Approximate Solution of Mathieu's Differential Equation

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By

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This paper presents a method for the approximate solution of the differential equations of the Mathieu-Hill type.

This method is based on the analytical method of the Periodically Interrupted Electric Circuits.

The second-order differential equations with periodic coefficients, considered in this paper, are represented by the general form: $d^2y/dz^2+f(z)y=0$, where $f(z)$ is a single-valued periodic function of fundamental period z_T, when $f(z)=a+16q\cos 2z$, it is known as Mathieu's differential equation.

Based on the procedure in this paper, the periodic function $f(z)$ is subdivided into m functions, $f_1(z),\cdots,f_r(z),\cdots,f_m(z)$, each of which has a different interval z_r, $(r=1, 2,\cdots,m)$ for one period z_T of $f(z)$. Namely the function $f_r(z)$ represents the linear approximation of $f(z)$ in each interval, that is, $f_r(z)=2cz+d$, $0\leqq z\leqq z_r$, $(r=1, 2,\cdots,m)$ where the values of c and d are constant.

From this practical linear approximation, the present method is adequate for the determination of the approximate solution of the differential equations of the Mathieu-Hill type and this method has certain advantages, especially for the stability of the solution and also the transient solution.

The stability chart for Mathieu's differential equation is obtained and plotted for the ranges of $-3\leqq a\leqq 34$ and $0\leqq q\leqq 2$. This result is very well coincident with Ince's numerical one computed for the range of $q=0$ to 5.0.

The obtained solutions and their numerical results may be extensively accurate. And the procedure considered in this paper is useful for the mathematical analysis of a large class of physical problems.

1. Introduction

The mathematical analysis of various physical problems in many cases leads to linear differential equations with constant coefficients. However we often encounter types of physical problems whose mathematical procedures have to solve the linear differential equations with periodic functions or the nonlinear differential equations, for example, a boundary-value problem and an initial-value problem in which several important problems are involved: the propagation of wave, the parametric excitation of electric circuits, the

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theory of the stability of the solutions of certain nonlinear differential equations and so on. In these cases the linear second-order differential equations with periodic coefficients of the Mathieu-Hill type may represent a mathematical model of physical problems from a mathematical analysis point of view.

The Mathieu's differential equation takes the form

$$\frac{d^2y}{dz^2}+(a+16q\cos 2z)y=0,$$

where the parameters a and q are limited to real numbers.

The stability investigation of this equation has been extensively discussed in the mathematical literature[1] and many efforts have been especially made to contribute to a solution with period π or 2π which is said to be neutral, but may be regarded as a special cases of a stable solution.

These periodic solutions with period π or 2π are, by definition, called the Mathieu functions. In order that such solutions may exist, the coefficient a must be the definite value for a given q for each Mathieu function.

The characteristic curves showing the relation between a and q satisfying the Mathieu functions are called the stability chart for Mathieu's differential equation.

To obtain Mathieu functions, in general we must be required to solve an eigen-value problem of a homogeneous integral equation with symmetrical kernel, but it is generally difficult to find its solution. Therefore in practice the Mathieu functions may be given only in the special case where the eigen value and the eigen function of this integral equation could be the infinite power series of q when $|q|$ is sufficiently small.

As a consequense, the Mathieu functions are developed as the Fourier series (see Appendix 1) and also for a given q the difinite value of a is given by the power series of q for each Mathieu function.

It is therefore evident that the obtained Mathieu functions by those procedure may be not so accurate and that these approximate solution could be available only for a sufficiently small value of q.

The method of finding the unstable solutions of Mathieu's differential equation which is introduced by Whittaker is expedient only for small values of $|q|$.

It has been difficult to find the adequate method for the solution of Hill's differential equation and if the values of a and q are much larger ones, the method to solve Mathieu's differential equation is not found presently.

This paper presents a method for the approximate solution of the differential equations of the Mathieu-Hill type.

This method is based on the analytical method of Periodically Interrupted Electric Circuits.[2] Based on this new procedure, the transient and the steady-state solutions of the differential equations of the Mathieu-Hill type in general are easily derived and relatively is valuable and accurate as will be indicated in the following sections.

It is especially noteworthy that the stability criterion for Mathieu's differential equation is uniquely determined and that the relation between a and q of Mathieu's differential equation corresponding to the Mathieu functions, that is, the stability chart for Mathieu's differential equation is illustrated for the ranges of $-3 \leqq a \leqq 34$ and $0 \leqq q \leqq 2$.

2. The Use of the Analytical Method of Periodically Interrupted Electric Circuits

The second-order differential equations with periodic coefficients, considered in this paper, are represented by the general form:

$$\frac{d^2y}{dz^2}+f(z)y=0 \tag{1}$$

where $f(z)$ is a single-valued periodic function of fundamental period z_T as shown in Fig. 1.

If $f(z)$ is represented by a general series of the form

$$f(z)=a_0+2a_1\cos 2z+2a_2\cos 4z+\cdots, \tag{2}$$

then Eq. (1) is known as Hill's differential equation.

If $f(z)$ reduces to the simple form

$$f(z)=a+16q\cos 2z, \tag{3}$$

then Eq. (1) is known as Mathieu's differential equation.

Now Eq. (1) can be written of the matrix form

$$\begin{bmatrix} D & f(z) \\ -1 & D \end{bmatrix}\begin{bmatrix} \dot{y}(z) \\ y(z) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

where $D=d/dz$ and $\dot{y}(z)=dy(z)/dz$.

Based on the procedure in this paper, the periodic function $f(z)$ is subdivided into m functions, $f_1(z),\cdots,f_r(z),\cdots,f_m(z)$, each of which has a different interval z_r, $(r=1,2,\cdots)$ for one period z_T of $f(z)$ as shown in Fig. 1, where one period z_T of $f(z)$ is called as one stage and a duration of z_r as an r-th mode according to the circuit theories used in this paper.

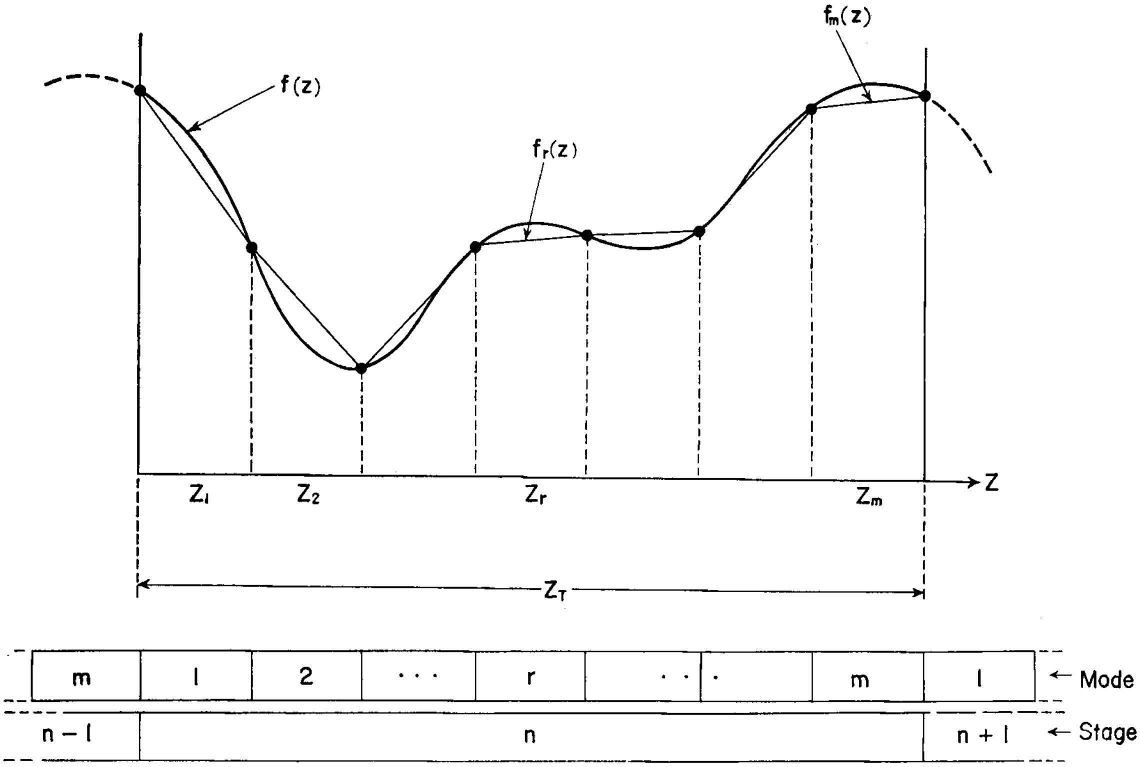


Fig. 1. Periodic coefficient $f(z)$ and its linear approximation $f_r(z)$.

The linear approximation of $f(z)$ in the r-th mode is expressed in the form

$$f_r(z) = \frac{f(\sum_{i=1}^{r} z_i) - f(\sum_{i=1}^{r-1} z_i)}{z_r} \cdot z + f(\sum_{i=1}^{r-1} z_i)$$
$$= 2cz + d \tag{5}$$

where

$$c = \frac{1}{2z_r}\Big\{ f(\sum_{i=1}^{r} z_i) - f(\sum_{i=1}^{r-1} z_i)\Big\},$$
$$d = f(\sum_{i=1}^{r-1} z_i)$$

and the origin of z in Eq. (5) is placed on the initial instant of the r-the mode as illustrated in Fig. 2.

Consequently, the differential equations of the Mathieu-Hill type can be set up as follows, for the r-th mode in the n-th stage

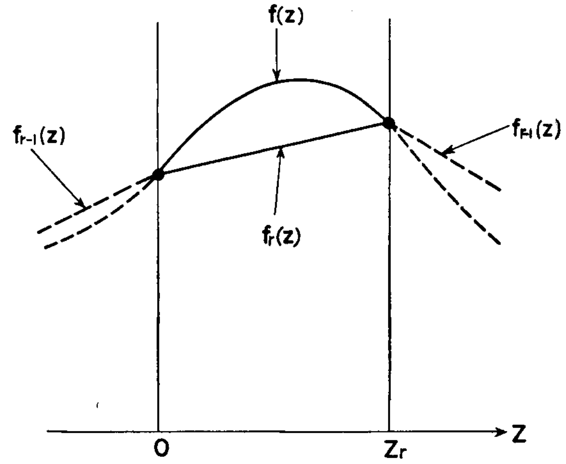


Fig. 2. Linear approximation $f_r(z)$ in r-th mode.

$$\begin{bmatrix} D & f_r(z) \\ -1 & D \end{bmatrix}\begin{bmatrix} \dot{y}_{nr}(z) \\ y_{nr}(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad 0 \leqq z \leqq z_r \tag{6}$$

where $y_{nr}(z)$ may be an approximate solution at the r-th mode in the n-th stage and the origin of z is also placed on the initial instant of the r-th mode and $f_r(z)$ is given by Eq. (5).

Eq. (6) is rewritten in the another form

$$\frac{d}{dt}\begin{bmatrix} \dot{y}_{nr}(z) \\ y_{nr}(z) \end{bmatrix} + \begin{bmatrix} 0 & f_r(z) \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \dot{y}_{nr}(z) \\ y_{nr}(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{7}$$

Hence Eq. (7) is the linear first-order differential equation with variable coefficients, the solution of Eq. (7) may be expressed in the matrix notation

$$\begin{aligned} \begin{bmatrix} \dot{y}_{nr}(z) \\ y_{nr}(z) \end{bmatrix} &= \exp\left\{-\int_0^z \begin{bmatrix} 0 & f_r(z) \\ -1 & 0 \end{bmatrix} dz\right\}\begin{bmatrix} \dot{y}_{nr}^{-0} \\ y_{nr}^{-0} \end{bmatrix} \\ &= \varepsilon^{[A(z)]} \cdot \begin{bmatrix} \dot{y}_{nr}^{-0} \\ y_{nr}^{-0} \end{bmatrix}, \\ & 0 \leqq z \leqq z_r,\ (r=1, 2, \cdots, m,\ n = 1, 2, \cdots) \end{aligned} \tag{8}$$

where $\begin{bmatrix} \dot{y}_{nr}^{-0} \\ y_{nr}^{-0} \end{bmatrix} \equiv \begin{bmatrix} \dot{y}_{n,r-1}(z_{r-1}) \\ y_{n,r-1}(z_{r-1}) \end{bmatrix}$ are the initial values at the r-th mode in the n-th stage and as a consequense of Eq. (5) we have

$$\begin{aligned} [A(z)] &= -\int_0^z \begin{bmatrix} 0 & f_r(z) \\ -1 & 0 \end{bmatrix} dz \\ &= \begin{bmatrix} 0 & -(cz^2+dz) \\ z & 0 \end{bmatrix}. \end{aligned} \tag{9}$$

Now the matrix exponential function of exp $\{[A(z)]\}$ in Eq. (8) will reduce to the available matrix form as follows.

The characteristic function of the matrix $[A(z)]$ is give by

$$\delta(\alpha) \equiv |\alpha[U]-[A(z)]| = 0 \tag{10}$$

where $[U]$ is a unit matrix, then substituting Eq. (9) into Eq. (10), we have

$$\begin{vmatrix} \alpha & (cz^2+dz) \\ -z & \alpha \end{vmatrix} = 0. \tag{11}$$

Obtaining α from Eq. (11), the latent roots $\alpha_1(z)$ and $\alpha_2(z)$ of the matrix $[A(z)]$ have the following values:

if $\quad cz+d \geqq 0,$

then

$$\left.\begin{aligned} \alpha_1(z) &= z\sqrt{cz+d}\, i \quad (\equiv \beta i) \\ \alpha_2(z) &= -z\sqrt{cz+d}\, i \quad (\equiv -\beta i) \end{aligned}\right\} \tag{12}$$

where $\quad i = -1,$

and if $\quad cz+d<0,$

then

$$\left.\begin{aligned} \alpha_1(z) &= z\sqrt{-(cz+d)} \quad (\equiv \gamma) \\ \alpha_2(z) &= -z\sqrt{-(cz+d)} \quad (\equiv -\gamma). \end{aligned}\right\} \tag{13}$$

Now by the use of the Sylvester expansion theorem (see Appendix 2), the matrix exponential function of exp $\{[A(z)]\} \equiv [\chi_r(z)]$ within the r-th mode in Eq. (8) is given by the following:

if $\quad cz+d \geqq 0,$

then

$$[\chi_r(z)] = \begin{bmatrix} \cos\beta & -(\sqrt{cz+d})\sin\beta \\ (\sqrt{cz+d})^{-1}\sin\beta & \cos\beta \end{bmatrix}, \tag{14}$$

and if $\quad cz+d<0,$

then

$$[\chi_r(z)] = \begin{bmatrix} \cosh\gamma & \sqrt{-(cz+d)}\sinh\gamma \\ (\sqrt{-(cz+d)})^{-1}\sinh\gamma & \cosh\gamma \end{bmatrix}. \tag{14'}$$

Hence as a consequence of Eq. (14) and Eq. (14′), Eq. (8) is rewritten in the form

$$\begin{bmatrix} \dot{y}_{nr}(z) \\ y_{nr}(z) \end{bmatrix} = [\chi_r(z)] \begin{bmatrix} \dot{y}_{nr}^{-0} \\ y_{nr}^{-0} \end{bmatrix}, \quad 0 \leqq z \leqq z_r. \tag{15}$$

The matrix of the initial values of Eq. (15) is evaluated by the following recurrence formulae in matrix notation, under the condition of being continious with respect to the initial values at the transition instant of each mode,

$$\begin{bmatrix} \dot{y}_{nr}^{-0} \\ y_{nr}^{-0} \end{bmatrix} = [\chi_{r-1}(z_{r-1})] \begin{bmatrix} \dot{y}_{n,r-1}^{-0} \\ y_{n,r-0}^{-0} \end{bmatrix}$$

$$= [\chi_{r-1}(z_{r-1})][\chi_{r-2}(z_{r-2})] \cdots [x(z_1)] \begin{bmatrix} \dot{y}_{n-1,1}^{-0} \\ y_{n-1,1}^{-0} \end{bmatrix}. \tag{16}$$

In addition, the following relation is satisfied

$$\begin{bmatrix} \dot{y}_{n-1,1}^{-0} \\ y_{n-1,1}^{-0} \end{bmatrix} = [B]^{n-1} \begin{bmatrix} \dot{y}_{11}^{-0} \\ y_{11}^{-0} \end{bmatrix} \tag{17}$$

where $[B]=[\chi_m(z_m)][\chi_{m-1}(z_{m-1})]\cdots[\chi_1(z_1)]$. (18)

Consequently, with above relations in mind, the general solution can be written as

$$\begin{bmatrix} \dot{y}_{nr}(z) \\ y_{nr}(z) \end{bmatrix} = [\chi_r(z)][\chi_{r-1}(z_{r-1})][\chi_{r-2}(z_{r-2})] \cdots [\chi_1(z_1)][B]^{n-1} \begin{bmatrix} \dot{y}_{11}^{-0} \\ y_{11}^{-0} \end{bmatrix},$$

$$0 \leqq z \leqq z_r, \ (r=1, 2, \cdots, m, \ n=1, 2, \cdots) \tag{19}$$

where $\dot{y}_{11}^{-0}$ and y_{11}^{-0} are the given initial values.

Thus we get finally the transient and the steady-state solutions subject to arbitrary initial conditions from Eq. (19).

We shall now discuss the stability of the solution under consideration.

The criterion is established under the presumption that the solution should be supposed to be stable so long as the factors containing the initial values of the system variables vanish away gradually from the solution as time goes.

Accordingly, in our case, the stability criterion may be achieved by examining the initial matrices of Eqs. (16) or (17), whose values must be limited ones for the stable solution.

As was well-defined in References 3, it is evident that the necessary and sufficient condition that the solution should be stable is that the absolute values of all the latent roots of $[B]$ should be less than unity, or in other words, that all the latent roots should lie inside the unit circle on the complex domain with its center at the origin.

Let λ_1 and λ_2 be the latent roots of 2×2 matrix $[B]$, the stability of the solution can be examined and summarized by the following procedures in the present case:

1). When $|\lambda_1|>|\lambda_2|$.

The powers of the matrix $[B]$ as following $[B]^{n-1}$ can be represented by the use of the Sylvester expansion theorem of the form

$$\lim_{n\to\infty}[B]^{n-1} \fallingdotseq \frac{\lambda_1^{n-1}\{\lambda_2[U]-[B]\}}{\lambda_2-\lambda_1}. \tag{20}$$

Therefore upon inspection of Eq. (20), the following three cases are derived for the stability criterion when $|\lambda_1|>|\lambda_2|$.

(a). If $|\lambda_1|<1$,

then
$$\lim_{n\to\infty}[B]^{n-1}=[0], \tag{21}$$

and the solution is stable.

(b). If $|\lambda_1|>1$,

then
$$\lim_{n\to\infty}[B]^{n-1}=[\infty], \tag{22}$$

and the solution is unstable.

(c). If $|\lambda_1|=1$,

then
$$\lim_{n\to\infty}[B]^{n-1}=\frac{\lambda_1^{n-1}\{\lambda_2[U]-[B]\}}{\lambda_2-\lambda_1}, \tag{23}$$

and the solution represents the equilibrium state between the stable and the unstable ones, or it is said to be neutral and regarded as a special case of a stable solution.

2). When $|\lambda_1|=|\lambda_2|$ and $\lambda_1\neq\lambda_2$.

Accordingly in a similar way, in this case we have

$$\lim_{n\to\infty}[B]^{n-1}=\text{Real part of }\frac{2\lambda_1^{n-1}\{\lambda_2[U]-[B]\}}{\lambda_2-\lambda_1}, \tag{24}$$

then the stability criterion is given by the following three cases.

(a). If $|\lambda_1|=|\lambda_2|<1$,
then the solution is stable.

(b). If $|\lambda_1|=|\lambda_2|>1$,
then the solution is unstable.

(c). If $|\lambda_1|=|\lambda_2|=1$,
then the solution is neutral.

3). When $|\lambda_1|=|\lambda_2|$ and $\lambda_1=\lambda_2$.

By the use of the generalized Sylvester expansion theorem,[4] the matrix $[B]^{n-1}$ in this case is written of the form

$$\lim_{n\to\infty}[B]^{n-1}=-(n-1)\lambda_1^{n-2}\{\lambda_1[U]-[B]\}. \tag{25}$$

Similarly, by inspecting Eq. (25), we can easily find the same stability criterion in this case as the conditions of (a), (b) and (c) in the former case 2).

3. The Stability of Mathieu's Differential Equation and Numerical Results of its Transient Solutions

In this section in order to illustrate the method given in the previous section, the stability of Mathieu's differential equation is examined and the numerical results of the stability chart and the transient solutions of Mathieu's differential equation are calculated by a digital computer.

3.1. The stability and the stability chart

The periodic coefficient $f(z)$ with period π of Mathieu's differential equation is written of the form

$$f(z) = a + 16q \cos 2z\,. \tag{26}$$

Now in the case of the linear approximation of $f(z)$ as Eq. (5), we let

$$z_1 = z_2 = \cdots = z_m = \pi/m \ (\equiv z_0)\,, \tag{27}$$

then the matrix $[B]$ is expressed by

$$[B] = [\chi_m(z_0)][\chi_{m-1}(z_0)] \cdots [\chi_1(z_0)] \tag{28}$$

where the values of $[\chi_r(z_0)]$, $(r=1, 2, \cdots, m)$, are obtained from Eqs. (5), (14) and (14').

Solving the characteristic function of the matrix $[B]$

$$\delta(\lambda) \equiv |\lambda[U]-[B]| = 0\,, \tag{29}$$

the latent roots $\lambda = \lambda_1$ and λ_2 are derived and then the stability of the solution in the present case is easily determined by means of Eqs. (20) through (25) as the preceding procedures.

Here we can determine the stability of the solutions by the use of the Digital Computers (KDC-I) and (NEAC-2101) according to the flow chart as shown in Fig. 3.

It is important to determine how to choose the value m. In general its value should be much larger in proportion to the value q. And also on the narrow regions of the stable or the unstable state of the solutions and on the the close regions of the neutral state of the solutions, m must become a larger value because of obtaining highly accurate results of the stability of the solutions.

In our case in order to determine the characteristic curves showing the relation between a and q, that is, the stability chart for Mathieu's differential equation, we first calculate the latent roots λ as the value $m=20$ to plot the outline of the characteristic curves and then let m be 40, the stability chart is obtained more precisely as illustrated in Fig. 4.

Here it should be remarked that the accumulate error of each component of the matrix $[B]$ is generally considered to be increased in proportion to the value m, but in our case this error hardly appears in computation of the digital computers.

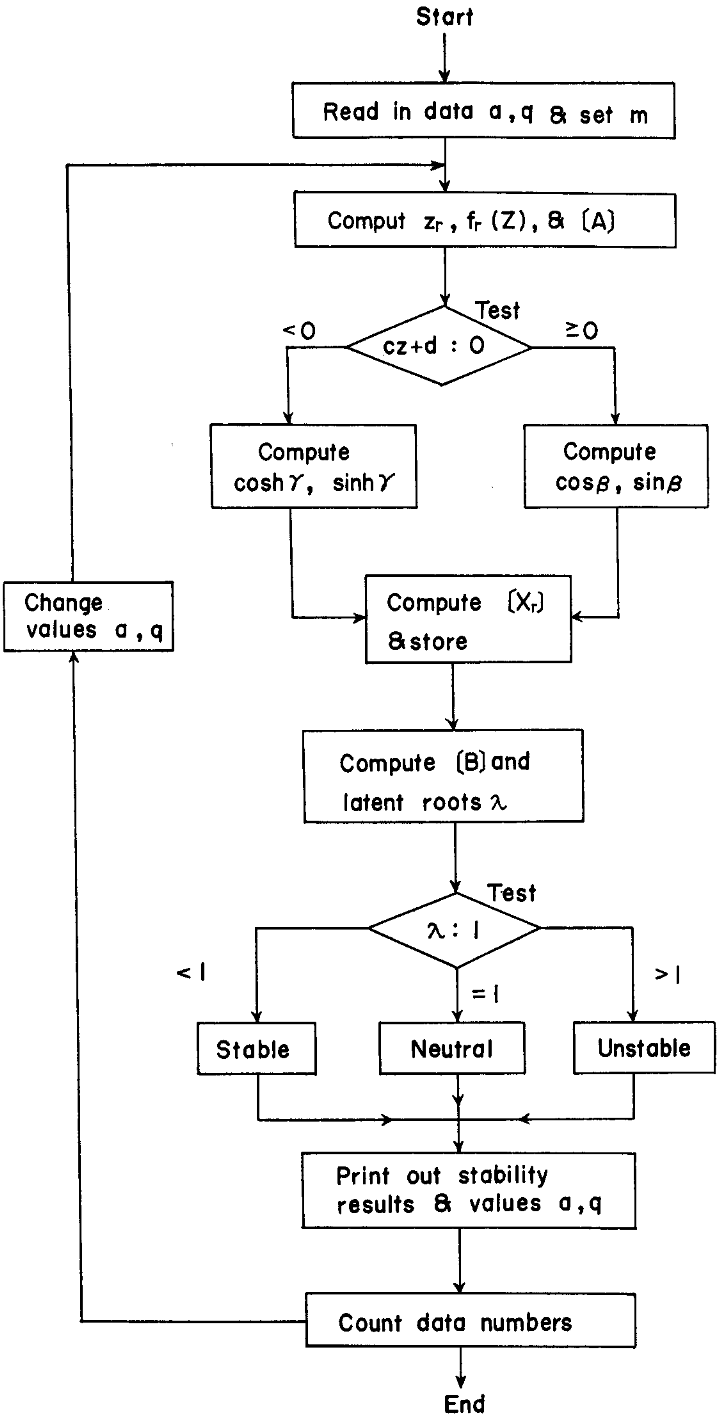


Fig. 3. Flow chart for stability of Mathieu's differential equation.

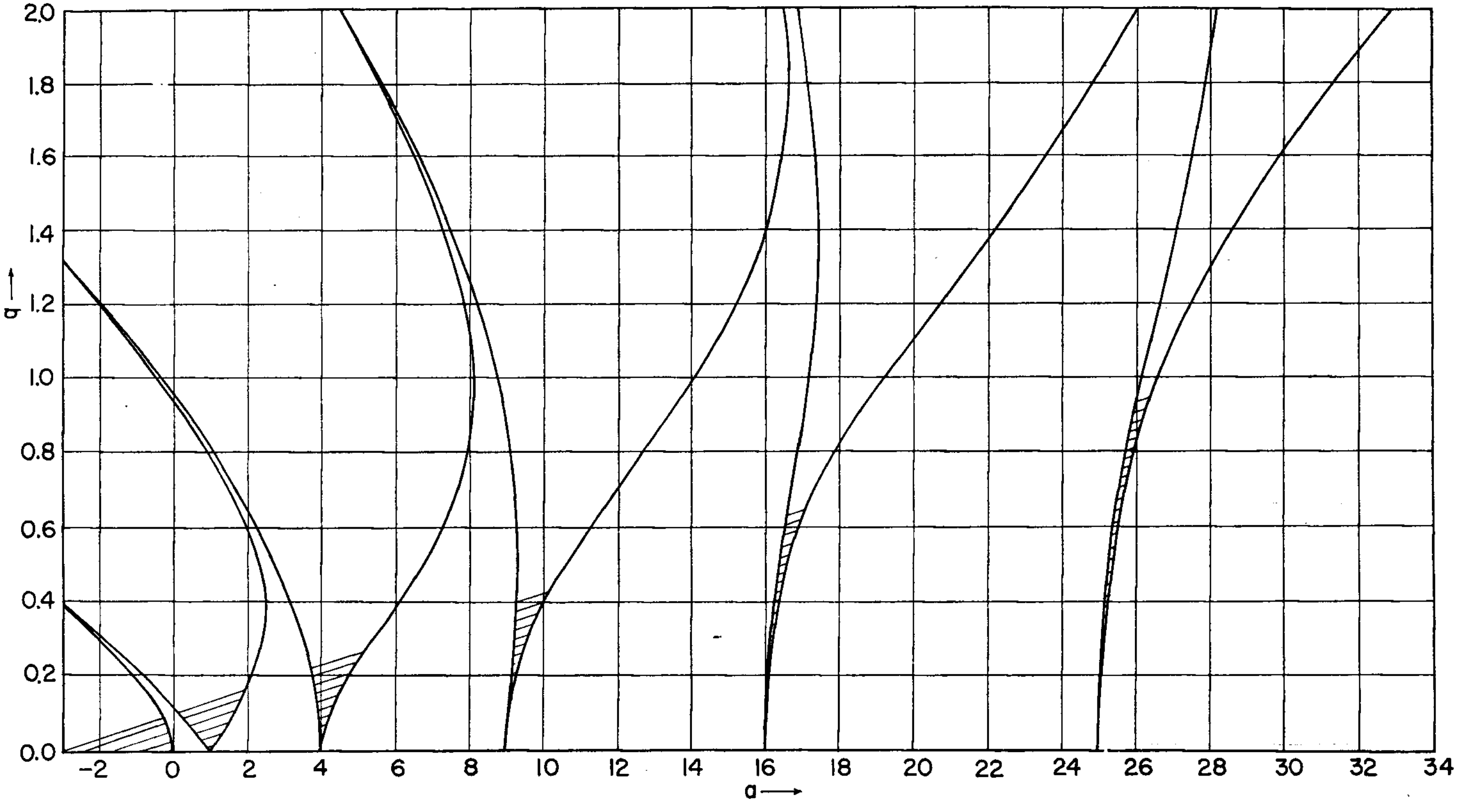


Fig. 4. Stability chart for Mathieu's differential equation.

Though Fig. 4 shows the stability chart only for the range of $a=-3$ to 34 and $q=0$ to 2.0, it could be obtainable for other range of a and q by the same procedure presented in this paper.

In view of Fig. 4 it is noteworthy that the relation between a and q, that is, the thick-line curves, illustrates the each Mathieu function (see Appendix 1).

These characteristic curves divide the plane into regions of stability and instability, that is, when a point (a, q) lies on the unstable region, shaded in part, interposed between curves, Mathieu's differential equation has an unstable solution, and when a point (a, q) lies on the remaining region, a stable solution results.

Now the stability chart computed by Ince[5] only for the range of $q=0$ to 1.25 is illustrated for the sake of comparison with the results in this paper as shown in Fig. 5 where the thick-line curves are plotted from Ince's numerical results[6] and the fine-line ones, that is, $a_{c1}, a_{s1}, \cdots, a_{s3}, \cdots$, are calculated by Eq. (A. 4) in Appendix 1.

As is well-seen from the comparison of our results in Fig. 4 with Ince's ones in Fig. 5, one closely coincides with another.

In addition as showing the variation of the characteristic (or latent) roots λ for the range of $a=3$ to 6 when $q=0.2$ as shown in Fig. 6, it could be obviously mentioned from inspection of Fig. 6 that the stability of the solutions may be easily determined by the method in this paper because of the

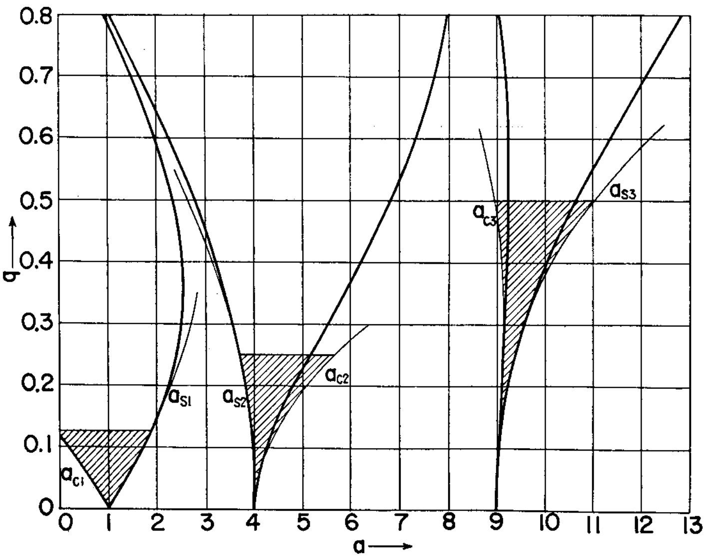


Fig. 5. Ince's stability chart for Mathieus differential equation.

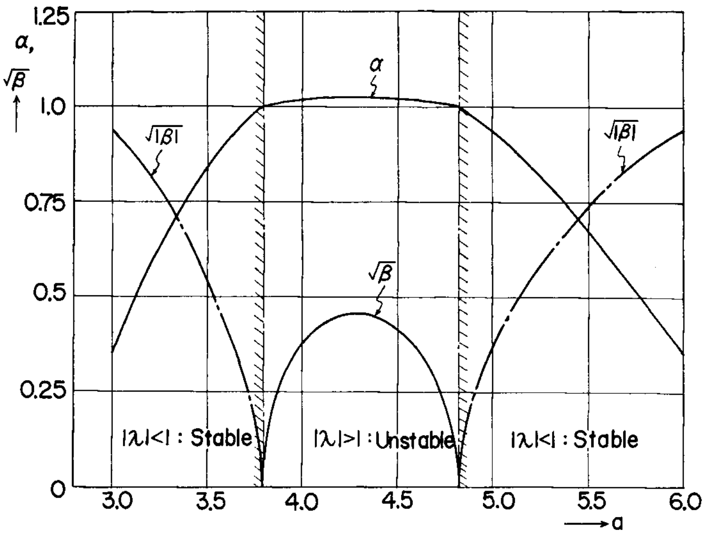


Fig. 6. Variation of characteristic root $\lambda=\alpha\pm\sqrt{\beta}$ vs. a when $q=0.2$.

remarkable and different changes of the characteristic roots λ at the vicinity of the neutral solutions.

3.2. The transient solutions

In this section it is mainly investigated that the transient solutions of Mathieu's differential equation are calculatcd for several values of a and q and that these numerical results coincide with ones of the stability chart in the preceding section.

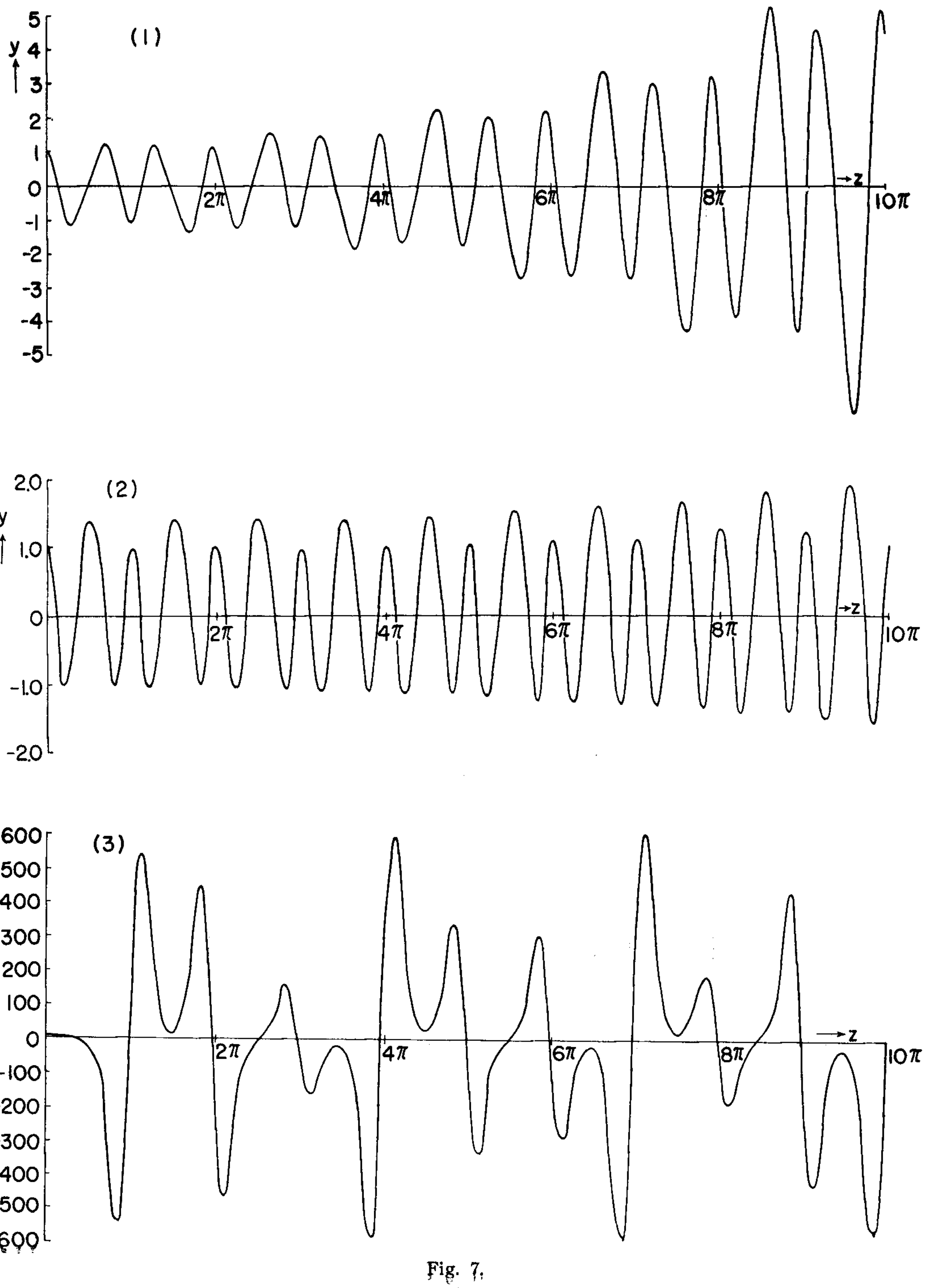
Let one period or one stage of $f(z)$ be subdivided into m narrow equal intervals, that is, $z_0=\pi/m$, the transient solutions of Mathieu's differential equation at the r-th mode in the n-th stage are expressed of the following form, from Eq. (20)

$$\begin{bmatrix} \dot{y}_{nr}(z) \\ y_{nr}(z) \end{bmatrix} = [\chi_r(z)][\chi_{r-1}(z_0)]\cdots[\chi_1(z_0)][B]^{n-1}\begin{bmatrix} \dot{y}_{\overline{11}}^{0} \\ y_{\overline{11}}^{0} \end{bmatrix}, \quad 0\leqq z\leqq z_0,\ (r=1,2,\cdots,m,\ n=1,2,\cdots) \tag{30}$$

where $$[B]=[\chi_m(z_0)][\chi_{m-1}(z_0)]\cdots[\chi_1(z_0)].$$

Thus if the values $[\chi_r(z_0)]$, $(r=1,2,\cdots,m)$ could be calculable, the transient solutions can be easily computed by the use of Eq. (30) for any given initial conditions of $\dot{y}_{\overline{11}}^{0}$ and $y_{\overline{11}}^{0}$.

Some numerical examples of the transient solutions of Mathieu's differential equation are shown in Fig. 7 where the given data a, q, m and initial values $\dot{y}_{\overline{11}}^{0}$, and $y_{\overline{11}}^{0}$, and also the stability results are given by Table 1, but



(1)
y
5
4
3
2
1
0
-1
-2
-3
-4
-5
2π
4π
6π
8π
10π
z
(2)
2.0
1.0
0
-1.0
-2.0
(3)
600
500
400
300
200
100
-100
-200
-300
-400
-500
-600

Fig. 7.

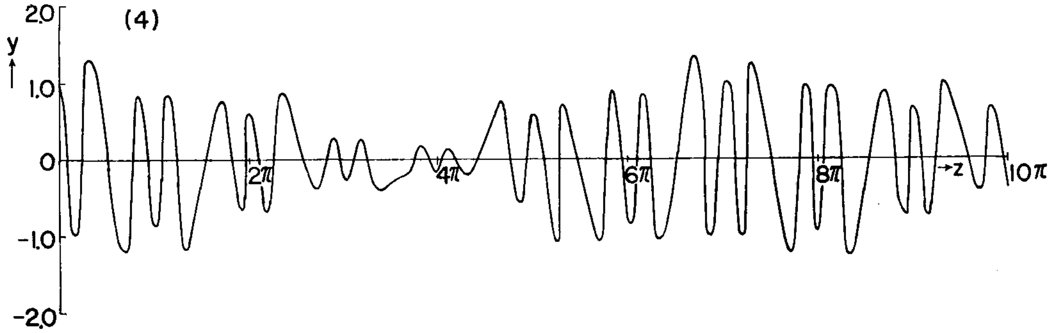


Fig. 7. Transient solutions of Mathieu's differential equation.

Table 1. Coefficients a and q, initial values and m

Fig. number in Fig. 7	a	q	$\dot{y}_{11}^{0}$	y_{11}^{0}	m	Stability	$\|\lambda\|$
(1)	9.5	0.4	0	1	40	unstable	1.24
(2)	16.6	0.55	0	1	40	unstable	1.05
(3)	−1.5	1.145	0	1	40	stable	0.907
(4)	27.8	1.9	0	1	40	stable	0.907

the absolute values $|\lambda|$ in Table 1 indicate the largest roots of λ obtained by solving the characteristic function of $[B]$. Those results in a good coincidence with the criterion of the stability chart illustrated in Fig. 4.

4. Simulation Results by an Analog Computor

Simulating Mathieu's differential equation by an analog computer in order to get its solutions or its stability, it may be said not to be adequate from a precise point of view, but it has several advantages to find directly various solutions in short time.

Mathieu's differential equation of the form

$$\frac{d^2y}{dz^2}+(a+16q\cos 2z)y=0 \tag{31}$$

can be rewritten in the form of the machine equation

$$\frac{d^2Y}{dT^2}=-\left(\frac{a}{\beta^2}+\frac{16}{\beta^2}q\cos\frac{2}{\beta}T\right)Y \tag{32}$$

where assumed the scale facter α for y and the time-scale facter β for z have been chosen so that, respectively

$$Y=\alpha y \quad \text{and} \quad T=\beta z. \tag{33}$$

Here putting $X=\cos(2/\beta)T$,

another machine equation is given by

$$\frac{d^2X}{dT^2} = -\left(\frac{2}{\beta}\right)T. \tag{34}$$

As a consequence of Eqs. (32) and (34), the block diagram of simulating Mathieu's differential equation is shown in Fig. 8.

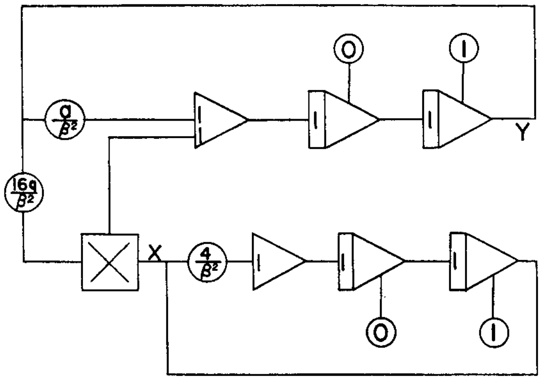


Fig. 8. Block diagram of simulating Mathieu's differential equation.

Fig. 9 shows the transient solutions simulated by the Analog Computer (MELCOM EA-7304) where the figuer numbers (1), (2) and (4) of Fig. 9 correspond respectively to each one of Fig. 7.

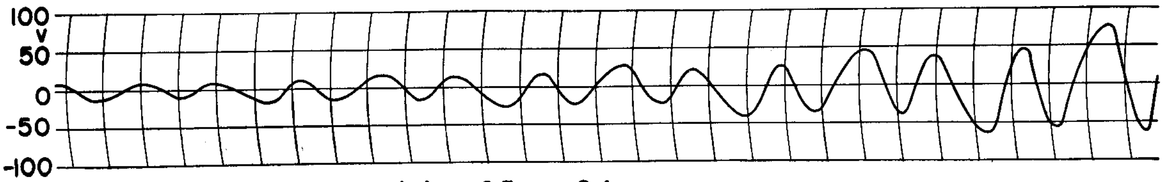
Comparing the theoretical results with the simulated ones, the results named by figure mumbers (1) and (2) in Fig. 9 indicate a close coincidence with the corresponding ones in Fig. 7. However the analogue simulation named (4) in Fig. 9 shows a similar oscillation mode to the theoretical result in Fig. 7, but its amplitude is different from each other.

These phenomena are supposed as follows:

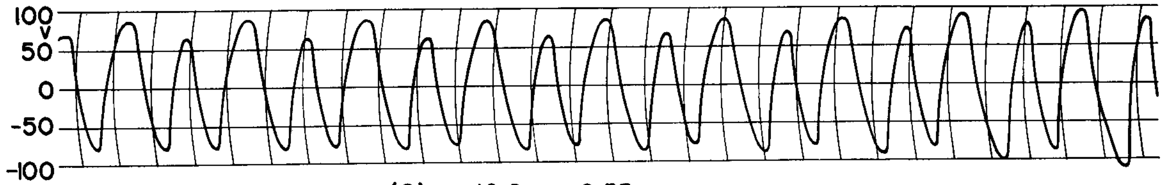
Since the solutions in the former, that is, figure numbers (1) and (2) in Fig. 9, exist in an extent region of the stable state, the variations of the solutions are insensitive to the changes of a or q. On the other hand, in the latter of (4) the simulated result varies sensitively due to a slight amount of the changes a and q because of the narrow stable region and yet the simulation accuracy is not so precise from a machine precision point of view.

The simulated result named (3) in Fig. 9 shows such a case where the solution largely varies with only the variation of $q=27.8$ to 28.0 as compared to (4) in Fig. 9.

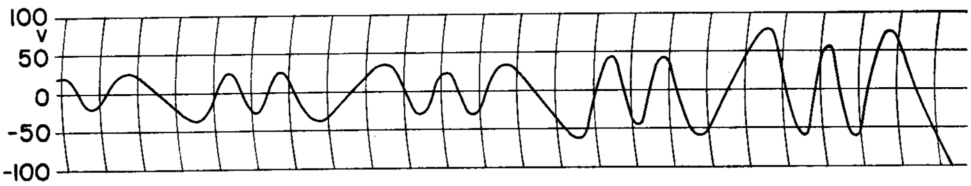
For other values of a and q, the stability and the transient solution in the preceding section are examined by the simulation ones by means of the



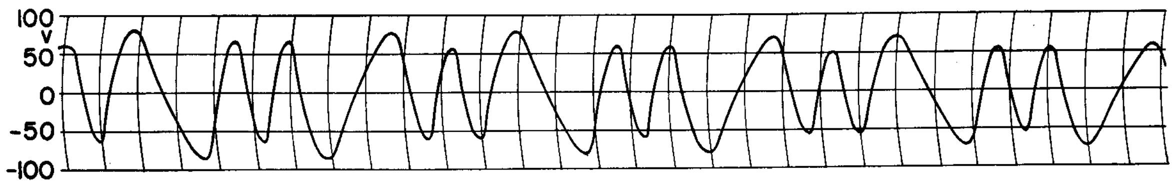
(1) a=9.5 , q=0.4



(2) a=16.6 , q=0.55



(3) a=28.0 , q=1.9



(4) a=27.8 , q=1.9

Fig. 9. Transient solutions simulated by the analog computer,

analog computer and both results are made sure to indicate a good coincidence to each other.

5. Conclusion

Based on the analytical method of the Periodically Interrupted Electric Circuits, the general method presented in this paper has demonstrated the study of stability and obtaining the approximate solutions of the differential equations of the Mathieu-Hill type.

The stability chart for Mathieu's differential equation, calculated by the use of the digital computer, is closely coincident with Ince's numerical result[6] computed for the range of $q=0$ to 5.0, and in addition the stability chart shown in the section 3 of this paper is extensively obtained for the larger values of a and q.

When the values of a and q become much larger, it is necessary for the value m to choose the larger one, and to use the higher-speed and precise digital computer is also important to rapidly get accurate results.

Comparing the method presented in this article with Mathieu's or Whittaker's one, it is noteworthy that the trasient and the steady-state solutions considered in this paper could be obtained for any values of a and q, and these become much accurate by putting m to be larger on the procedure given in this paper.

The accumulated error in digital computation for our objects has been scarcely appreciated, but how to introduce the error on assuming the linear approximation of $f(z)$ shall have to be clarified.

Though the investigation in this article is mainly on the Mathieu's differential equation, it is evident that the Hill's differential equation or the higher-order linear differential equations with variable coefficients would be studied by the method presented in this paper.[7]

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Appendix 1. Mathieu functions

The periodic solutions of the Mathieu's differential equation with period π or 2π are, by definition, called the Mathieu functions.

In order that such solutions may exist, a must have one of an infinite sequence of functions of q. When q is zero, the solutions required are

$$\left.\begin{array}{l} 1,\ \cos z,\ \cos 2z, \cdots, \\ \quad \sin z,\ \sin 2z, \cdots \end{array}\right\} \tag{A.1}$$

for the corresponding values of $a=n^2$, $(n=0, 1, 2, \cdots)$.

For other values of q, the Mathieu functions are denoted by

$$\left.\begin{array}{r} ce_0(z, q),\ ce_1(z, q),\ ce_2(z, q), \cdots, \\ se_1(z, q),\ se_2(z, q), \cdots, \end{array}\right\} \tag{A.2}$$

and those Mathieu functions reduce respectively to cos nx and sin nx when $q \to 0$.

The Fourier series for the Mathieu functions are written of the forms

$$\left.\begin{array}{l} ce_{2n}(z, q) = \sum\limits_{r=0}^{\infty} A_{2r}(q) \cos 2rz \\ ce_{2n+1}(z, q) = \sum\limits_{r=1}^{\infty} A_{2r+1}(q) \cos (2r+1)z \\ se_{2n}(z, q) = \sum\limits_{r=1}^{\infty} B_{2r}(q) \sin 2rz \\ se_{2n+1}(z, q) = \sum\limits_{r=1}^{\infty} B_{2r+1}(q) \sin (2r+1)z \end{array}\right\}. \tag{A.3}$$

In these series A and B are functions q. If $|q|$ is sufficiently small, these coefficients and accordingly the Mathieu functions are developed as the power series of q, but very little is known on the convergence of series for the Mathieu functions.

For a given q, the value of a is definite for each Mathieu function, and is called the "characteristic number" of the corresponding Mathieu function.

Following Mathieu and whittaker, the characteristic number denoted by a_{cn} and a_{sn} corresponding to $ce_n(z, q)$ and $se_n(z, q)$ respectively are given by the following expansions.

$$\left.\begin{aligned}
a_{c0} &= -32q^2+224q^4-\frac{2^{10}\cdot 29}{9}q^6+\cdots,\\
a_{c1} &= 1-8q-8q^2+8q^3-\frac{8}{3}q^4+\cdots,\\
a_{s1} &= 1+8q-8q^2-8q^3-\frac{8}{3}q^4+\cdots,\\
a_{c2} &= 4+\frac{80}{3}q^2-\frac{6104}{27}q^4+\cdots,\\
a_{s2} &= 4-\frac{16}{3}q^2+\frac{40}{27}q^4+\cdots,\\
a_{c3} &= 9+4q^2-8q^3+\frac{13}{5}q^4+\cdots,\\
a_{s3} &= 9+4q^2+8q^3+\frac{13}{5}q^4+\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}\right\} \tag{A. 4}$$

The Ince's numerical result showing the relation between a and q in Section 3 are obtained by the use of the recurrence-relations between the coefficients of Eq. (A. 3).

The solutions of the Mathieu's differential equation in the unstable region were investigated by Whittaker and the reader will consult References 1 about these problems.

Appendix 2. Sylvester Expansion Theorem

This theorem states that, if the m latent roots of $[A]$, viz. $a_1, a_2, \cdots, a_m$ are all distinct and complex in general, and if $F([A])$ is a function of $[A]$, then we have

$$F([A]) = \sum_{r=1}^{m} F(a_r)[K(a_r)] \tag{A. 5}$$

where

$$[K(a_r)] = \prod_{s\neq r}^{s=1,\cdots,m} \frac{a_s[U]-[A]}{a_s-a_r}$$

and $[U]$ is a unit matrix.

The generalized Sylvester expansion theorem states that, if a_r is an s_r-ple latent root of a square matrix $[A]$ of order m, and provided that the corresponding characteristic matrix has full degeneracy for that root, and

$$s_1+s_2+\cdots+s_i = m,$$

then

$$F([A]) = \sum_{r=1}^{i} F(a_r)[K(a_r)]^{(s_r)} \tag{A. 6}$$

where

$$[K(a_r)]^{(s_r)} = \prod_{k\neq r}^{k=1,\cdots,i} \frac{(a_k[U]-[A])^{s_k}}{(a_k-a_r)^{s_k}}.$$