A Remark on a Non-linear Hyperbolic Equation of First Order

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By

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This remark is concerned with an existence theorem of the generalized solution of the Canchy problem in the large

$$u_t+(u^2/2)_x=g(x)u^2$$
$$u(0, x)=u_0(x) \qquad t\geq 0, \ -\infty<x<+\infty ,$$

where $g(x)$ is any continuously differentiable function with compact support and $u_0(x)$ is any bounded measurable function.

1. Introduction

Since J.M. Burgers derived a mathematical model for the theory of turbulence in 1940[1], many authors have investigated the following type of equations.

$$(1) \qquad \frac{\partial u}{\partial t}+\frac{f(t, x, u)}{\partial x}+g(t, x, u)=0 .$$

Concerning to the generalised solution for the Cauchy problem (1), (3), for example, O.A. Oleinik[2] proved the existence in the large and some kind of uniqueness under the following assumptions.

a) $f_{uu}\geq\delta>0$ where δ is a constant,

b) $\max_{\substack{t,x \\ |u|\leq v}}|g(t, x, u)|\leq^{\exists}h(v)$ such that $\int^{\infty}\frac{dv}{h(v)}=\infty$.

Now we will indicate a possibility of extending this result in the case where b) is not satisfied. We are considering the following equation

$$(2) \qquad \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\frac{u^2}{2}=g(x)u^2$$

with the Cauchy data:

$$(3) \qquad u(x, 0)=u_0(x) .$$

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We suppose the following conditions.

(4) $g(x)$ is a continuously differentiable function with compact support in $-\infty<x<+\infty$.

(5) $u_0(x)$ is a bounded measurable function.

Theorem. Cauchy problem (2), (3) has a generalized solution in $0\leq t<+\infty$ which satisfies Oleinik's uniqueness condition.

2. Cauchy Problem for a Simple Discontinuous Initial Data

Let $u_-(x)$ be a function with continuous bounded derivative on $a\leq x\leq 0$, and $u_+(x)$ a function with continuous bounded derivative on $0\leq x\leq b$.
We call $u_0(x)$ defined below a simple discontinuous initial data.

$$u_0(x)=\begin{cases} u_-(x) & a\leq x<0 \\ u_+(x) & 0\leq x\leq b\,. \end{cases} \tag{6}$$

Lemma 1. The Cauchy problem (2), (6) has a generalized solution on some domain D, which is defined below.

Proof. The local existence of the solution is shown by the method of characteristics in 3). Here we only have to make clear the domain D where the generalized solution is defined by this method.

Consider the system of ordinary differential equations.

$$\frac{dx}{dt}=u\,,\quad \frac{du}{dt}=g(x)u^2\,. \tag{7}$$

Solving these equations by the following initial value, where ξ runs in $[a, 0]$ and $[0, b]$,

$$x(0,\xi)=\xi\,,\qquad u(0,\xi)=u_0(\xi)\,. \tag{8}$$

we have the solution of the form

$$\begin{cases} \int_\xi^{x(t,\xi)} e^{-\int_\xi^\zeta g(\eta)d\eta}\,d\zeta=u_0(\xi)t \\ u(t,\xi)=u_0(\xi)e^{\int_\xi^{x(t,\xi)}g(\eta)d\eta}\,. \end{cases} \tag{9}$$

If $u_-(0)<u_+(0)$ we supplement another initial data:

$$x(0,\varepsilon)=0,\quad u(0,\varepsilon)=u_-(0)+\varepsilon(u_+(0)-u_-(0))\qquad 0\leq\varepsilon\leq 1 \tag{10}$$

and solving (7), (10), we have

$$(11)\qquad \begin{cases} \displaystyle\int_0^{x(t,\varepsilon)} e^{-\int_0^\zeta g(\eta)d\eta} d\zeta = u(0, \varepsilon)t \\ u(t, \varepsilon) = u(0, \varepsilon)e^{\int_0^{x(t,\varepsilon)} g(\eta)d\eta} . \end{cases}$$

We will see that these functions $\{u(t, \xi), x(t, \xi)\}$, $\{u(t, \varepsilon), x(t, \varepsilon)\}$ generate a generalized solution. First we determine t_1 the value for which the following inequality is true.

$$(12)\qquad \frac{\partial x(t, \xi)}{\partial \xi} = e^{\int_\xi^{x(t,\xi)} g(\eta)d\eta}\left(1 - tg(\xi)u_0(\xi) + t\frac{\partial u_0}{\partial \xi}\right) > 0 \qquad \text{for} \quad 0 \leq t \leq t_1 .$$

Remark. $\dfrac{\partial x(t, \varepsilon)}{\partial \varepsilon} = (u_+(0) - u_-(0))te^{\int_0^x g(\eta)d\eta} > 0$ for all t

by the definition of $x(t, \varepsilon)$.

Denoting $x_-(t, \xi)$, $u_-(t, \xi)$ the solutions of (7), (8) for $a \leq \xi \leq 0$, and $x_+(t, \xi)$, $u_+(t, \xi)$ for $0 \leq \xi < b$, we consider two sets of functions $\{x_-(t, \xi), u_-(t, \xi)\}$ $\{x_+(t, \xi), u_+(t, \xi)\}$ on the sets

$$D_- = \{(t, x):\ 0 \leq t \leq t_1,\ x_-(t, a) \leq x \leq x_-(t, 0)\}$$
$$D_+ = \{(t, x):\ 0 \leq t \leq t_1,\ x_+(t, 0) \leq x \leq x_+(t, b)\},$$

respectively and if $u_-(0) < u_+(0)$, we also consider the functions $\{u(t, \varepsilon), x(t, \varepsilon)\}$ on $D_0\{(t, x):\ 0 \leq t < +\infty,\ x(t, 0) < x < x(t, 1)\}$. By (12) we know that these sets of functions define local genuine solutions of (2) $u_-(t, x)$, $u_+(t, x)$ and $u(t, \varepsilon)$ on D_-, D_+ and D_0 respectively. Now we proceed to discuss 2 cases.

i) $u_-(0) < u_+(0)$.

We can construct generalized solution by defining

$$u(t, x) = \begin{cases} u_-(t, x) & \text{for} \quad D_- \\ u(t, \varepsilon) & \text{for} \quad D_0 \\ u_+(t, x) & \text{for} \quad D_+ \end{cases}$$

and so we have $D = \{0 \leq t \leq t_1\} \cap \{D_- \cup D_0 \cup D_+\}$

ii) $u_-(0) > u_+(0)$.

In this case, D_- and D_+ have a common domain D_1. We have to determine a shock line $x = x^*(t)$ in this common domain D_1 by the following differential equation.

$$(13)\qquad \frac{dx^*}{dt} = \frac{1}{2}\left\{u_-(t, x^*) + u_+(t, x^*)\right\}, \quad x^*(0) = 0 .$$

Taking $u_+(t, x)$ in the right hand side of $x^*(t)$ and $u_-(t, x)$ in the left hand side of $x^*(t)$ in D_1, we obtain a generalized solution in $D=\{(t, x): 0\leq t\leq t_1, x_-(t, a)\leq x\leq x_+(t, b)\}$ which satisfies the Rankine-Hugoniot condition. We can see easily in both cases, these constructed generalized solutions satisfy Oleinik's uniqueness condition, if we note that $\partial u/\partial x=\frac{\partial u}{\partial \xi}\Big/\frac{\partial x}{\partial \xi}$ or $\frac{\partial u}{\partial \varepsilon}\Big/\frac{\partial x}{\partial \varepsilon}$.

3. Local Solution for Pieceweise Constant Initial Data

Now we consider the following piecewise constant function derived from our initial data $u_0(x)$:

$$u^h(0, x) = \begin{cases} \frac{1}{h}\int_{kh}^{(k+1)h} u_0(\xi)d\xi \quad \text{on} \quad kh\leq x<(k+1)h\,, \\ \text{where} \quad h>0, k=0, \pm 1, \cdots \text{ and } |x|\leq\left[\frac{1}{h}\right], \\ 0 \quad \text{for} \quad |x|>\left[\frac{1}{h}\right]. \end{cases} \tag{14}$$

Taking this function as initial data for (2), we have

Lemma 2. There exists the generalized solution of (2), (14) in $0\leq t\leq t_1$, where t_1 is independent of $h(0<h\leq h_0)$ and depends only on $\sup_{-\infty<x<+\infty}|u_0(x)|$.

Proof. We note that $|u^h(0, x)|\leq \sup_{-\infty<x<+\infty}|u_0(x)|=u_0$. First following S. K. Godunov[4] and J. Glimm[5], using lemma 1 on each discontinuity $x=kh$, we can construct the generalized solution $u^h(t, x)$ of (2) in some interval $0\leq t\leq t^*$. Here we note that each local genuine solution of (2) $u^{(k)}(t, x)$ initiated on $t=0$, $kh\leq x<(k+1)h$ is defined on $\{(t, x): 0\leq t\leq t_1, x^{(k)}(t, kh)\leq x<x^{(k)}(t, (k+1)h)\}$ and bounded-continuous with its derivative, because t_1 is defined by (12) i.e.

$$t_1=\frac{1}{\bar{g}u_0}, \tag{15}$$

where $\bar{g}=\max_{-\infty<x<\infty}|g(x)|$.

If it happens that $t^*<t_1$, we must continue the generalized solution $u^h(t, x)$ beyond $t=t^*$, where the neighboruing decompositions of discontinuities influence each other at $t=t^*$, $x=x^*$. By virtue of the smoothness of each $u^{(k)}(t, x)_{k=0,\pm1,\cdots}$ on $t=t^*$, $u^h(t, x)$ is piecewise bounded continuous with its derivative, therefore using once more lemma 1 on each discontinuity point we have a generalized solution of (2), (14) on $0\leq t\leq t^{**}(>t^*)$, where t^{**} is limited by new influence at (t^{**}, x^{**}) of neighbouring decompositions of discontinuities on $t=t^*$.

Repeating use of lemma 1 we can construct the generalized solution $u^h(t, x)$

of (2) (14) on $0 \leq t \leq t_1$ because the local genuine solutions $u^{(k)}(t, x)$ are determined for $0 \leq t \leq t_1$, and the number of the influence points of the neighbouring decompositions is finite by (13) with the boundedness of $u^{(k)}(t, x)$ and the fact $u^{(k)}(t, x) = 0$ for large $|k|$. q.e.d.

4. Approximating Solution

The continuation of the generalized solution beyond $t = t_1$ has some difficulty which comes from the many-valuedness of the function $u^{(k)}(t, x)$ which is defined by the solution of the characteristics and for $0 \leq t \leq t_1$ has also been local genuine solution of (2), therefore we use a slight modification of the method of S. K. Godunov[4], i.e. we change the value on $t = t_1$ of the generalized solution $u^h(t, x)$ obtained for $0 \leq t \leq t_1$ to the below-defined value $u^h(t_1 + 0, x)$, which is considered as the initial data of the equation (2) for $t \geq t_1$.

Definition:

$\{u^h(t, x)\}$ $0 < h < h_0$ is called an approximating solution on $0 \leq t \leq T$ ($^{\forall} T > 0$) of the equation (2) with the initial data (3), if it satisfies the following:

i) There exist $t_1, \cdots, t_{n-1}$ such that $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$ and on each region $\{t_i < t < t_{i+1}, -\infty < x < +\infty\}$ $i = 0, \cdots, n-1$, $u^h(t, x)$ is the piecewise continuous and bounded generalized solution of (2) with some initial data $u^h(t_i + 0, x)_{i=0,1,\cdots n-1}$ that satisfy the following:

ii) for $i = 0, 1, \cdots, n-1$, $^{\forall} X > 0$

$$\int_{|x| \leq X} |u^h(t_i - 0, x) - u^h(t_i + 0, x)| \, dx \leq c \cdot h ,$$

where c is a constant independent of h, may depend on X and

$$u^h(t_0 - 0, x) = u_0(x) , \quad u^h(t_0 + 0, x) = u^h(0, x) .$$

Lemma 3. If we have the approximating solution of (2), (3) $\{u^h(t, x)\}$ that is uniformly bounded on $0 \leq t \leq T$, compact in each space $L^1(0 \leq t \leq T, |x| \leq X)$ $^{\forall} X > 0$ and uniformly satisfies the Oleinik condition on each region $t_i < t < t_{i+1}$ $i = 0, 1, \cdots, n-1$, then we can have the generalized solution of (2), (3) as the a.e. limit ($h \to 0$) of $u^h(t, x)$.

Proof. By the definition on each region $t_i < t < t_{i+1}$, $u^h(t, x)$ is the generalized solution i.e. it fulfills for arbitrary smooth finite function $\varphi(t, x)$

$$\iint_{t_i < t < t_{i+1}} \left[u^h \frac{\partial \varphi}{\partial t} + \frac{(u^h)^2}{2} \frac{\partial \varphi}{\partial x} + g \cdot (u^h)^2 \varphi \right] dt\, dx +$$

$$+\int_{t=t_i \text{ or } t_{i+1}} [u^h(t_i+0, x)\varphi(t_i, x)-u^h(t_{i+1}-0, x)\varphi(t_{i+1}, x)]dx = 0\,.$$

Summimg up these equalities $i=0, 1, \cdots, n-1$ gives

$$\iint_{0<t<T}\left[u^h\frac{\partial\varphi}{\partial t}+\frac{(u^h)^2}{2}\frac{\partial\varphi}{\partial x}+g\cdot(u^h)^2\cdot\varphi\right]dt\,dx+ \tag{16}$$
$$+\sum_{i=0}^{n-2}\int_{t=t_i}(u^h(t_i+0, x)-u(t_i-0, x))\varphi(t_i, x)dx+$$
$$+\int_{t=0}u_0(x)\varphi(0, x)dx-\int_{t=T}u^h(T, x)\varphi(T, x)dx = 0\,.$$

By virtue of the assumption of lemma 3 we can choose a subsequence $\{u^{h_j}(t, x)\}_{j=1,2\cdots}$ converging a.e. to some bounded measurable function $u(t, x)$, and so we pass to the limit $h_j\to 0$ in (16), using the definition (ii), then we have the following identity for arbitrary smooth function $\varphi(t, x)$ vanishing on large $|x|$ and $t=T$.

$$\iint_{0<t<T}\left[u\frac{\partial\varphi}{\partial t}+\frac{u^2}{2}\frac{\partial\varphi}{\partial x}+gu^2\varphi\right]dx\,dt+\int_{t=0}u_0(x)\varphi(0, x)dx = 0\,.$$

Furthermore we see that all sequences $\{u^h(t, x)\}$ converge to the same limit $u(t, x)$ that satisfies the Oleinik condition on each region $t_i<t<t_{i+1}$ $i=0, 1, 2, \cdots, n-1$, because under this condition the uniqueness of the generalized solution is assured. q.e.d.

Now for the existence in the large of the generalized solution of (2), (3) it is sufficient to see that there is an approximating solution $\{u^h(t, x)\}$ $0<h<h_0$ of (2), (3) satisfying the assumptions of lemma 3.

5. The Existence and Convergence of an Approximating solution

Lemma 4. For $0\leq t\leq^{V} T$ there exists the approximating solution $\{u^h(t, x)\}$ $0<h<h_0$ of (2), (3) that also satisfies the following:

$$|u^h(t, x)|\leq u_0 e^{\int_{-\infty}^{\infty}|g(\eta)|d\eta}\equiv M \tag{17}$$
$$\text{where}\quad u_0=\sup_{-\infty<x<\infty}|u_0(x)|$$

$$\frac{u^h(t, x_1)-u^h(t, x_2)}{x_1-x_2}<\frac{c}{t-t_i} \tag{18}$$
$$\text{for}\quad t_i<t<t_{i+1}\quad i=0, 1, \cdots, n-1\,,$$

where c is independent of h.

Proof. On the strip $0 \leq t < t_1$ $u^h(t, x)$, constructed in lemma 2 is the generalized solution of (2) with the initial $u^h(0, x)$ defined by (14), and by (9) or (11) has the form:

$$u^h(t, x) = u^h(t, \xi) = u^h(0, \xi)\, e^{\int_\xi^x g(\eta)d\eta}$$

(19) where $\int_\xi^x e^{-\int_\xi^\zeta g(\eta)d\eta} d\zeta = u^h(0, \xi)t$ or

$$u^h(t, x) = u^h(t, \varepsilon) = u^h(0, \varepsilon)\, e^{\int_{kh}^x g(\eta)d\eta},$$

(20) where $\int_{kh}^x e^{-\int_{kh}^\zeta g(\eta)d\eta} d\zeta = u^h(0, \varepsilon)t$,

therefore we have the estimate on $0 \leq t < t_1$

$$|u^h(t, x)| \leq u_0 e^{\left|\int_\xi^x |g(\eta)|d\eta\right|} \tag{21}$$

where $\xi = \xi(t, x)$ is defined by (19) or (20).

We turn to $t_1 \leq t \leq t_2$ where t_2 is defined below. We define two piecewise constant functions $u^h(t_1, x)$, $u^h(t_1+0, x)$ as follows:

$$u^h(t_1, x) = \frac{1}{h}\int_{kh}^{(k+1)h} u^h(t_1-0, \eta)d\eta \qquad \text{for} \quad x \in I_k, \tag{22}$$

where $h>0$, $k=0, \pm 1, \pm 2, \cdots$, $I_k=[kh, (k+1)h)$ and then

i) on all intervals I_k where $u^h(t_1, x)>0$

$$u^h(t_1+0, x) = \min\{\max\{u^h(t_1-0, kh), 0\}, u^h(t_1, x)\}, \tag{23.1}$$

ii) on all intervals I_k where $u^h(t_1, x)<0$

$$u^h(t_1+0, x) = \max\{\min\{u^h(t_1-0, (k+1)h), 0\}, u^h(t_1, x)\}, \tag{23.2}$$

iii) on all intervals I_k where $u^h(t_1, x)=0$

$$u^h(t_1+0, x) = 0. \tag{23.3}$$

By these definitions we have

$$\text{either} \quad 0 \leq u^h(t_1+0, x) \leq \max\{u^h(t_1-0, kh), 0\} \tag{24.1}$$

or

$$0 \geq u^h(t_1+0, x) \geq \min\{u^h(t_1-0, (k+1)h), 0\}. \tag{24.2}$$

Also by the definition (23) and the Oleinik condition for $0<t<t_1$, especially for $t=t_1-0$, that is,

$$\frac{u^h(t_1-0, x_1)-u^h(t_1-0, x_2)}{x_1-x_2} < \frac{c}{t_1}, \tag{25}$$

we have the following for $k=0, \pm 1, \cdots$

$$\int_{I_k} |u^h(t_1+0, x)-u^h(t_1-0, x)|\,dx<\frac{1}{2}\cdot\frac{c}{t_1}h^2 . \tag{26}$$

Thus on these bases by using lemma 2 we can construct the generalized solution $u^h(t, x)$ for $t_1<t<t_2$, where t_2 is defined analogous as t_1:

$$t_2-t_1 = \frac{1}{\bar{g}M}, \quad M = u_0 e^{\int_{-\infty}^{+\infty}|g(\eta)|d\eta} \tag{27}$$

Furthermore we can estimate $u^h(t, x)$ for $t_1<t<t_2$ analogously as (21) by the explicit solution of the characteristics equation.

Either
$$\begin{aligned} 0<u^h(t, x) &= u^h(t_1+0, \xi)\, e^{\int_\xi^x g(\eta)d\eta} \\ &\leq u^h(t_1-0, kh)\, e^{\int_{kh}^x |g(\eta)|d\eta} \leq u_0 e^{\int_{\xi_0}^{kh}|g(\eta)|d\eta} e^{\int_{kh}^x |g(\eta)|d\eta} \\ &\leq u_0 e^{\int_{\xi_0}^x |g(\eta)|d\eta} \quad \text{with} \quad \xi_0<kh\leq\xi<x , \end{aligned}$$

or

$$\begin{aligned} 0>u^h(t, x) &= u^h(t_1+0, \xi)\, e^{\int_\xi^x g(\eta)d\eta} \\ &\geq -u_0 e^{-\int_{\xi_0}^{(k+1)h}|g(\eta)|d\eta} e^{-\int_{(k+1)h}^x |g(\eta)|d\eta} \\ &= -u_0 e^{-\int_{\xi_0}^x |g(\eta)|d\eta} \quad \text{with} \quad x>\xi\geq(k+1)h>\xi_0 \end{aligned}$$

or $u^h(t, x)=0$ holds,
where ξ and ξ_0 are defined by (19) or (20)

$$\int_\xi^x e^{-\int_\xi^\zeta g(\eta)d\eta} d\zeta = u^h(t_1+0, \xi)(t-t_1) , \tag{28}$$

hence $\exists k$, $\xi\in I_k$ and

$$\int_{\xi_0}^{kh} e^{-\int_{\xi_0}^\zeta g(\eta)d\eta} d\zeta = u_0(\xi_0)t_1 \quad \text{or} \tag{29.1}$$

$$\int_{\xi_0}^{(k+1)h} e^{-\int_{\xi_0}^\zeta g(\eta)d\eta} d\zeta = u_0(\xi_0)t_1 \tag{29.2}$$

Thus we have

$$|u^h(t, x)| \leq u_0 e^{\left|\int_{\xi_0}^x |g(\eta)|d\eta\right|} \qquad \text{for} \quad 0\leq t\leq t_2 , \tag{30}$$

where ξ_0 is defined by (28), (29).
Moreover the same argument as the last part of 2. gives the Oleinik condition for $t_1<t\leq t_2$:

$$\frac{u^h(t, x_1) - u^h(t, x_2)}{x_1 - x_2} < \frac{c}{t - t_1}, \tag{31}$$

where c is independent of h and depends only on M.
Repeating the same argument for $t \geq t_2$ we have $t_1 < t_2 < t_3 < \cdots < t_n = T$, which are defined by

$$t_{i+1} - t_i = \frac{1}{\bar{g}M} \qquad i = 0, 1, \cdots, n-1 \tag{32}$$

where $M = u_0 e^{\int_{-\infty}^{\infty} |g(\eta)| d\eta}$, and on each $t_i < t < t_{i+1}$ $i = 0, 1, \cdots, n-1$, $u^h(t, x)$ is the desired generalized solution satisfying (17), (18).

Lemma 5. For ${}^{\forall}X > 0$, any small $\alpha > 0$, $t_{i+1} > {}^{\forall}t''$, ${}^{\forall}t' \geq t_i + \alpha$, $i = 0, 1, \cdots, n-1$ we have

$$\int_{-X}^{X} |u^h(t', x) - u^h(t'', x)| dx < c|t'' - t'|, \tag{33}$$

where c is a constant independent of h and dependent on X and α.

Proof. $u^h(t, x)$ has uniformly locally bounded variation on $t = \text{const.} \notin [t_i, t_i + \alpha]$ with respect to x by virtue of (18) by the same argument of lemma 3 in 2).

We see (33) as follows;

We note that $u^h(t, x)$ is piecewise continuous with its derivative and its discontinuous line $x = x_p(t)$ $p = 0, \pm 1, \cdots, \pm P(h)$ is the solution of (13):

$$\frac{dx_p}{dt} = \frac{1}{2}\{u^h(t, x_p - 0) + u^h(t, x_p + 0)\}, \quad \text{therefore}$$

$$\left|\frac{dx_p}{dt}\right| \leq M, \tag{34}$$

where M is in (27), therefore, for a fixed constant $c_1 > M$ and $\Delta X = c_1(t'' - t')$, $t'' > t'$ all discontinuous lines of $u^h(t, x)$ cross over the straight line segment joining (t'', x), $(t', x + \Delta X)$ from the left side to the right side of the latter.

$$\begin{aligned}
&\int_{-X}^{X} |u^h(t'', x) - u^h(t', x)| dx \leq \\
&\leq \int_{-X}^{X} |u^h(t'', x) - u^h(t', x + \Delta X)| dx + \int_{-X}^{X} |u^h(t', x + \Delta X) - u^h(t', x)| dx \\
&\leq \int_{-X}^{X} c_1(t'' - t') \left|\frac{\partial u}{\partial \zeta}\right| dx + \sum_p \int_{X_p}^{Y_p} |u^h(t, x_p - 0) - u^h(t, x_p + 0)| dx + \\
&+ \int_{-X}^{X} (u(t', x + \Delta X) - u(t', x)) dx + c2X\Delta X,
\end{aligned}$$

where $\frac{\partial u}{\partial \zeta}$ is the derivative of $u^h(t, x)$ along the line joining (t'', x), $(t', x+\Delta X)$, $x=x_p(t)$ is the discontinuous line of $u^h(t, x)$, $X_p=x_p(t')$, $Y_p=x_p(t'')+\Delta X$, $(|Y_p-X_p| \leq 2\ \Delta X)$ and c is a constant appearing in (18),

$$\leq c_2(t''-t')+\sum_p \int_{X_p}^{Y_p} |u^h(t, x_p-0)-u^h(t, x_p+0)|\,dx+$$
$$+2M\Delta X+2cX\Delta X\,,$$

where $c_2=c_1 \max \left|\frac{\partial u}{\partial \zeta}\right| 2X$, in which maximum is taken in $t_i+\alpha \leq t \leq t_{i+1}$, $x \neq x_p$ and exists by the bounded differentiability of $u^{(k)}(t, x)$ except $t=t_i+0$

$$\leq c(t''-t')+\sum_p \int_{X_p}^{Y_p} |u^h(t, x_p-0)-u^h(t, x_p+0)|\,dx$$

by $\quad \Delta X = c_1(t''-t')\,.$

Now we estimate the last term by dividing the interval $[t', t'']$ with the width h, $t' = t^1 < t^2 < \cdots < t^{\left[\frac{t''-t'}{h}\right]+1} \geq t''$. Considering $u^h(t, x_p \pm 0)$ in $t^j \leq t < t^{j+1}$ gives

$$|u^h(t, x_p-0)-u^h(t^j, \xi^-)| \leq c|t-t^j|$$
$$|u^h(t, x_p+0)-u^h(t^j, \xi^+)| \leq c|t-t^j|\,,$$

where

$$\int_{\xi^\mp}^{x_p\mp 0} e^{-\int_{\xi^\mp}^{\zeta} g(\eta)d\eta} d\zeta = u^h(t^j, \xi^\mp)(t-t^j)\,, \tag{35}$$

and $\quad u^h(t, x_p \mp 0) = u^h(t^j, \xi^\mp)\, e^{\int_{\xi^\mp}^{x_p\mp 0} g(\eta)d\eta}\,.$

Therefore $\int_{X_p}^{Y_p} |u^h(t, x_p-0)-u^h(t, x_p+0)|\,dx$

$$= \sum_j \int_{x_p(t^j)+c_1(t^j-t')}^{x_p(t^{j+1})+c_1(t^{j+1}-t')} |u^h(t, x_p-0)-u^h(t, x_p+0)|\,dx$$
$$\leq \sum_j \left\{ \int |u^h(t, x_p-0)-u^h(t^j, \xi^-)|\,dx+ \right.$$
$$\left. +\int |u^h(t^j, \xi^-)-u^h(t^j, \xi^+)|\,dx+\int |u^h(t^j, \xi^+)-u^h(t, x_p+0)|\,dx \right\}$$
$$\leq \sum_j 2c(t-t^j)\{c_1(t^{j+1}-t^j)+x_p(t^{j+1})-x_p(t^j)\}+$$
$$+\sum_j \{c_1(t^{j+1}-t^j)+x_p(t^{j+1})-x_p(t^j)\}.\ \underset{x\in[x_p(t^j)-\varepsilon_1,\, x_p(t^j)+\varepsilon_2]}{\text{Variation}}\ u^h(t^j, x)$$

where

$\varepsilon_1=\xi^-(t^{j+1}, x_p(t^{j+1})-0)$, $\varepsilon_2=\xi^+(t^{j+1}, x_p(t^{j+1})+0)$ defined by (35).

At last we have

$$\sum_p \int_{X_p}^{Y_p} |u^h(t, x_p-0) - u^h(t, x_p+0)|\,dx = \sum_{p,j} \int | \quad " \quad |\,dx \leq$$

$$\leq \sum_{p,j} \{c(t^{j+1}-t^j)^2 + (t^{j+1}-t^j)\,|x_p(t^{j+1}) - x_p(t^j)| +$$

$$+ \text{Variation } u^h(t^j, x)(c(t^{j+1}-t^j) + |x_p(t^{j+1}) - x_p(t^j)|)\} \leq$$

$$\leq c(t''-t') + \sum_p hc(t''-t') +$$

$$+ \sum_j \underset{[-X, X+\Delta X]}{\text{Variation}}\, u^h(t^j, x)\cdot c(t^{j+1}-t^j) \leq c(t''-t'),$$

where the summation for p is the same order as $\frac{1}{h}$, where c is independent of h and dependent only on M, X and α. q.e.d.

Lemma 4, 5 give the properties of $u^h(t, x)$ assumed in lemma 3 by the same argument as in 2), therefore we can conclude as the foregoing theorem.

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