A Method to Optimize the Stability of a Linear Dynamic System : II. With Equality Constraints

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By

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A numerical method to optimize the stability of a linear dynamic system which was described in the last paper is generalized to the case of equality constraints in this report. The process is an application of the "steepest-descent method" as well, and it is inclusive of the procedure given in the last paper as a simple case. Hence, with application of this method, problems which are of practical interest but are difficult to treat with the previous method are expected to be solvable. A numerical example is presented for maximizing the damping of the Dutch-Roll mode of motion of an airplane.

1. Introduction

The method of gradients or "the steepest-descent method" was described for optimizing the stability of a linear dynamic system in the last report[1]. However, in many problems of practical interest, the coefficients of the characteristic equation of a linear system are not only functions of control variables but also those of state variables, i.e. the subsidiary conditions are in general given by the following equations,

$$g_l(\beta_k, \gamma_l) = 0$$

where β_k $(k=1, 2, \cdots, M)$ and γ_l $(l=1, 2, \cdots, N)$ are M control variables and N state variables respectively.

If the functions g_l are given analytically and solved for N state variables in terms of M control variables, the state variables can be eliminated and the problem is reduced to one without constraints, which was previously described in the last report. However, such an approach will not be practical in many applications, and, therefore, a general procecure applicable to the problems is stated in this report.

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2. Optimization of the Stability Problems with Equality Constraints

The characteristic equation of a linear system is in general expressed by

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{1}$$

where the coefficients a_i are given by

$$a_i = f_i(\gamma_l) \qquad \begin{matrix} i = 0, 1, \cdots, n-1 \\ l = 1, 2, \cdots, N \end{matrix} \tag{2}$$

and γ_l are the "state variables". The state variables are also related to other variables by the following known equations,

$$g_l(\beta_k, \gamma_l) = 0 \tag{3}$$

where β_k $(k=1, 2, \cdots, M)$ are the "control variables" which are free to choose.

Roots of the characteristic equation are in general expressed by

$$\lambda_j = n_j \pm i\omega_j \tag{4}$$

if the roots are real, $\omega_j = 0$. Since the stability criterion is the value of the real part of the roots*, i.e. n_j in Eq. (4), then in order to optimize the stability, $|n_j|$ should be maximized or n_j should be minimized, because n_j is negative for the stable state. Since λ_j or n_j are the functions of the coefficients a_i, or γ_l from Eq. (2), then it is necessary to determine β_k so as to minimize $n_j(\gamma_l)$ subject to the constraints of Eq. (3).

This method starts with the starting values of the control variables β_k^*. In order to determine the change in $n(\gamma_l)$ for a small perturbation of the control variables $d\beta_k$, consider first the quantity

$$\Phi = n(\gamma_l) + \sum_{l=1}^{N} \mu_l \cdot g_l(\beta_k, \gamma_l) \tag{5}$$

where μ_l are Lagrange multipliers. Hence,

$$\begin{aligned} d\Phi &= dn \\ &= \left[\frac{\partial n}{\partial \gamma_1} \cdots \frac{\partial n}{\partial \gamma_N}\right] \begin{pmatrix} d\gamma_1 \\ \vdots \\ d\gamma_N \end{pmatrix} + [\mu_1 \cdots \mu_N] \begin{pmatrix} \frac{\partial g_1}{\partial \gamma_1} & \cdots\cdots & \frac{\partial g_1}{\partial \gamma_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_N}{\partial \gamma_1} & \cdots\cdots & \frac{\partial g_N}{\partial \gamma_N} \end{pmatrix} \begin{pmatrix} d\gamma_1 \\ \vdots \\ d\gamma_N \end{pmatrix} \\ &\quad + [\mu_1 \cdots \mu_N] \begin{pmatrix} \frac{\partial g_1}{\partial \beta_1} & \cdots\cdots & \frac{\partial g_1}{\partial \beta_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_N}{\partial \beta_1} & \cdots\cdots & \frac{\partial g_N}{\partial \beta_M} \end{pmatrix} \begin{pmatrix} d\beta_1 \\ \vdots \\ d\beta_M \end{pmatrix} \end{aligned}$$

* See Appendix

$$= \left(\frac{\partial n}{\partial \gamma_1} + \sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \gamma_1}\right) d\gamma_1 + \cdots + \left(\frac{\partial n}{\partial \gamma_N} + \sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \gamma_N}\right) d\gamma_N$$
$$+ \left(\sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \beta_1}\right) d\beta_1 + \cdots + \left(\sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \beta_M}\right) d\beta_M \tag{6}$$

where the partial derivatives are evaluated at the starting point.

Now let us choose μ_1 through μ_N so that the coefficients of $d\gamma_l$ $(l=1, 2, \cdots, N)$ vanish, i.e.

$$\begin{gathered} \frac{\partial n}{\partial \gamma_1} + \mu_1 \frac{\partial g_1}{\partial \gamma_1} + \cdots + \mu_N \frac{\partial g_N}{\partial \gamma_1} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial n}{\partial \gamma_N} + \mu_1 \frac{\partial g_1}{\partial \gamma_N} + \cdots + \mu_N \frac{\partial g_N}{\partial \gamma_N} = 0 \end{gathered} \tag{7}$$

or the Lagrange multipliers are determined by

$$\begin{gathered} \mu_1 \cdot \Delta = \begin{vmatrix} -\frac{\partial n}{\partial \gamma_1} & \frac{\partial g_2}{\partial \gamma_1} & \cdots\cdots & \frac{\partial g_N}{\partial \gamma_1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial n}{\partial \gamma_N} & \frac{\partial g_2}{\partial \gamma_N} & \cdots\cdots & \frac{\partial g_N}{\partial \gamma_N} \end{vmatrix} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_N \cdot \Delta = \begin{vmatrix} \frac{\partial g_1}{\partial \gamma_1} & \cdots\cdots & \frac{\partial g_{N-1}}{\partial \gamma_1} & -\frac{\partial n}{\partial \gamma_1} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial g_1}{\partial \gamma_N} & \cdots\cdots & \frac{\partial g_{N-1}}{\partial \gamma_N} & -\frac{\partial n}{\partial \gamma_N} \end{vmatrix} \end{gathered} \tag{8}$$

where

$$\Delta = \begin{vmatrix} \frac{\partial g_1}{\partial \gamma_1} & \cdots\cdots & \frac{\partial g_N}{\partial \gamma_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial \gamma_N} & \cdots\cdots & \frac{\partial g_N}{\partial \gamma_N} \end{vmatrix} \tag{9}$$

This reduces Eq. (6) to the expression

$$dn = \left(\sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \beta_1}\right) d\beta_1 + \cdots + \left(\sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \beta_M}\right) d\beta_M \tag{10}$$

For the steepest descent, the following value of the positive definite quadratic form is defined,

$$(dP)^2 = \sum_{k=1}^{M} w_k (d\beta_k)^2 \tag{11}$$

where w_k $(k=1, 2, \cdots, M)$ are the positive weighting numbers. To maximize

$|dn|$ for a small perturbation $d\beta_k$ under a constraint condition given by Eq. (11), the similar process to the previous calculation is applied again, i.e. consider the quantity

$$d\Psi = \left(\sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \beta_1}\right) d\beta_1 + \cdots + \left(\sum_{l=1}^{N} \mu_l \frac{\partial g_l}{\partial \beta_M}\right) d\beta_M + \nu\left[(dP)^2 - \sum_{k=1}^{M} w_k (d\beta_k)^2\right] \tag{12}$$

where μ_1 through μ_N are given by Eq. (8) and ν is another Lagrange multiplier. The maximum $|dn|$ is then obtained when

$$\frac{d\Psi}{d\beta_k} = \left(\mu_1 \frac{\partial g_1}{\partial \beta_k} + \cdots + \mu_N \frac{\partial g_N}{\partial \beta_k}\right) - 2\nu w_k \cdot d\beta_k = 0 \qquad (k = 1, 2, \cdots, M) \tag{13}$$

Accordingly,

$$d\beta_k = \frac{1}{2\nu w_k}\left(\mu_1 \frac{\partial g_1}{\partial \beta_k} + \cdots + \mu_N \frac{\partial g_N}{\partial \beta_k}\right) \tag{14}$$

Substituting Eq. (14) into Eq. (11), and solving in ν,

$$\frac{1}{2\nu} = \pm (dP)\left[\frac{1}{w_1}\left(\sum_{l=1}^{N} \frac{\partial g_l}{\partial \beta_1}\right)^2 + \cdots + \frac{1}{w_M}\left(\sum_{l=1}^{N} \frac{\partial g_l}{\partial \beta_M}\right)^2\right]^{-1/2} \tag{15}$$

Therefore, substituting Eq. (15) into Eq. (14),

$$d\beta_k = -|dP| \frac{\dfrac{1}{w_k}\left(\sum_{l=1}^{N} \mu_l \dfrac{\partial g_l}{\partial \beta_k}\right)}{\left[\dfrac{1}{w_1}\left(\sum_{l=1}^{N} \dfrac{\partial g_l}{\partial \beta_1}\right)^2 + \cdots + \dfrac{1}{w_M}\left(\sum_{l=1}^{N} \dfrac{\partial g_l}{\partial \beta_M}\right)^2\right]^{1/2}} \tag{16}$$

Since dn should be negative, or n is to be decreased, $-$ sign is used in Eq. (16).

If the functions $g_l(\beta_k, \gamma_l)$ are given analytically and Eq. (3) are solved for the state variables in terms of the control variables, i.e.

$$\gamma_l = G_l(\beta_k) \tag{17}$$

Eq. (3) is reduced to

$$g_l = \gamma_l - G_l(\beta_k) = 0 \tag{18}$$

or

$$\frac{\partial g_l}{\partial \gamma_m} = \begin{cases} 1 & : \ l = m \\ 0 & : \ l \neq m \end{cases} \tag{19}$$

Hence, by Eq. (8) and (18),

$$\mu_l = -\frac{\partial n}{\partial \gamma_l}, \qquad \frac{\partial g_l}{\partial \beta_k} = -\frac{\partial \gamma_l}{\partial \beta_k} \tag{20}$$

Consequently, the state variables are eliminated and Eq. (16) is identical with the result which was given in the previous report.

For the next step, $\beta_k^* + d\beta_k$ are the starting values and the same procedure is repeated. The process should be repeated several times until the gradient dn/dP or

$$\frac{dn}{dP} = -\left[\frac{1}{w_1}\left(\sum_{l=1}^{N}\frac{\partial g_l}{\partial \beta_1}\right)^2 + \cdots + \frac{1}{w_M}\left(\sum_{l=1}^{N}\frac{\partial g_l}{\partial \beta_M}\right)^2\right]^{1/2} \tag{21}$$

is nealy zero or the absolute values of all real parts of the roots become roughly the same. The maximum stability of the system will then be obtained.

3. Example—Optimization of the Dutch-Roll Stability of an Airplane

As a numerical example, the optimization problem of the Duch-Roll stability of a light airplane is considered. Since the lateral equations of motion of an airplane are in general expressed by simultaneous linear differential equations in 3 variables, and the characteristic equation is the quartic, then it is exprssed by

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{22}$$

where the coefficients a_i are[2)]

$$\begin{aligned}
a_3 &= -\left(\frac{C_{y\beta}}{2\mu} + \frac{C_{lp}}{i_A} + \frac{C_{nr}}{i_c}\right) \\
a_2 &= \frac{1}{i_A i_c}(C_{nr}C_{lp} - C_{np}C_{lr}) + \frac{C_{n\beta}}{i_c} + \frac{C_{y\beta}}{2\mu}\left(\frac{C_{nr}}{ic} + \frac{C_{lp}}{i_A}\right) \\
a_1 &= \frac{1}{2\mu}\cdot\frac{C_{y\beta}}{i_A i_c}(C_{lr}C_{np} - C_{nr}C_{lp}) + \frac{1}{i_A i_c}(C_{l\beta}C_{np} - C_{n\beta}C_{lp}) \\
&\qquad - \frac{1}{2\mu}\cdot\frac{1}{i_A}C_{L0}C_{l\beta} \\
a_0 &= \frac{1}{2\mu}\cdot\frac{C_{L0}}{i_A i_c}(C_{l\beta}C_{nr} - C_{n\beta}C_{lr})
\end{aligned} \tag{23}$$

For the simplification of calculation, small quantities, e.g. small stability derivatives, product of inertia, etc., are left out in this expression.

According to the expressions described in the previous section, the state variables γ_l are, therefore, as follows:

$$\begin{aligned}
\gamma_1 &= C_{y\beta} & \gamma_7 &= C_{lr} \\
\gamma^2 &= C_{n\beta} & \gamma_8 &= i_A \\
\gamma_3 &= C_{l\beta} & \gamma_9 &= i_c \\
\gamma_4 &= C_{nr} & \gamma_{10} &= \mu \\
\gamma_5 &= C_{lp} & \gamma_{11} &= C_{L0} \\
\gamma_6 &= C_{np} & &
\end{aligned} \tag{24}$$

i.e. $M=11$. The control variables β_k which are freely chosen in the design of an airplane are considered as follows:

$$\begin{aligned}
\beta_1 &= S_V/S_V^* : \text{ vertical tail area}\\
\beta_2 &= l_V/l_V^* : \text{ vertical tail length}\\
\beta_3 &= z_V/z_V^* : \text{ vertical tail height}\\
\beta_4 &= \Gamma/\Gamma^* : \text{ effective dihedral angle}
\end{aligned} \tag{25}$$

i.e. $N=4$. In the above expressions the variable quantities are nondimensionalized for the convenience of the following calculation. By the definition of the state variables, the constraining equations are*

$$\begin{aligned}
g_1 &= \gamma_1-K_1\beta_1-K_1' = 0\\
g_2 &= \gamma_2-K_2\beta_1\beta_2-K_2' = 0\\
g_3 &= \gamma_3-K_3\beta_1\beta_3-K_3'\beta_4-K_3'' = 0\\
g_4 &= \gamma_4-K_4\beta_1\beta_2^2-K_4' = 0\\
g_5 &= \gamma_5-K_5 = 0\\
g_6 &= \gamma_6-K_6 = 0\\
g_7 &= \gamma_7-K_7 = 0\\
g_8 &= \gamma_8-K_8 = 0\\
g_9 &= \gamma_9-K_9 = 0\\
g_{10} &= \gamma_{10}-K_{10} = 0\\
g_{11} &= \gamma_{11}-K_{11} = 0
\end{aligned} \tag{26}$$

where K_1 through K_3'' are assumed to be constants. Strictly speaking, g_5 through g_{11} are the functions of the control variables too, but the influences are considered so small that they are neglected and $\gamma_5, \cdots, \gamma_{11}$ are assumed to be constant in this calculation.

The steepest-descent method described in the last section is, therefore, applied as follows:**

1) control variables (starting conditions)

$$\beta_1^* = S_V/S_V^* = 1 \qquad \beta_3^* = z_V/z_V^* = 1$$
$$\beta_2^* = l_V/l_V^* = 1 \qquad \beta_4^* = \Gamma/\Gamma^* = 1$$

2) equations of constraints

$$\gamma_1^* = -0.157-0.245\,\beta_1^* = -0.402$$
$$\gamma_2^* = -0.055+0.0955\,\beta_1^*\beta_2^* = 0.0405$$

* This example is not so suitable for the case presented in this report, for this example can be computed by the procedure described in the last report. However, since it is easy to understand the procedure given here by this example, it was chosen as an illustrative problem.

** The numerical data of a typical light airplane are employed in this example.

$$\gamma_3^* = 0.0365 - 0.085\,\beta_4^* - 0.034\,\beta_1^*\beta_3^* = -0.0825$$
$$\gamma_4^* = -0.0085 - 0.074\,\beta_1^*\beta_2^{*2} = -0.0825$$
$$\gamma_5^* = -0.490$$
$$\gamma_6^* = -0.0505$$
$$\gamma_7^* = 0.101$$
$$\gamma_8^* = 0.032$$
$$\gamma_9^* = 0.0615$$
$$\gamma_{10}^* = 16.26$$
$$\gamma_{11}^* = 0.405$$

The characteristic equation, Eq. (22), was solved by the use of NEAC-2101 at the Department of Applied Mathematics and Physics, Kyoto University, and 4 roots of the characteristic equation at the starting condition are obtained as follows:

$$\lambda_1 = -0.001277 \ : \ \text{spiral mode}$$
$$\lambda_2 = -0.4828 \ : \ \text{rolling mode}$$
$$\lambda_3, \lambda_4 = -0.02230 \pm 0.1620i \ : \ \text{Dutch-Roll mode}$$

Hence, the real value to be optimized or minimized is

$$n = -0.02230$$

First of all, substituting $\Delta\gamma_1 = -0.040$, $\Delta\gamma_2 = \cdots = \Delta\gamma_{11} = 0$ into the characteristic equation, it was solved again. By the small change of n or Δn,

$$\frac{\Delta n}{\Delta\gamma_1} = \left(\frac{\partial n}{\partial\gamma_1}\right)^* = 0.015$$

The same procedure was repeated for the case of $\Delta\gamma_2, \Delta\gamma_3, \cdots$ respectively, and $\left(\frac{\partial n}{\partial\gamma_2}\right)^*, \cdots$ were obtained as follows:

$$\left(\frac{\partial n}{\partial\gamma_2}\right)^* = -0.0395$$
$$\left(\frac{\partial n}{\partial\gamma_3}\right)^* = -0.091$$
$$\left(\frac{\partial n}{\partial\gamma_4}\right)^* = 0.2315$$

By Eq. (26),

$$\frac{\partial g_i}{\partial\gamma_j} = \begin{cases} 1 : & i = j \\ 0 : & i \neq j \end{cases}$$

and, therefore, by Eq. (8),

$$\mu_1 = -\left(\frac{\partial n}{\partial\gamma_1}\right)^*$$
$$\cdots\cdots\cdots\cdots$$

$$\mu_{11} = -\left(\frac{\partial n}{\partial \gamma_{11}}\right)^*$$

Furthermore, in Eq. (10) $\frac{\partial g_i}{\partial \beta_j}$ are calculated by Eq. (26), i.e.

$$\left(\frac{\partial g_1}{\partial \beta_1}\right)^* = -\frac{\partial \gamma_1}{\partial \beta_1} = 0.245$$

$$\left(\frac{\partial g_2}{\partial \beta_1}\right)^* = -\frac{\partial \gamma_2}{\partial \beta_1} = -0.0955\,\beta_2{}^*$$

....

Employing those results, the following expression was computed,

$$D = \left[\frac{1}{w_1}\left(\mu_1\frac{\partial g_1}{\partial \beta_1}+\cdots+\mu_{11}\frac{\partial g_{11}}{\partial \beta_1}\right)^2+\cdots+\frac{1}{w_4}\left(\mu_1\frac{\partial g_1}{\partial \beta_4}+\cdots+\mu_{11}\frac{\partial g_{11}}{\partial \beta_4}\right)^2\right]^{1/2}$$
$$= 0.04444$$

where $w_1=\cdots=w_4=1$ are the weighting numbers. When $|dP|=0{\cdot}1$,

$$d\beta_1 = -\frac{|dP|}{D}\left[\frac{1}{w_1}\left(\mu_1\frac{\partial g_1}{\partial \beta_1}+\cdots+\mu_{11}\frac{\partial g_{11}}{\partial \beta_1}\right)^2\right] = 0.0483$$

$$d\beta_2 = -\frac{|dP|}{D}\left[\frac{1}{w_2}\left(\mu_1\frac{\partial g_1}{\partial \beta_2}+\cdots+\mu_{11}\frac{\partial g_{11}}{\partial \beta_2}\right)^2\right] = 0.0855$$

$$d\beta_3 = -\frac{|dP|}{D}\left[\frac{1}{w_3}\left(\mu_1\frac{\partial g_1}{\partial \beta_3}+\cdots+\mu_{11}\frac{\partial g_{11}}{\partial \beta_3}\right)^2\right] = -0.0070$$

$$d\beta_4 = -\frac{|dP|}{D}\left[\frac{1}{w_4}\left(\mu_1\frac{\partial g_1}{\partial \beta_4}+\cdots+\mu_{11}\frac{\partial g_{11}}{\partial \beta_4}\right)^2\right] = -0.0174$$

For the second step, the starting values of the control variables are therefore

$$\beta_1 = \beta_1{}^*+d\beta_1 = 1.0483$$
$$\beta_2 = \beta_1{}^*+d\beta_2 = 1.0855$$
$$\beta_3 = \beta_3{}^*+d\beta_3 = 0.9930$$
$$\beta_4 = \beta_4{}^*+d\beta_4 = 0.9826$$

The characteristic equation was solved for those β_k, and the roots are, at the second point,

$$\lambda_1 = -0.001052$$
$$\lambda_2 = -0.4826$$
$$\lambda_3,\ \lambda_4 = -0.02702\pm 0.1798\,i$$

i.e. the real part of the complex roots is

$$n = -0.02702$$

that is, the stability of the Dutch-Roll mode was augmented about 21% by one step of the calculation.

The same procedure was repeated several times, and the results are shown in Table 1 and Fig. 1. The figure shows clearly that the stability of the Dutch-Roll mode is augmented remarkably by the application of this method, i.e. the damping or the time to half amplitude at the starting point is 3.12 sec. in this example, but it is 1.55 sec. after 5 times of iteration of the computation. The combination of control variables corresponding to this last damping mode is as follows:

$$S_V/S_V^* = 1.196 \qquad z_V/z_V^* = 0.975$$

$$l_V/l_V^* = 1.342 \qquad \Gamma/\Gamma^* = 0.942$$

That is, it is found that the vertical tail area and the vertical tail length should be larger and the effective dihedral angle should be smaller than the original configuration.

Table 1.

	Control variables	Characteristic roots	Dutch-Roll mode	
			Time to ½ amplitude	Period
(1)	$\beta_1=S_V/S_V^*=1$ $\beta_2=l_V/l_V^*=1$ $\beta_3=Z_V/Z_V^*=1$ $\beta_4=\Gamma/\Gamma^*=1$	$\lambda_1=-0.001277$ $\lambda_2=-0.4828$ $\left.\begin{matrix}\lambda_3\\\lambda_4\end{matrix}\right\}=-0.02230\pm0.1620i$	(sec.) 3.12	(sec.) 3.92
(2)	$\beta_1=1.0483$ $\beta_2=1.0855$ $\beta_3=0.9930$ $\beta_4=0.9826$	$\lambda_1=0.001052$ $\lambda_2=-0.4826$ $\left.\begin{matrix}\lambda_3\\\lambda_4\end{matrix}\right\}=-0.02702\pm0.1798i$	2.58	3.53
(3)	$\beta_1=1.0970$ $\beta_2=1.1712$ $\beta_3=0.9870$ $\beta_4=0.9680$	$\lambda_1=-0.000950$ $\lambda_2=-0.4827$ $\left.\begin{matrix}\lambda_3\\\lambda_4\end{matrix}\right\}=-0.03229\pm0.1986i$	2.16	3.19
(4)	$\beta_1=1.1463$ $\beta_2=1.2569$ $\beta_3=0.9810$ $\beta_4=0.9542$	$\lambda_1=-0.000909$ $\lambda_2=-0.4826$ $\left.\begin{matrix}\lambda_3\\\lambda_4\end{matrix}\right\}=-0.03823\pm0.2159i$	1.82	2.94
(5)	$\beta_1=1.1963$ $\beta_2=1.3424$ $\beta_3=0.9753$ $\beta_4=0.9419$	$\lambda_1=-0.000903$ $\lambda_2=0.4826$ $\left.\begin{matrix}\lambda_3\\\lambda_4\end{matrix}\right\}=-0.04485\pm0.2327i$	1.55	2.73

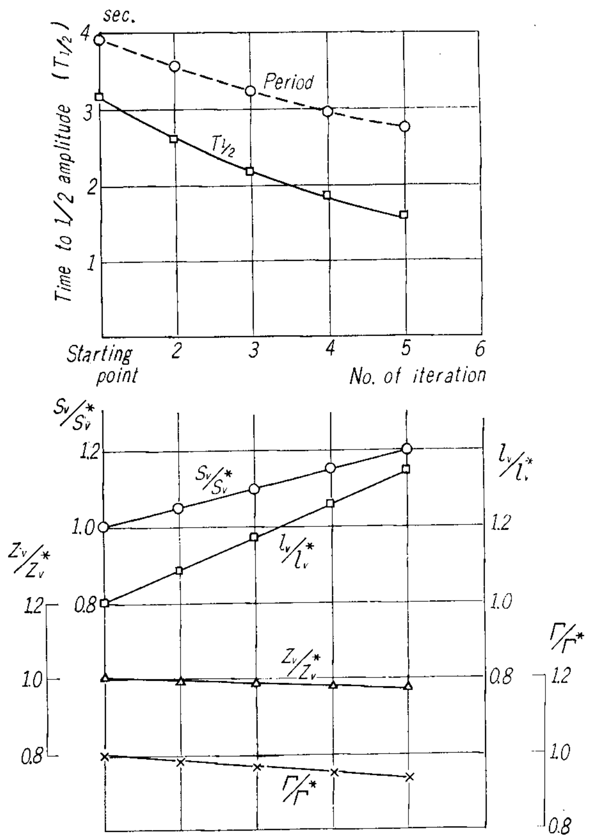

Fig 1. An example of steepest-descent computation.

4. Summary

A numerical method for optimizing the stability of a linear system subject to subsidiary conditions is presented. The practical procedure is described for the problems in which the subsidiary conditions are given as constraining equations in control and state variables. The process is an application of the "steepest-descent method" too, and is similar to that of the last report. Consequently, the procedure described in the last report is involved as a simple case in which the functions are given analytically and the set of constraining equations can be solved for the state variables in terms of the control variables.

A numerical example is presented for maximizing the damping of the Dutch-Roll mode of an airplane, and it is found that, to obtain the better stability of this mode, the vertical tail area and the vertical tail length should be larger and the effective dihedral angle should be smaller than the original configuration.

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Nomenclature

a_i : coefficients of characteristic equations
C_{L0} : lift coefficient (steady state)
$C_{y\beta}, C_{n\beta}, C_{l\beta}$, etc.: aerodynamic stability derivatives
i_A, i_c : moments of inertia about X– and Z–axis
l_V : vertical tail length
$N_{1/2}$: cycle to $\frac{1}{2}$ amplitude
S_V : vertical tail area
$T_{1/2}$: time to $\frac{1}{2}$ amplitude
w_k : weighting numbers
z_V : vertical tail height
β_k : control variables
Γ : dihedral angle
γ_l : state variables
λ_j : roots of characteristic equations ($\lambda_j = n_j \pm i\omega_j$)
μ : relative density factor
μ_l, ν : Lagrange multipliers

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Appendix—On the Stability Criterion of a Linear Dynamic System, Especially of the Oscillatory Mode

In general, the values of the real part of the characteristic roots are taken as the stability criterion of a linear system, i.e. when the roots of the characterisitc equation are expressed by $\lambda_j = n_j \pm i\omega_j$ (if the roots are real, $w_j = 0$) and $|n_1| > |n_2|$ or $n_1 < n_2$, the motion corresponding to n_1 is considered more stable than that of n_2. The analysis described in this report is dependent upon this criterion, too.

This criterion is quite right in the case of aperiodic mode of motion, but in the case of oscillatory mode another criterion is sometimes considered more reasonable. More specifically, since the real part of the characteristic root is related to the time to $\frac{1}{2}$ amplitude, i.e.

$$T_{1/2} = \frac{0.69}{|n|} \text{ sec.}$$

then large $|n|$ corresponds to small $T_{1/2}$, and the disturbed motion will disappear more quickly. This situation is the same in the case of the oscillatory mode as well, but for the oscillatory motion the frequency ω should be taken into consideration together with n.

For instance, by a flight test it is reported that, even if the time to $\frac{1}{2}$ amplitude is smaller, when the frequency ω is larger simultaneously the airplane motion is not recognized as more stable. This means that pilots, crews or passengers are so sensitive to the acceleration of motion that the cycle to $\frac{1}{2}$ amplitude $N_{1/2}$ is important as well as the time to $\frac{1}{2}$ amplitude $T_{1/2}$.

Consequently, in such a special case as the airplane motion, we suggest that the cycle to $\frac{1}{2}$ amplitude is the more reasonable criterion of the system's stability. More specifically, since

$$N_{1/2} = 0.110 \frac{\omega}{|n|}$$

then $\frac{\omega}{|n|}$ should be the smallest in order to minimize $N_{1/2}$ or optimize the stability. When the frequency ω is constant, this criterion is identical with the ordinary one, because $|n|$ should still be maximized.