \\ 8/12 \\ 4/12 \end{bmatrix},$$

$$c_1 = -5/12, \quad c_2 = -11/12.$$

$$x_1' = x_3, \qquad x_2' = x_5.$$

Next, we compute the value of $(\bar{\alpha})$.

The element of subgroup $H_1$ and $H_2$

| | |
|---|---|
| $\bar{\alpha}_1^T = (9/12, 1/12, 8/12)$ | $\bar{\alpha}_2^T = (3/12, 7/12, 8/12)$ |
| $2\bar{\alpha}_1^T = (6/12, 2/12, 4/12)$ | $2\bar{\alpha}_2^T = (6/12, 2/12, 4/12)$ |
| $3\bar{\alpha}_1^T = (3/12, 3/12, 0)$ | $3\bar{\alpha}_2^T = (9/12, 9/12, 0)$ |
| $4\bar{\alpha}_1^T = (0, 4/12, 8/12)$ | $4\bar{\alpha}_2^T = (0, 4/12, 8/12)$ |
| $5\bar{\alpha}_1^T = (9/12, 5/12, 4/12)$ | $5\bar{\alpha}_2^T = (3/12, 11/12, 4/12)$ |
| $6\bar{\alpha}_1^T = (6/12, 6/12, 0)$ | $6\bar{\alpha}_2^T = (6/12, 6/12, 0)$ |
| $7\bar{\alpha}_1^T = (3/12, 7/12, 8/12)$ | $7\bar{\alpha}_2^T = (9/12, 1/12, 8/12)$ |
| $8\bar{\alpha}_1^T = (0, 8/12, 4/12)$ | $8\bar{\alpha}_2^T = (0, 8/12, 4/12)$ |
| $9\bar{\alpha}_1^T = (9/12, 9/12, 0)$ | $9\bar{\alpha}_2^T = (2/12, 3/12, 0)$ |
| $10\bar{\alpha}_1^T = (6/12, 10/12, 8/12)$ | $10\bar{\alpha}_2^T = (6/10, 10/12, 8/12)$ |
| $11\bar{\alpha}_1^T = (3/12, 11/12, 2/12)$ | $11\bar{\alpha}_2^T = (9/12, 5/12, 2/12)$ |
| $12\bar{\alpha}_1^T = (0, 0, 0)$ | $12\bar{\alpha}_2^T = (0, 0, 0)$ |

Therefore we obtain the value of $m_j$, $j=1, 2$, as follows.

$$8\bar{\alpha}_1 = \bar{\alpha}_0, \quad 8\bar{\alpha}_2 = \bar{\alpha}_0,$$

so, $\quad m_1 = 8, \quad m_2 = 8.$

The calculation of $(\bar{\alpha})$

First stage computation.

$$\Psi_1(\bar{\alpha}_1) = \max[\Psi_1(0)+c_1, \Psi(\bar{\alpha}_1)] = c_1$$
$$\Psi_1(2\bar{\alpha}_1) = \max[\Psi(\bar{\alpha}_1)+c_1, \Psi(2\bar{\alpha}_1)] = 2c_1$$
$$\Psi_1(3\bar{\alpha}_1) = \max[\Psi(2\bar{\alpha}_1)+c_1, \Psi(3\bar{\alpha}_1)] = 3c_1$$
$$\Psi_1(4\bar{\alpha}_1) = \max[\Psi(3\bar{\alpha}_1)+c_1, \Psi(4\bar{\alpha}_1)] = 4c_1$$
$$\Psi_1(5\bar{\alpha}_1) = \max[\Psi(4\bar{\alpha}_1)+c_1, \Psi(5\bar{\alpha}_1)] = 5c_1$$

$\Psi_1(6\bar{\alpha}_1) = \max[\Psi(5\bar{\alpha}_1)+c_1, \Psi(6\bar{\alpha}_1)] = 6c_1$

$\Psi_1(7\bar{\alpha}_1) = \max[\Psi(6\bar{\alpha}_1)+c_1, \Psi(7\bar{\alpha}_1)] = 7c_1$

$\Psi_1(8\bar{\alpha}_1) = \max[\Psi(7\bar{\alpha}_1)+c_1, \Psi(8\bar{\alpha}_1)] = 8c_1$

Second stage computation.

$\Psi_2(\ \bar{\alpha}_2) = \max[\Psi_2(\ 0\ )+c_2, \Psi_1(\ \bar{\alpha}_2)] = c_2 = -11/12$

$\Psi_2(2\bar{\alpha}_2) = \max[\Psi_2(\ \bar{\alpha}_2)+c_2, \Psi_1(2\bar{\alpha}_2)] = 2c_1 = -10/12$

$\Psi_2(3\bar{\alpha}_2) = \max[\Psi_2(2\bar{\alpha}_2)+c_2, \Psi_1(3\bar{\alpha}_2)] = 2c_1+c_2 = -21/12$

$\Psi_2(4\bar{\alpha}_2) = \max[\Psi_2(3\bar{\alpha}_2)+c_2, \Psi_1(4\bar{\alpha}_2)] = 4c_1 = -20/12$

$\Psi_2(5\bar{\alpha}_2) = \max[\Psi_2(4\bar{\alpha}_2)+c_2, \Psi_1(5\bar{\alpha}_2)] = 4c_1+c_2 = -31/12$

$\Psi_2(6\bar{\alpha}_2) = \max[\Psi_2(5\bar{\alpha}_2)+c_2, \Psi_1(6\alpha_2)] = 6c_1 = -30/12$

$\Psi_2(7\bar{\alpha}_2) = \max[\Psi_2(6\bar{\alpha}_2)+c_2, \Psi_1(7\bar{\alpha}_2)] = c_1 = -5/12$

$\Psi_2(8\bar{\alpha}_2) = \max[\Psi_2(7\bar{\alpha}_2)+c_2, \Psi_1(8\bar{\alpha}_2)] = c_1+c_2 = -16/12$

Then, we have

$\Psi_n(\bar{\alpha}_0) = -16/12.$

We retrieve $\{x_j\}$ of $\Psi_n(\bar{\alpha}_0)$ as follows;
From Table 5,

$\Psi_2(\bar{\alpha}_0) = -16/12$

$\Psi_2(8\bar{\alpha}_2-\bar{\alpha}_2) = \Psi_2(7\bar{\alpha}_2).$

Then, $x_2' = 1.$

Now, the value of $i(2, 7\alpha\ )$ is 1.
Then we compute the following value;

$\Psi_2(7\bar{\alpha}_2-\bar{\alpha}_1) = \Psi_2(0).$

Table 5

| | $\Psi_1(\ )$ | $x_1'$ | $i(1,\ )$ | $\Psi_2(\ )$ | $x_1'+x_2'$ | $i(2,\ )$ |
|---|---|---|---|---|---|---|
| $0=0$ | 0 | 0 | | 0 | 0 | |
| $\bar{\alpha}_1=7\bar{\alpha}_2$ | −5/12 | 1 | 1 | −5/12 | 1 | 1 |
| $2\bar{\alpha}_1=2\bar{\alpha}_2$ | −10/12 | 2 | 1 | −10/12 | 2 | 1 |
| $3\bar{\alpha}_1=$ | −15/12 | 3 | 1 | — | | |
| $4\bar{\alpha}_1=4\bar{\alpha}_2$ | −20/12 | 4 | 1 | −20/12 | 4 | 1 |
| $5\bar{\alpha}_1=$ | −25/12 | 5 | 1 | | | |
| $6\bar{\alpha}_1=6\bar{\alpha}_2$ | −30/12 | 6 | 1 | −30/12 | 6 | 1 |
| $7\bar{\alpha}_1=\ \bar{\alpha}_2$ | −35/12 | 7 | 1 | −11/12 | 1 | 2 |
| $8\bar{\alpha}_1=8\bar{\alpha}_2$ | −40/12 | 8 | 1 | −16/12 | 2 | 2 |
| $=3\bar{\alpha}_2$ | | | | −21/12 | 3 | 2 |
| $=5\bar{\alpha}_2$ | | | | −31/12 | 5 | 2 |

So, $x_1' = 1$.

Therefore the computation of retrieving $\{x_1\}$ stops.
Now we have found the value of the element of the set $\{x_1'\}$ which constructs $\Psi_2(\bar{\alpha}_0)$ as follows;

$$x_3 = 1, \quad x_5 = 1.$$

Finally we check if $X_1$ is nonnegative or not.

$$X_1 = \alpha_0 - \alpha_1 x_3 - \alpha_2 x_5$$
$$= \begin{bmatrix} 1 \\ 20/12 \\ 16/12 \end{bmatrix} - \begin{bmatrix} -1/4 \\ 1/12 \\ -8/12 \end{bmatrix} - \begin{bmatrix} 3/12 \\ -5/12 \\ 8/12 \end{bmatrix} \geqq 0$$

Therefore, we obtain the following solution of the integer linear programming problem (p1);

$$x_1 = 1,\ x_2 = 2,\ x_3 = 1,\ x_4 = 2,\ x_5 = 1,\ x^0 = -11.$$

## 5. Remarks

This paper has described the algorithm for solving the integer linear programming problem, which is based on Gomory's rounding algorithm.

It is noted that Gomory's method requires $nd$ times computation. On the other hand, this new algorithm requires only $\Sigma \| H_j \|$ times computation, where, $\| H_j \|$ denotes the cardinal number of the set $H_j$, $n$ is the number of nonbasic variables of the problem and $d$ is the determinant of the coefficient matrix of the basic variables.

As mentioned before, $\| H_j \|$ is equal to or less than $d$ and

$$\| H_j \| \leqq \| H_{j+1} \|,\ j = 1, 2, \cdots, n.$$

Therefore, we get the following relation

$$nd \geqq \sum_{j=1}^{n} \| H_j \|.$$

Consequently, from the above relation, this new algorithm is superior to Gomory's for solving the integer programming problem.

In particular, if the problem has the property that $d \gg n$, then, $\sum_{j=1}^{n} \| H_j \|$ is considerbly less than $n\ d$. Therefore, if the problem has the above characteristic, the new algorithm is more efficient for solving the problem than Gomory's.

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