# A Network-Topological Study on Statical Analysis of Rigid Framed Structures



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For the numerical analysis of complex structures with high redundancy by use of the digital computer, there arise a number of difficulties to solve the problems on an account of the fact that the increase of the rank of matrice involved requires much storage and time to obtain numerical results. A network-topological formulation of structural analysis leads effective algorithm to solve these problems. The alternative derivation of Spiller's formula and a memory-saving algorithm is considered in this paper.

## 1. Introduction

Recently, Branin[1] and Spiller[2] have developed independently a network-topological formulation of structures which are eminently suited for programming on a digital computer. These two formulations differ from each other with respect to ordinate system, with reference to which the forces and displacement of the structure are specified. In the formulation by Spiller, a local coordinate system is used, while Branin uses a global coordinate system in his formulation. The former formulation provides considerably greater numerical accuracy in computation, while the latter one is conceptually simple, but requires much time in transforming variables into a local coordinate system.

In this paper, Spiller's formulas are derived from Branin's formulas through matrix transformation into a local coordinte system in order to make clear the relationships between these two formulations. A memory-saving and effective algorithm for computer analysis of rigid framed structures is derived by applying Graph theory and Householder's formula[3].

## 2. Coordinate System and Transformation Matrix

Let a global coordinate system fixed in space and denoted by right-handed Cartesian axes, $x, y, z$ arbitrarily oriented. (Fig. 1) For each branch $i$, there is also a local coordinate system, fixed in the member represented by a right-handed Cartesian axes $\xi_i$, $\eta_i$, $\zeta_i$ chosen such that $\xi_i$ has the same orientation on the directed line connecting the initial and final nodes of branch $i$. (Fig. 1)

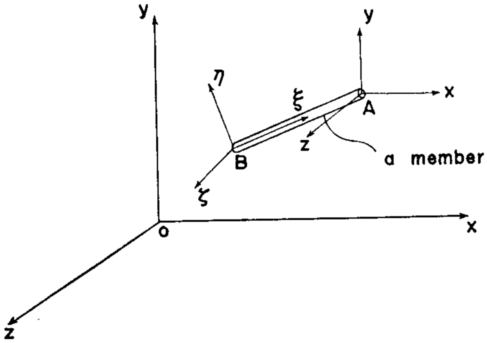


Fig. 1. Coordinate System.

(1) $O$-$xyz$: A global coordinate system with origin at $O$ which is arbitrarily chosen in space

(2) $A$-$xyz$: A global coordinate system with origin at $A$ which includes all nodes before deformation

(3) $B$-$\xi\eta\zeta$: A local coordinate system with origin at $B$ which is an initial point of all branches

(4) $C$-$\xi\eta\zeta$: A local coordinate system with origin at $C$ which is an initial point of all tree branches

An arbitrary force $\boldsymbol{P}$ is specified as a six-dimensional vector ($P_\xi$, $P_\eta$, $P_\zeta$, $m_\xi$, $m_\eta$, $m_\zeta$) in the local coordinate system. Similarly, an arbirary displacement $\boldsymbol{u}$ is specified as a six-dimensional vector ($u_\xi$, $u_\eta$, $u_\zeta$, $\theta_\xi$, $\theta_\eta$, $\theta_\zeta$) in the local coordinate system.

$\boldsymbol{P}_A$ and $\boldsymbol{u}_A$ denote the vector force and displacement at $A$ represented in a local coordinate system.

$\tilde{\boldsymbol{P}}_A$ and $\tilde{\boldsymbol{u}}_A$ denote the force vector and the displacement vector at $A$ represented in a global coordinte system, respectively.

Then these are related to each other as follows[1)]

$$\tilde{\boldsymbol{P}}_A = \boldsymbol{T}_{A0}\boldsymbol{P}_A \tag{1}$$

$$\tilde{\boldsymbol{u}}_A = [\boldsymbol{T}_{A0}^{-1}]^t\boldsymbol{u}_A \tag{2}$$

where $\boldsymbol{T}_{A0}$ is a $6\times6$ transformation matrix describing the position and orientation of the local coordinate axes at $A$ in terms of the global coordinate axes at $O$.

$\boldsymbol{T}_{A0}$ is given as follows

$$\boldsymbol{T}_{A0}=\begin{bmatrix}\boldsymbol{A} & 0\\ \boldsymbol{AX} & \boldsymbol{A}\end{bmatrix} \tag{3}$$

in which

$$\boldsymbol{A}=\begin{bmatrix}\lambda_\xi & \mu_\xi & \nu_\xi\\ \lambda_\eta & \mu_\eta & \nu_\eta\\ \lambda_\zeta & \mu_\zeta & \nu_\zeta\end{bmatrix} \tag{4}$$

and

$$\boldsymbol{X}=\begin{bmatrix}0 & z_A & -y_A\\ -z_A & 0 & x_A\\ y_A & -x_A & 0\end{bmatrix} \tag{5}$$

$(\lambda_\xi, \mu_\xi, \nu_\xi)$ : the direction cosines of $\xi$ axis relative to $x, y, z$ axes
$(x_A, y_A, z_A)$: the global coordinates of the point $A$

## 3. Alternative Derivation of Spiller's Formula

In Spiller's formula, some quantities are expressed in one coordinate system and the rest of them are expressed in another coordinate system as given in the following table. To make clear the relation between the Spiller's method and the Branin's method, the former is derived from the latter in this paragraph.

Table

| | Quantities | Coordinate System |
|---|---|---|
| Node | $\boldsymbol{P}$, $\boldsymbol{u}$ | A-xyz |
| Branch | $\boldsymbol{R}$, $\boldsymbol{r}$, $\boldsymbol{V}$, $\boldsymbol{v}$, $\boldsymbol{F}$, $\boldsymbol{K}$ | B-$\xi\eta\zeta$ |
| Mesh | $\boldsymbol{p}$, $\boldsymbol{U}$ | C-$\xi\eta\zeta$ |

(1) Transformation matrice

$\boldsymbol{T}_{A0}$, $\boldsymbol{T}_{B0}$ and $\boldsymbol{T}_{C0}$ are the transformation matrice describing the position and orientation of $A$-$xyz$ coordinate system, $B$-$\xi\eta\zeta$ coordinate system and $C$-$\xi\eta\zeta$ coordinate system in terms of $O$-$xyz$ coordinate system.

$$\boldsymbol{T}_{A0}=[\boldsymbol{T}^i_{A0}], \qquad \boldsymbol{T}_{B0}=[\boldsymbol{T}^i_{B0}], \qquad \boldsymbol{T}_{C0}=[\boldsymbol{T}^i_{C0}]$$

$$\boldsymbol{T}^i_{A0}=\begin{bmatrix}\boldsymbol{I} & 0\\ \boldsymbol{X}_i & \boldsymbol{I}\end{bmatrix}, \qquad \boldsymbol{T}^i_{B0}=\begin{bmatrix}\boldsymbol{A}_i & 0\\ \boldsymbol{A}_i\boldsymbol{X}_i & \boldsymbol{A}_i\end{bmatrix}$$

$$\boldsymbol{T}_{B0}=\begin{bmatrix}(\boldsymbol{T}_{B0})_T\\ (\boldsymbol{T}_{B0})_L\end{bmatrix}, \qquad \boldsymbol{T}_{C0}=[(\boldsymbol{T}_{B0})_L]$$

The subscript $T$ indicates tree members and the subscript $L$ link members.

(2) Displacement method

By continuity, the induced member distortions in the global coordinate system are given as[1)]

$$\tilde{D} = \tilde{V} + \tilde{A}\tilde{u} \tag{6}$$

The same quantities may be expressed in terms of the local coordinate system; namely from eq's (2) and (6) we have

$$T^t_{B0} D = T^t_{B0} V + \tilde{A} T^t_{A0} u$$

which yields to

$$D = V + Au \tag{7}$$

where $A$ is called a the modified incidence matrix given as

$$A = [T^{-1}_{B0}]^t \tilde{A} T^t_{A0} \tag{8}$$

Using the stiffness form of the load-deflection relationship, the induced member forces are written as

$$S = KD = KV + KAu \tag{9}$$

By the equilibrium at the joints, the joint forces are given as follows

$$P = A^t S = A^t K V + A^t K A u \tag{10}$$

From eq. (10) the joint displacements are found to be

$$u = (A^t K A)^{-1}(P - A^t K V) \tag{11}$$

and back-substituting into eq's (7) and (9) yields the induced member displacements and the induced member forces as

$$D = A(A^t K A)^{-1} P + [I - A(A^t K A)^{-1} A^t K] V \tag{12}$$

$$S = KA(A^t K A)^{-1} P + [K - KA(A^t K A)^{-1} A^t K] V \tag{13}$$

(3) Force Method

Using the equilibrium law, the induced member forces expressed in the global coordinate system are found to be

$$\tilde{S} = \tilde{R} + \tilde{C}\tilde{P} \tag{14}$$

Expressing eq. (14) in $B$-$\xi\eta\zeta$ coordinate system by means of transformations it yields to

$$\begin{aligned} S &= R + T_{B0} \tilde{C} T^{-1}_{C0} p \\ &= R + Cp \end{aligned} \tag{15}$$

in which $\boldsymbol{C}$ is given as follows

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{T}_{B0}\tilde{\boldsymbol{C}}\boldsymbol{T}_{C0}^{-1} = \boldsymbol{T}_{B0}\begin{bmatrix}\tilde{\boldsymbol{C}}_T \\ \tilde{\boldsymbol{C}}_L\end{bmatrix}\boldsymbol{T}_{C0}^{-1} \\ &= \boldsymbol{T}_{B0}\begin{bmatrix}-\boldsymbol{B}_T\boldsymbol{A}_L^t \\ \boldsymbol{I}\end{bmatrix}[(\boldsymbol{T}_{B0})_L]^{-1} \\ &= \begin{bmatrix}-\tilde{\boldsymbol{B}}_T\tilde{\boldsymbol{A}}_L^t \\ \boldsymbol{I}\end{bmatrix} \end{aligned} \tag{16}$$

where $\boldsymbol{B}_T$ is modified node-to-datum path matrix and is givn as follows

$$\begin{aligned} \boldsymbol{B}_T &= (\boldsymbol{T}_{B0})_T\tilde{\boldsymbol{B}}_T(\tilde{\boldsymbol{T}}_{A0})^{-1} \\ &= (\boldsymbol{A}_T^{-1}) \end{aligned} \tag{17}$$

Using the flexibility form of the load-deflection relationship, the induced member distortions may be written as

$$\boldsymbol{D} = \boldsymbol{F}\boldsymbol{S} = \boldsymbol{F}(\boldsymbol{R}+\boldsymbol{C}\boldsymbol{p}) \tag{18}$$

By continuity the induced member distorsions must be equated to the applied mesh distortions so that

$$\boldsymbol{U} = \boldsymbol{C}^t\boldsymbol{D} = \boldsymbol{C}^t\boldsymbol{F}\boldsymbol{R}+\boldsymbol{C}^t\boldsymbol{F}\boldsymbol{C}\boldsymbol{p} \tag{19}$$

Solving for the mesh forces gives

$$\boldsymbol{p} = (\boldsymbol{C}^t\boldsymbol{F}\boldsymbol{C})^{-1}(\boldsymbol{U}-\boldsymbol{C}^t\boldsymbol{F}\boldsymbol{R}) \tag{20}$$

and back-substituting into eq's (15), (18) yields the induced member forces and distortions

$$\boldsymbol{S} = \boldsymbol{C}(\boldsymbol{C}^t\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{C}^t\boldsymbol{V}+[\boldsymbol{I}-\boldsymbol{C}(\boldsymbol{C}^t\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{C}^t\boldsymbol{F}]]\boldsymbol{R} \tag{21}$$

$$\boldsymbol{D} = \boldsymbol{F}\boldsymbol{C}(\boldsymbol{C}^t\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{C}^t\boldsymbol{V}+[\boldsymbol{F}-\boldsymbol{F}\boldsymbol{C}(\boldsymbol{C}^t\boldsymbol{F}\boldsymbol{C})^{-1}\boldsymbol{C}^t\boldsymbol{F}]\boldsymbol{R} \tag{22}$$

## 4. Tearing and Interconnecting of Structures

If a complex structure is treated as one system, the rank of matrice concerned increases. Then, many difficulties are given arise to in matrix computation by means of a digital computer. In this section, the structure is divided into two substructures which are analysed as independent ones under the same load condition as in the given structure and the solution for the given structure is obtained by interconnecting the results which are found in the independent substructures. The given structure is shown in Fig. 2 and divided into two systems as in Fig. 3.

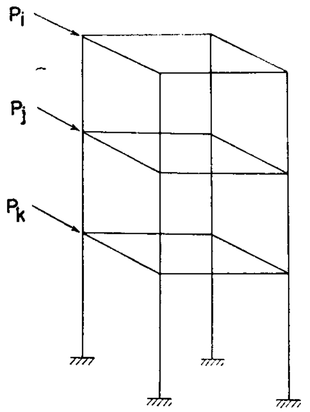


Fig. 2. Given space framed structure.

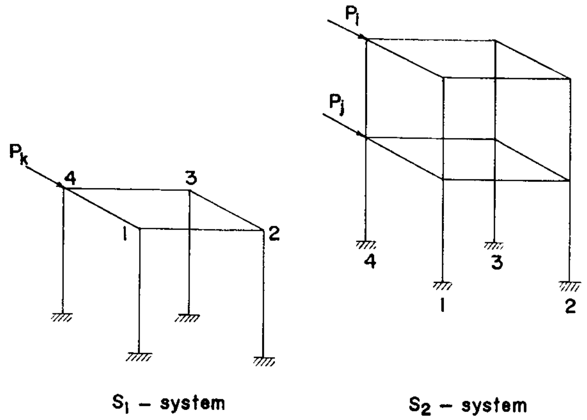


Fig. 3. Division of structure into two substructures.

Modified incidence matrix is written as

$$A = \begin{bmatrix} A_1 & 0 \\ E & A_2 \end{bmatrix} \tag{23}$$

Subscript indicates the number of substructure. Now $S_2$ system is divided into two systems, $S_{21}$ and $S_{22}$. (Fig.4) Then, the following modified incidence matrice are obtained

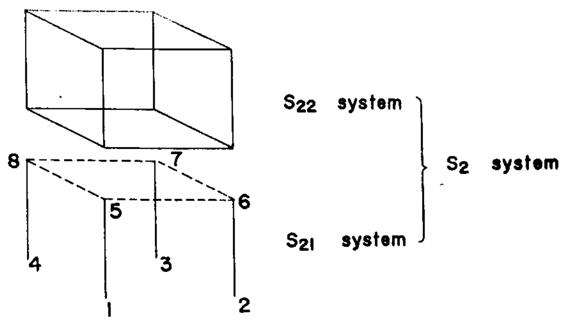


Fig. 4. Division of $S_2$ system.

$$E = \begin{bmatrix} 0 & e_1 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} e_2 & 0 \\ a_1 & a_2 \end{bmatrix} \tag{24}$$

in which $e_1$, $e_2$ are modified incidence matrice indicating how $S_{21}$ system is connected to $S_1$ system and $S_{22}$ system repsectively.

The primitive stiffness matrix and the primitive flexibility matrix of the structure are written as

$$\boldsymbol{K}=\begin{bmatrix}\boldsymbol{K}_1 & 0\\ 0 & \boldsymbol{K}_2\end{bmatrix} \qquad \boldsymbol{K}_2=\begin{bmatrix}\boldsymbol{k}_{21} & 0\\ 0 & \boldsymbol{k}_{22}\end{bmatrix} \tag{25}$$

$$\boldsymbol{F}=\begin{bmatrix}\boldsymbol{F}_1 & 0\\ 0 & \boldsymbol{F}_2\end{bmatrix} \qquad \boldsymbol{F}_2=\begin{bmatrix}\boldsymbol{f}_{21} & 0\\ 0 & \boldsymbol{f}_{22}\end{bmatrix} \tag{26}$$

in which subscripts indicate the system number.

Joint stiffness matrice of $S_1$ and $S_2$ system are given as follows

$$(\boldsymbol{A}_1^t\boldsymbol{K}_1\boldsymbol{A}_1)^{-1}=\begin{bmatrix}\boldsymbol{Z}_d & \boldsymbol{Z}_c\\ \boldsymbol{Z}_b & \boldsymbol{Z}_a\end{bmatrix} \tag{27}$$

$$(\boldsymbol{A}_2^t\boldsymbol{K}_2\boldsymbol{A}_2)^{-1}=\begin{bmatrix}\boldsymbol{Z}_h & \boldsymbol{Z}_g\\ \boldsymbol{Z}_f & \boldsymbol{Z}_c\end{bmatrix} \tag{28}$$

in which $\boldsymbol{Z}_a\sim\boldsymbol{Z}_h$ are the submatrix of $(\boldsymbol{A}_1^t\boldsymbol{K}_1\boldsymbol{A}_1)^{-1}$ and that of $(\boldsymbol{A}_2^t\boldsymbol{K}_2\boldsymbol{A}_2)^{-1}$ which are consisting of points where $S_{21}$ system is connected to $S_1$ and $S_{22}$ system respectively.

Using eq's (23), (25), the joint stiffness matrix of the given structure is found to be

$$\begin{aligned}\boldsymbol{A}^t\boldsymbol{K}\boldsymbol{A} &= \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{E}^t\\ 0 & \boldsymbol{A}_2^t\end{bmatrix}\begin{bmatrix}\boldsymbol{K}_1 & 0\\ 0 & \boldsymbol{K}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{A}_1 & 0\\ \boldsymbol{E} & \boldsymbol{A}_2\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{A}_1^t\boldsymbol{K}_1\boldsymbol{A}_1 & 0\\ 0 & \boldsymbol{A}_2^t\boldsymbol{K}_2\boldsymbol{A}_2\end{bmatrix}+\begin{bmatrix}0\\ \boldsymbol{A}_2^t\end{bmatrix}[\boldsymbol{K}_2]\,[\boldsymbol{E},\,0]\\ &\quad+\begin{bmatrix}\boldsymbol{E}^t\\ 0\end{bmatrix}[\boldsymbol{K}_2]\,[\boldsymbol{E},\,\boldsymbol{A}_2]\end{aligned}$$

Successive application of the Householder's formula yields to

$$(\boldsymbol{A}^t\boldsymbol{K}\boldsymbol{A})^{-1}=\begin{bmatrix}\begin{matrix}\boldsymbol{Z}_d, & \boldsymbol{Z}_c\\ \boldsymbol{Z}_b, & \boldsymbol{Z}_a\end{matrix} & 0\\ 0 & \begin{matrix}\boldsymbol{Z}_h, & \boldsymbol{Z}_g\\ \boldsymbol{Z}_f, & \boldsymbol{Z}_e\end{matrix}\end{bmatrix}-\begin{bmatrix}\begin{bmatrix}\boldsymbol{Z}_c\\ \boldsymbol{Z}_a\end{bmatrix}\boldsymbol{d}_1[\boldsymbol{Z}_b,\,\boldsymbol{Z}_a], & \begin{bmatrix}\boldsymbol{Z}_c\\ \boldsymbol{Z}_a\end{bmatrix}\boldsymbol{d}_2[\boldsymbol{Z}_h,\,\boldsymbol{Z}_g]\\ \begin{bmatrix}\boldsymbol{Z}_h\\ \boldsymbol{Z}_f\end{bmatrix}\boldsymbol{d}_2^t[\boldsymbol{Z}_b,\,\boldsymbol{Z}_a], & \begin{bmatrix}\boldsymbol{Z}_h\\ \boldsymbol{Z}_f\end{bmatrix}\boldsymbol{d}_3[\boldsymbol{Z}_h,\,\boldsymbol{Z}_g]\end{bmatrix} \tag{29}$$

in which

$$\begin{aligned}\boldsymbol{d}_1 &= \boldsymbol{e}_1^t\boldsymbol{K}_e(\boldsymbol{I}-\boldsymbol{e}_2\boldsymbol{Z}_h\boldsymbol{e}_2^t\boldsymbol{k}_{21})\,\boldsymbol{e}_1 \qquad \boldsymbol{d}_2=\boldsymbol{e}_1^t\boldsymbol{K}_e\boldsymbol{e}_2\\ \boldsymbol{d}_3 &= \boldsymbol{e}_2^t\boldsymbol{k}_{21}\boldsymbol{e}_1\boldsymbol{Z}_a\boldsymbol{e}_1^t\boldsymbol{K}_e\boldsymbol{e}_2\\ \boldsymbol{K}_e &= (\boldsymbol{e}_1\boldsymbol{Z}_a\boldsymbol{e}_1^t+\boldsymbol{e}_2\boldsymbol{Z}_h\boldsymbol{e}_2^t\boldsymbol{k}_{21}\boldsymbol{e}_1\boldsymbol{Z}_a\boldsymbol{e}_1^t+\boldsymbol{f}_{21})^{-1}\end{aligned} \tag{30}$$

The first term of eq. (29) is the joint flexibility matrice of $S_1$ and $S_2$ systems, while the second term is corrections which are to be added when $S_1$ system is reunited with $S_2$ system. The system may be expanded through successive application of eq. (29).

## 5. Further Considerations

A memory-saving and effective algorithm in computer analysis is derived through the application of eq. (29). When thermal expansion and support movement are neglected, the joint displacement due to the external joint load $\boldsymbol{P}$ is given as follows

$$\boldsymbol{D} = (\boldsymbol{A}^t\boldsymbol{K}\boldsymbol{A})^{-1}\boldsymbol{P} \tag{31}$$

Then

$$\begin{Bmatrix} \boldsymbol{D}_1 \\ \boldsymbol{D}_1^* \\ \boldsymbol{D}_2^* \\ \boldsymbol{D}_2 \end{Bmatrix} = (\boldsymbol{A}^t\boldsymbol{K}\boldsymbol{A})^{-1} \begin{Bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_1^* \\ \boldsymbol{P}_2^* \\ \boldsymbol{P}_2 \end{Bmatrix} \tag{32}$$

where the subscripts represent the system number and the asterisk represent the points where $S_{12}$ system is connected to $S_1$ and $S_2$ system.

Substituting eq. (29) into eq. (32) we have

$$\begin{aligned} \boldsymbol{D}_1 &= \bar{\boldsymbol{D}}_1 - \boldsymbol{Z}_c(\boldsymbol{d}_1\bar{\boldsymbol{D}}_1^* + \boldsymbol{d}_2\bar{\boldsymbol{D}}_2^*) \\ \boldsymbol{D}_1^* &= \bar{\boldsymbol{D}}_1^* - \boldsymbol{Z}_n(\boldsymbol{d}_1\bar{\boldsymbol{D}}_1^* + \boldsymbol{d}_2\bar{\boldsymbol{D}}_2^*) \\ \boldsymbol{D}_2^* &= \bar{\boldsymbol{D}}_2^* - \boldsymbol{Z}_h(\boldsymbol{d}_2^t\bar{\boldsymbol{D}}_1^* - \boldsymbol{d}_3\bar{\boldsymbol{D}}_2^*) \\ \boldsymbol{D}_2 &= \bar{\boldsymbol{D}}_2 - \boldsymbol{Z}_f(\boldsymbol{d}_2^t\bar{\boldsymbol{D}}_1^* - \boldsymbol{d}_3\bar{\boldsymbol{D}}_2^*) \end{aligned} \tag{33}$$

in which

$$\begin{Bmatrix} \bar{\boldsymbol{D}}_1 \\ \bar{\boldsymbol{D}}_1^* \\ \bar{\boldsymbol{D}}_2^* \\ \bar{\boldsymbol{D}}_2 \end{Bmatrix} = \begin{vmatrix} \boldsymbol{Z}_d & \boldsymbol{Z}_c & & \\ \boldsymbol{Z}_b & \boldsymbol{Z}_a & \multicolumn{2}{c}{0} \\ & & \boldsymbol{Z}_h & \boldsymbol{Z}_g \\ \multicolumn{2}{c}{0} & \boldsymbol{Z}_f & \boldsymbol{Z}_e \end{vmatrix} \begin{vmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_1^* \\ \boldsymbol{P}_2^* \\ \boldsymbol{P}_2 \end{vmatrix} \tag{34}$$

$\bar{\boldsymbol{D}}$ represents the joint displacement of $S_1$ and $S_2$ systems which are analysed as independent ones under the same loading condition as the specified one.

From eq. (33) it is clear that the joint displacement of the complete structure is determined by the results which are obtained when each substructure is analysed separately under the action of specified external loading. The procedure for the analysis of each substructure can be treated as the chain job in programming for the computer analysis. The size of matrice involved depends on the size of each substructure. It is necessary for interconnection by computer that $\bar{\boldsymbol{D}}$ and [$\boldsymbol{Z}_c$, $\boldsymbol{Z}_a$,

$\boldsymbol{Z}_h$, $\boldsymbol{Z}_f$] are stored. Then eq. (29) will lead to a memory-saving and effective algorithm if the overall structure is properly divided.

## 6. Conclusion

In this investigation the Spiller's formulas are obtained by the Branin's formulas through the transformation into the local coordinate system. And a memory saving and effective alogorithm for the computer analysis is derived through application of the Graph theory and the Householder's formula. The size of each substructure may be determined according to the capacity of the computer.

### References

1) Fenves & Branin; Proc. ASCE, Vol. 89, ST, 1963
2) DiMaggio & Spiller; Proc. ASCE, Vol. 91, EM, 1965
3) A.S. Householder; Jour. Soc. Ind. & Appl. Math., Vol. 5, No. 3, 1957

## Appendix

1. Graph theory

(1) Branch-node incidence matrix, $\bar{\boldsymbol{A}}$

The ij-th element of $\bar{\boldsymbol{A}}$ is defined as follows

$$\bar{a}_{ij} = (1, -1, 0)$$

if the i-th branch is positively, negatively, not incident on the j-th node. If we divide the graph into the tree branches and the links, or tree-complement, and properly rearrange the rows of $\boldsymbol{A}_T$, $\boldsymbol{A}_L$ may be written as follows

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_T \\ \boldsymbol{A}_L \end{bmatrix}$$

where subscripts represent the tree branches and the links of the graph.

(2) Node-to-datum path matrix, $\bar{\boldsymbol{B}}_T$

$\bar{\boldsymbol{B}}_T$ is defined in the tree of the graph. The $ij$-th element of $\bar{\boldsymbol{B}}_T$ is given as follows

$$\bar{b}_{ij} = (1, -1, 0)$$

if the $i$-th branch is positively, negatively, not included in the $j$-th node-to -datum path.

The relation between $\bar{\boldsymbol{A}}_T$ and $\bar{\boldsymbol{B}}_T$ is written as

$$(\bar{A}_T)^{-1} = \bar{B}_T^t$$

(3) Branch-mesh matrix, $\bar{C}$

We define the $ij$-th element of $\bar{C}$ as follows

$$\bar{c}_{ij} = (1, \ -1, \ 0)$$

if the $i$-th branch is positively, negatively, not included in the $j$-th basic mesh. If the graph is devided into the tree and the links and properly rearrange the rows of $C$, $C$ may be given as follows

$$\bar{C} = \begin{bmatrix} \bar{C}_T \\ \bar{C}_L \end{bmatrix} = \begin{bmatrix} \bar{C}_T \\ I_L \end{bmatrix}$$

in which $I_L$ is $L_L$ a unit matrix. Then we have the following relations.

$$\bar{A}^t\bar{C} = \bar{C}^t\bar{A} = 0$$

$$\bar{C}_T = -\bar{B}_T\bar{A}_L^t$$

2. Modifications of $\bar{A}$, $\bar{B}_T$ and $\bar{C}$ matrice

(1) Modified branch-node incidence matrix $A$

The $rj$-th element of $A$ is modified as follows

$$a_{ij} = (T_i, \ -T_i^*, \ 0)$$

if $\bar{a}_{ij}=(1, \ -1, \ 0)$

in which

$$T_i = \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix} \qquad T_i^* = \begin{bmatrix} A_i & M_i \\ 0 & A_i \end{bmatrix}$$

$$M_i = \begin{bmatrix} 0 & 0 & 0 \\ l_i\lambda_\zeta^i, & l_i\mu_\zeta^i, & l_i\nu_\zeta^i \\ -l_i\lambda_\eta^i, & -l_i\mu_\eta^i, & -l_i\nu_\eta^i \end{bmatrix}$$

where $\mathbf{0}$ is $6\times6$ zero matrix

(2) Modified note-to-datum path matrix, $B_T$

The $ij$-th element of $B_T$ is given as follows

$$b_{ij} = (T_i, \ -\tilde{T}_i, \ \mathbf{0})$$

if $\bar{b}_{ij}=(1, \ -1, \ 0)$ and

$$\tilde{T}_i = \begin{bmatrix} A_i & \mathbf{0} \\ M_i & A_i \end{bmatrix}$$

We have also the following relationship

$$\boldsymbol{A}_T^{-1} = \boldsymbol{B}_T^t$$

(3) Modified branch-mesh matrix, $\boldsymbol{C}$

$\boldsymbol{C}$ is defined as follows

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_T \\ \boldsymbol{C}_L \end{bmatrix} = \begin{bmatrix} -\boldsymbol{B}_T\boldsymbol{A}_L^t \\ \boldsymbol{I}_L \end{bmatrix}$$

3. Householder's formula

The formula is expressed as

$$(\boldsymbol{Z}+\boldsymbol{U}^t\boldsymbol{K}\boldsymbol{V})^{-1} = \boldsymbol{Z}^{-1}-\boldsymbol{Z}^{-1}\boldsymbol{U}^t(\boldsymbol{K}^{-1}+\boldsymbol{V}\boldsymbol{Z}^{-1}\boldsymbol{U}^t)^{-1}\boldsymbol{V}\boldsymbol{Z}^{-1}$$

in which $\boldsymbol{Z}$ and $\boldsymbol{K}$ are nonsingular square matrix.