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On the Flexural Rigidity of Reinforced Concrete Beams

By

Hiroshi MUGURUMA* and Shiro MORITA*

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It is general that the ordinary straight-line theory does not meet with success for evaluating the flexural rigidity of reinforced concrete beams. The co-operation of concrete between two successive cracks with the tensile reinforcement should be taken into account for its exact calculation. In this study, from the tensile loading test results on axially reinforced concrete prisms the mechanical behaviour of co-operation between concrete and reinforcement in tensile zone of beam is primarily discussed; and the behaviour discussed is introduced into the development of the theoretical equation of flexural rigidity of cracked beams. The theory developed was verified by the experiments on simply supported reinforced concrete beams. Furthermore, some prospective way of clarifying the opening-up of crack width as well as the spacing between two successive cracks of beams is also discussed on the basis of tensile behaviour of axially reinforced concrete prisms.

Introduction

Recently, high-strength steel is increasingly used as concrete reinforcement with higher allowable stresses than those for mild steel. The use of higher allowable stresses for tensile reinforcement results in larger deflection as well as larger opening width of cracks of beam under the service load. Therefore, the design procedure of checking the deflections and the crack width must become weighty to keep them in the limited values at service loads.

For calculating the deflection of reinforced concrete beams, it is essential to know the relationship between applied moment and flexural rigidity of the section. There are many theoretical or experimental studies on this problem,[1][2][3][4] but no sufficient results are obtained from them. In this study, the concept of the effective tensile reinforcement, such as some portion of the concrete around the tensile reinforcement is effective to resist the tensile force even after beam cracking, was introduced for the evaluation of bonding effects and the practical theory for calculating the flexural rigidity was derived.

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The confirmation of proposed theory was also carried out by means of the flexural tests on several rectangular beams.

1. General Procedure for Calculating Moment-Rigidity Relationship.

Before the formation of cracks, the flexural rigidity of a reinforced concrete beam section can be calculated reasonably by using whole concrete sectional area with transformed area of steel reinforcement. The equation can be written as the form

$$EI = E_c I_c + E_s I_s \tag{1}$$

After cracking, it decreases gradually with increase of applied bending moment and is essentially affected by the elasto-plastic behaviour of concrete in the compression zone as well as the magnitude of co-operation of concrete between two successive cracks with tensile reinforcement. Fig. 1 shows the longitudinal strain distributions in the beam subjected to pure bending moment. At the cracked section the whole tensile force is carried by the tensile reinforcement, because the contribution of concrete between the neutral axis and the interior end of crack is so small as to be negligible. At the uncracked section between two successive cracks the tensile force is carried by both concrete and reinforcement due to the bond between them. Consequently, as shown in Fig. 1 the tensile force in reinforcement becomes maximum at the section through crack, and minimum at the center between two successive cracks. Of course, the compressive stress in concrete at the extreme fibre as well as the level of neutral axis also varies in correspondence to the transition of tensile force in reinforcement. And the hypothesis that a plane section remains plane after bending can be reasonably applied to the average strains of concrete and reinforcement and not to the actual strains of them shown in Fig. 1. Thus, the apparent flexural rigidity of beam section after cracking should be defined by

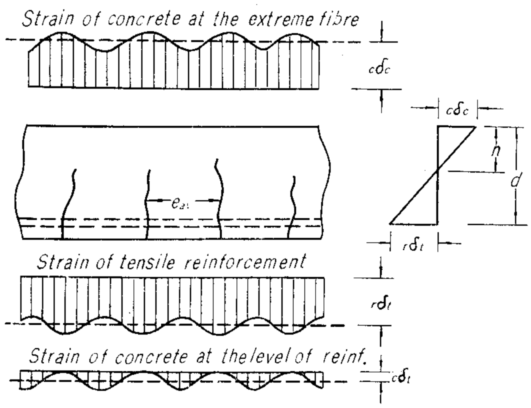


Fig. 1. Longitudinal strain distributions in reinforced concrete beam.

$$EI = \frac{M}{\varphi} = \frac{d}{{}_r\delta_t + {}_c\delta_c} M \tag{2}$$

The values of average strains ${}_r\delta_t$ and ${}_c\delta_c$ for given moment can be solved from two stress equilibrium equations, the equilibrium of inner forces and the equilibrium of inner and applied moments. Denoting the compressive stress-strain relation of concrete and the nominal stress-strain relation of tensile reinforcement embedded in concrete by

$${}_c\sigma_z = f({}_c\sigma_z) \tag{3}$$

and

$${}_r\sigma_t = g({}_r\delta_t) \tag{4}$$

respectively, two equilibrium equations for the rectangular section shown in Fig. 2 can be written as follows:

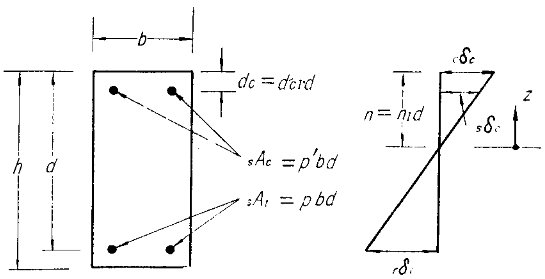


Fig. 2. Strain distribution in rectangular reinforced concrete beam section.

$$\frac{nb}{{}_c\delta_c} \int_0^{{}_c\delta_c} f({}_c\delta_z)\, d\,{}_c\delta_z + p'bd\,{}_s\delta_c E_s = pbd\,{}_r\sigma_t \tag{5}$$

$$M = b \int_0^n f({}_c\delta_z) z dz + \frac{(n-d_c)^2}{d-n} {}_r\delta_t E_s p'bd + pbd(d-n) \cdot g({}_r\delta_t) \tag{6}$$

where,

$$n = \frac{{}_c\delta_c}{{}_r\delta_t + {}_c\delta_c} d \tag{7}$$

In the derivation of Eqs. (5) and (6), the stress in the compressive reinforcement is assumed as to still remain in the elastic range. If the stress-strain relations generally represented by Eqs. (3) and (4) are surely known, the flexural rigidity of beam section can be analysed by the above mentioned method even if the cracks whould occur in the beam.

2. Stress-strain Relationships of Concrete and Tensile Reinforcement Embedded in Concrete.

Many equations have been proposed for the stress-strain curve of concrete subjected to compression; for example, parabola, cubic parabola and exponential function, etc. In this study, the cubic parabolic equation is used.

$$_c\sigma_z = E_0\,{}_c\delta_z + B_c\delta_z^2 + C_c\delta_z^3 \tag{8}$$

where, E_0 is the initial tangent modulus and B and C are the constants to be determined from the tests.

The reason for choosing this equation is that the cubic parabola having the simple form can be easily integrated, and also the ascending portion of the stress-strain curve can be represented by it more exactly than by any other forms. The descending part of the stress-strain curve is not necessary in this study, because the main purpose of this study is to discuss the moment rigidity relationship before the beam reaches its yielding moment, and also for the beam adequately designed it is common that at beam yielding the compressive fibre strain of concrete is smaller than or equal to the strain corresponding to its compressive strength.

For evaluating the stress-strain relation of tensile reinforcement, the cooperation of concrete between two successive cracks should be considered. For this purpose, it is assumed that after the formation of cracks some portion of concrete around the tensile reinforcements still resists effectively against the tensile force, that is, the whole tensile force in tensile zone is carried by the tensile reinforced concrete member consisting of the tensile reinforcement and the effective area of concrete around it. Such an effective area of concrete is determined so as to have the same centroid as the reinforcement as shown in Fig. 3. Hereafter, such a tensile reinforced concrete member is called the effective tensile reinforcement in this study.

According to the axially loading test results, the relation between nominal stress and average strain of effective tensile reinforcement is given by the

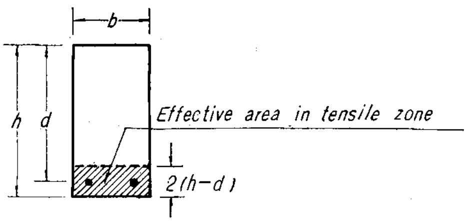


Fig. 3. Assumption of effective tensile reinforcement.

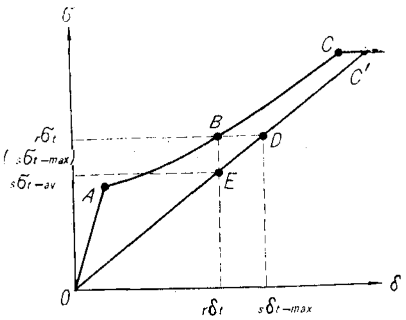


Fig. 4. Stress-strain relation of effective tensile reinforcement.

curve OABC in Fig. 4, where the nominal stress is defined as the stress that the whole tensile force is assumed to be carried by the reinforcement only. Before cracking, the tensile force is carried by both concrete and reinforcement After the initial crack occurs at Point A, the effective tensile reinforcement. shows the elasto-plastic behavior due to the rapid increase of strain, and the stress distribution in embedded reinforcement becomes similar to that in the beam, that is, as shown in Fig. 1 the maximum tensile stress occurs at the cracked section and the minimum at the center of two successive cracks. The stress-strain curve of the reinforcement itself is also illustrated in Fig. 4 (Curve OEDC′). From Fig. 4 it is apparent that Point D corresponding to Point B gives the maximum stress and strain in embedded reinforcement which occur at the cracked section and also Point E the average stress and strain of embedded reinforcement itself.

On the other hand, the difference between the maximum and minimum stresses actually occurring in the embedded reinforcement, ${}_s\sigma_{t-\max}-{}_s\sigma_{t-\min}$, has direct relation to the bonding effect between concrete and reinforcement. However, it is difficult to measure the maximum difference, ${}_s\sigma_{t-\max}-{}_s\sigma_{t-\min}$, directly from the actual tests on effective tensile reinforcement, while the difference between the maximum and the average stresses in reinforcement, ${}_s\sigma_{t-\max}-{}_s\sigma_{t-av}$, can be obtained from the test results. So, in this study the following relation is assumed between them.

$${}_s\sigma_{t-\max}-{}_s\sigma_{t-av} = k_1({}_s\sigma_{t-\max}-{}_s\sigma_{t-\min}) \tag{9}$$

According to the paper presented by A. Johnson, the coefficient depends on the bond characteristics between concrete and reinforcement, and varies from 1/3 to 2/3. Of course, the longitudinal tensile stress in concrete at the middle cross section between two successive cracks will be either less than or equal to the tensile strength of concrete. Let k_2 be the co-efficient less than or equal to unity.

$${}_sA_t({}_s\sigma_{t-\max}-{}_s\sigma_{t-\min}) = k_2\cdot{}_c\sigma_{tB}\cdot{}_cA_t \tag{10}$$

Hence, from Eqs. (9) and (10), we obtain

$${}_s\sigma_{t-\max}-{}_s\sigma_{t-av} = k_1k_2\frac{{}_c\sigma_{tB}}{p_r} \tag{11}$$

The maximum tensile stress ${}_s\sigma_{t-\max}$ is equal to the nominal stress ${}_r\sigma_t$ defined previously. So, Eq. (11) becomes

$${}_r\sigma_t = E_s\cdot{}_r\delta_t+k_1k_2\frac{{}_c\sigma_{tB}}{p_r} \tag{12}$$

If the reasonable value of k_1k_2 can be obtained, the flexual behavior of cracked beam can be analysed reasonably by applying the relation of Eq. (12) to the tensile reinforcement. As mentioned after, the value of k_1k_2 can be obtained exactly from the axially loading test results on the effective tensile reinforcement.

3. Proposed Method for Calculating Flexural Rigidity of Cracked Beam.

Based on the stress-strain relations discussed in the previous section, the flexural rigidity on cracked beam can be analysed reasonably. Using the expressions of Eqs. (8) and (12) instead of Eqs. (3) and (4), respectively, the equilibrium equations (5) and (6) become.

$$\frac{1}{{}_r\delta_t+{}_c\delta_c}\left(\frac{E_0}{2}{}_c\delta_c^{\,2}+\frac{B}{3}{}_c\delta_c^{\,3}+\frac{C}{4}{}_c\delta_c^{\,4}\right)+p'E_s\left\{{}_c\delta_c-d_{c1}({}_r\delta_t+{}_c\delta_c)\right\}$$
$$=p\left(E_s\cdot{}_r\delta_t+k_1k_2\frac{{}_c\sigma_{tB}}{p_r}\right) \tag{13}$$

and

$$\frac{M}{bd^2}=\frac{E_0}{3}\left(\frac{{}_r\delta_t}{1-n_1}\right)n_1^{\,3}+\frac{B}{4}\left(\frac{{}_r\delta_t}{1-n_1}\right)^2 n_1^{\,4}+\frac{C}{5}\left(\frac{{}_r\delta_c}{1-n_1}\right)^3 n_1^{\,5}$$
$$+p'E_s\frac{(n_1-d_{c1})^2}{1-n_1}{}_r\delta_t+p\left(E_s\cdot{}_r\delta_t+k_1k_2\frac{{}_c\sigma_{tB}}{p_r}\right)(1-n_1) \tag{14}$$

Thus, substituting the values of ${}_c\delta_c$ and ${}_r\delta_t$ obtained from Eqs. (13) and (14) into Eq. (2), the apparent flexural rigidity can be calculated theoretically. The same procedure is applicable to T beam or others.

The typical moment-flexural rigidity relation calculated by above mentioned method is shown in Fig. 5. The initial rigidity (the amplitude of Point 0 in Fig. 5) can be evaluated from Eq. (1), where the initial tangent modulus of concrete should be used as the value of E_c. Point A denotes the critical rigidity at which the whole of the concrete and reinforcement sections still remains effective against the applied moment just before the initial crack occurs in the beam. The curve A'BC represents the result obtained from Eqs. (2), (13) and (14). Point A' gives the flexural rigidity at the apparent initial cracking moment which can be calculated by the use of nominal stress-strain relation of effective tensile reinforcement given in Fig. 4. The portion of A'B is an imaginary part

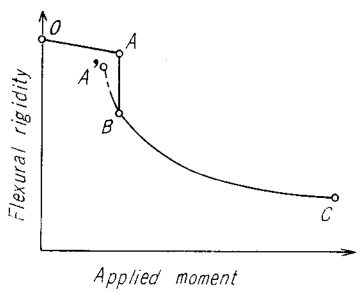


Fig. 5. Schematic moment-rigidity relation of reinforced concrete beam section.

caused by the assumption of effective tensile reinforcement. Also Point C represents the flexural rigidity at the beam yielding. Thus, the moment-flexural rigidity relation is illustrated by the curve OABC in Fig. 5.

4. Experimental Verification.

4.1. Tensile tests on axially reinforced concrete prisms.

For the purpose of obtaining the value of k_1k_2 in Eq. (12), tensile tests were performed on the axially reinforced concrete prisms shown in Fig. 6. The test specimens having the square cross section of three different dimensions were reinforced by one ♯5 (16 mm dia.) plain or deformed bar and two specimens were provided for each type of test prisms.

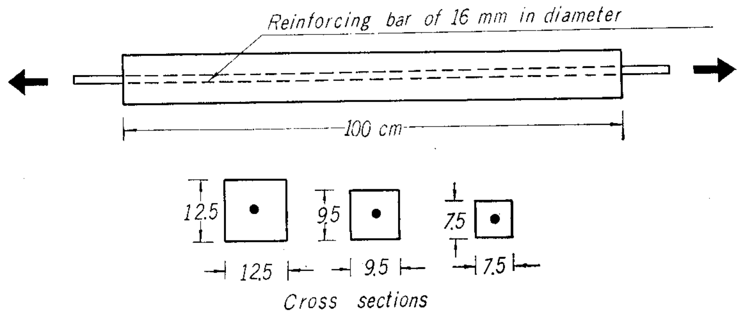


Fig. 6. Axially reinforced concrete prism specimens.

All specimens had been cured in the room at the relative humidity of about 100 per cent. Prior to the tests, they were removed in the laboratory and cured in natural air during two days. The tests were carried out at the age of 28 days. The compressive and tensile strengths of concrete were 322 kg/cm^2 and 28.5 kg/cm^2, respectively. The tensile load was applied to the specimen by pulling the reinforcing bar extending from both ends of specimen and the elongation of the embedded portion of the bar was measured by the special deformator.

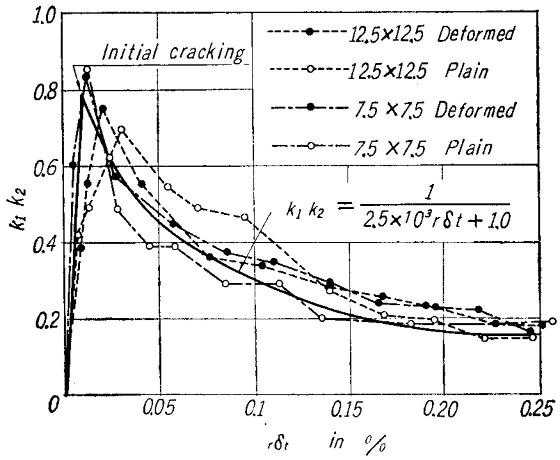


Fig. 7. Typical $k_1k_2 - {}_r\delta_t$ curves.

From the measurements the values of k_1k_2 were calculated. Typical results are illustrated in Fig. 7. From Fig. 7 it can be concluded that the $k_1k_2 - {}_r\delta_t$ relation does not depend on the type of reinforcement as well as the dimension of cross section of the specimen. Thus, the following empirical relation between the value of k_1k_2 and the average steel strain ${}_r\delta_t$ was obtained from the results after the formation of cracks.

$$k_1k_2 = \frac{1}{2.5\times 10^3\,{}_r\delta_t + 1} \tag{15}$$

The exact measuring of the average strain ${}_r\delta_t$ at initial cracking is very difficult in this axially loading test. Based on the previous studies,[5)6)] this can be presumed as 0.01% for the prism specimen subjected to pure tension. So, in this study the concrete tensile strain at initial cracking was assumed as 0.01% in the derivation of Eq. (15). From Eq. (15) the value of k_1k_2 at initial cracking, that is, the value coresponding to ${}_r\delta_t = 0.01\%$ becomes 0.8. Theoretically, it should be taken as 1.0. The difference between them seems to be caused by the facts that the tensile strength of concrete prism differs from that obtained by Brazilian test on cylinder specimen and the inner tensile stress takes place before loading in the concrete of prism specimen due to the shrinkage of concrete.

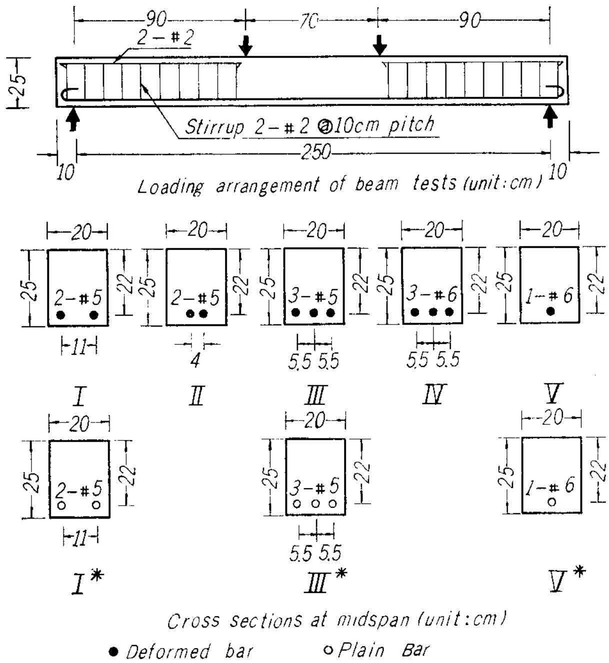


Fig. 8. Details of beam specimens.

4. 2. Bending tests of reinforced concrete beams.

Bending tests weie carried out on eight rectangular beams shown in Fig. 8, and the measured flexural rigidity for each beam was compared to the theoretical one calculated by the proposed method.

The beams were cast by concrete having the mix proportion of 1:2.35:3.52 by weight and the water-cement ratio of 58%. After casting, they were wet-cured until the age of loading test. At the age of 28 days the flexural loading tests were carried out by the hydraulic machine. The method of loading is shown in Fig. 8. The test results of the beams as well as the properties of materials used, are summarized in Table 1.

Table 1. Summary of beam test results.

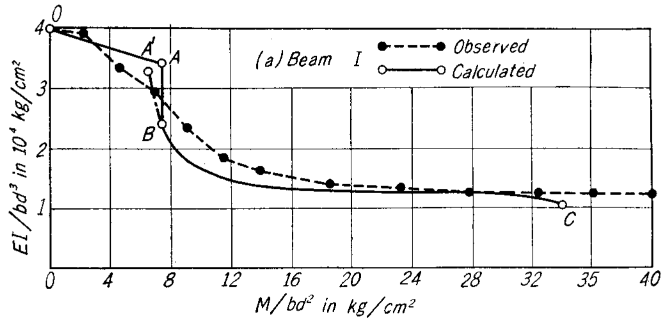
Beam Designation	Concrete					Reinforcement					Crack Moment t.m		Yield Moment t.m		Ultimate Moment t.m	
	$_c\sigma_{cB}$ kg/cm2	$_c\sigma_{tB}$ kg/cm2	E_0 $\times 10^5$ kg/cm2	B $\times 10^8$ kg/cm2	C $\times 10^{10}$ kg/cm2	No. & Size	type	p %	pr %	$_s\sigma_y$ kg/cm2	test	calc.	test	calc.	test	calc.
I	283	27.3	3.00	0.714	0.437	2-#5	Deform	0.90	3.30	4390	1.13	0.73	3.99	3.30	4.05	3.56
I*	283	27.3	3.00	0.714	0.437	2-#5	Plain	0.91	3.35	3410	0.90	0.73	3.37	2.62	3.37	2.85
II	255	26.3	2.50	0.465	0.760	2-#5	Deform	0.90	3.30	4500	0.68	0.68	3.96	3.42	4.03	3.63
III	283	27.3	3.00	0.714	0.437	3-#5	Deform	1.35	4.95	4450	0.72	0.80	5.76	5.13	5.76	5.20
III*	283	27.3	3.00	0.714	0.437	3-#5	Plain	1.37	5.03	3370	0.90	0.80	4.43	3.86	4.52	4.45
IV	255	26.3	2.50	0.465	0.760	3-#5	Deform	1.84	7.13	4660	1.28	0.82	7.38	7.15	7.38	7.24
V	227	21.3	2.50	0.800	0.575	1-#6	Deform	0.65	2.37	4700	0.45	0.56	2.70	2.62	2.93	2.77
V*	227	21.3	2.50	0.800	0.575	1-#6	Plain	0.65	2.28	3080	0.54	0.56	1.80	1.74	1.91	1.84

4. 3. Comparison between the measured and the theoretical flexural rigidity.

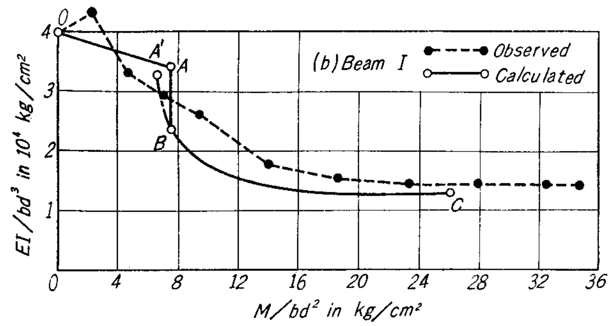
To obtain the flexural rigidity of beam section, the strain distribution of concrete at the compressive extreme fibre and that at the level of tensile reinforcement within the whole pure flexural span length of 70 cm were measured by the contact type strain meter in gauge length of 10 cm. From the measured strains the flexural rigidity was calculated. The typical results are shown in Fig. 9. In Fig. 9, the theoretical rigidity curves are also plotted by solid lines.

The curve A'BCD in Fig. 9 can be obtained from Eqs. (13), (14) and (2), where the value of k_1k_2 given by the empirical equation (15) was used. Also, the tensile strength of prism specimen was assumed to be 0.8. times of that obtained from the Brazilian tests on $\phi 15\times 30$ cm cylinder specimens. Additionally, in calculation of the theoretical rigidity at beam cracking (Point A in Fig. 9) the tensile stress-strain curve of concrete shown in Fig. 10 was used.

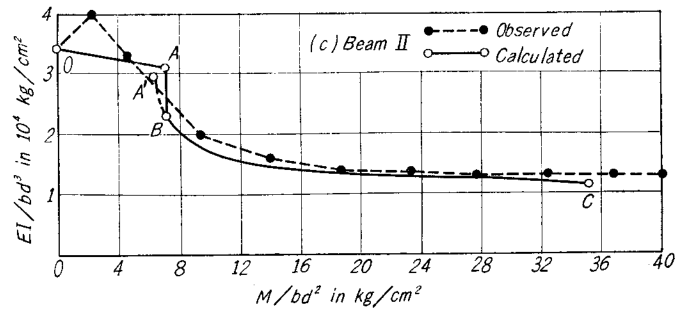
(a)


(a) Beam I
Observed
Calculated
O
A'
A
B
C
EI/bd^3 in 10^4 kg/cm^2
M/bd^2 in kg/cm^2
0 4 8 12 16 20 24 28 32 35 40
1 2 3 4

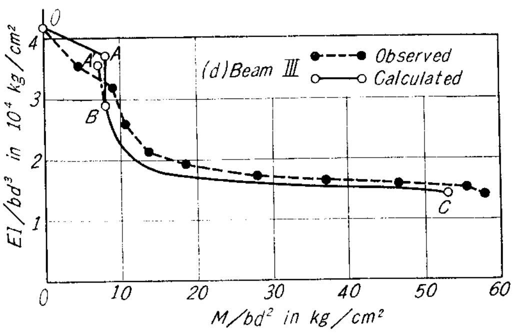
(b)


(b) Beam I
Observed
Calculated
O
A'
A
B
C
EI/bd^3 in 10^4 kg/cm^2
M/bd^2 in kg/cm^2
0 4 8 12 16 20 24 28 32 36
1 2 3 4

(c)


(c) Beam II
Observed
Calculated
O
A'
A
B
C
EI/bd^3 in 10^4 kg/cm^2
M/bd^2 in kg/cm^2
0 4 8 12 16 20 24 28 32 36 40
1 2 3 4

(d)


(d) Beam III
Observed
Calculated
O
A'
A
B
C
EI/bd^3 in 10^4 kg/cm^2
M/bd^2 in kg/cm^2
0 10 20 30 40 50 60
1 2 3 4

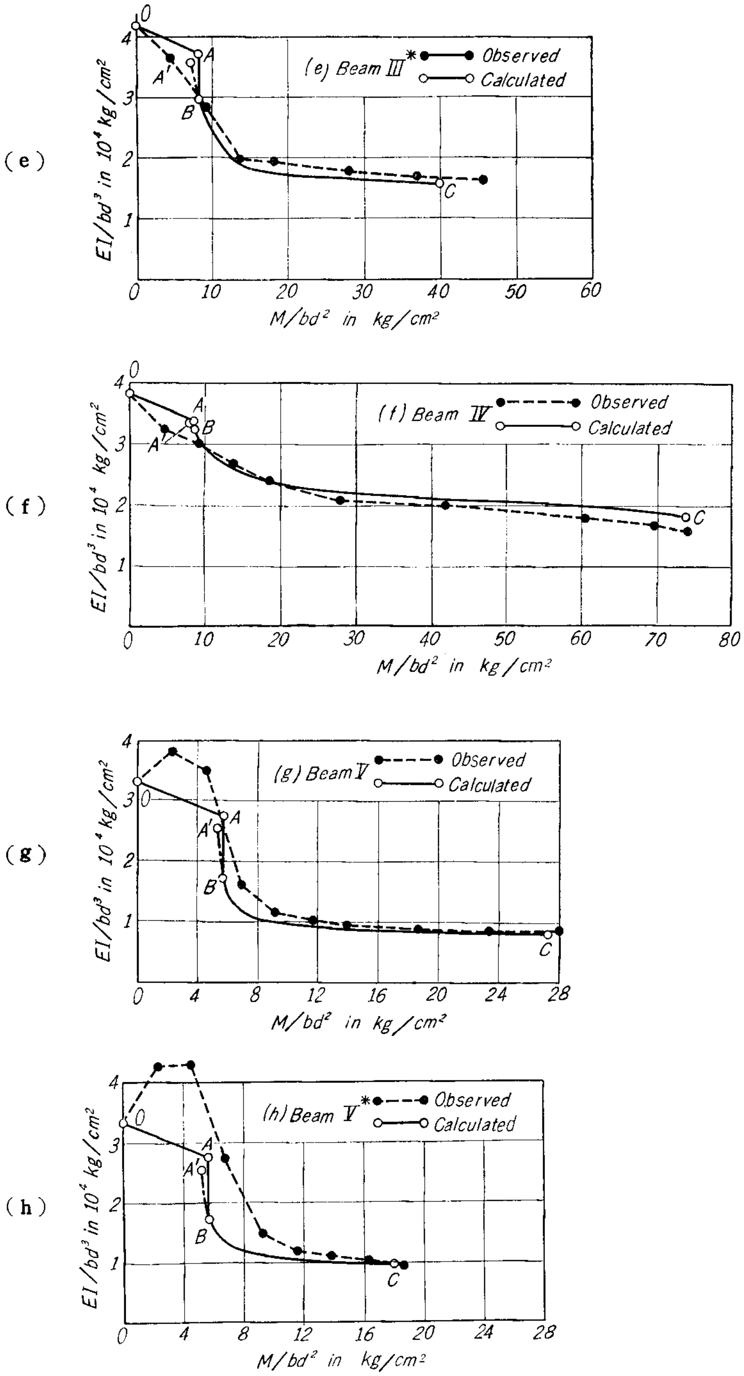


Fig. 9. Comparisons between calculated and observed moment-rigidity relationships $(a)\sim(h)$.

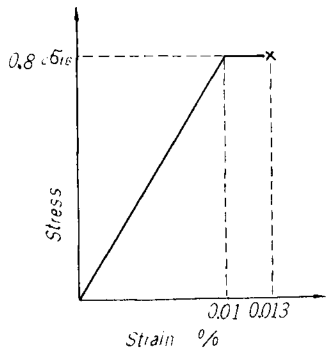


Fig. 10. Assumption of tensile stress-strain curve of concrete.

In Fig. 9 no obvious differences in the moment-rigidity relationship were observed in the tests on the beam reinforced by deformed bars and corresponding that by plain bars, which can be predicted from the axially loading test results on reinforced concrete prisms described in Section 4.1. Also, it is apparent from the results on Beams I and II that there are no significant influences of the horizontal distance between two tensile reinforcements upon the flexural rigidity.

Comparison between theoretical and experimental flexural rigidities showed that the former closed fairly well to the latter, excepting in the vicinity of the initial cracking moment. Thus, it can be concluded that the reasonable prediction of flexural rigidity of cracked beam can be done by the method proposed in this study.

In Fig. 11, the measured average strain of reinforcement and that of concrete at the extreme fibre in Beam I are plotted against the applied moment. The calculated value from the theory proposed in this study and that from the ordinary straight-line theory are also shown in Fig. 11. It is evident from Fig. 11 that after initial cracking the curves obtained from the ordinary straight-line theory differ from the measured curves, and so, the contribution of concrete in tensile zone should be taken into account in calculating the flexural rigidity of reinforced concrete beam.

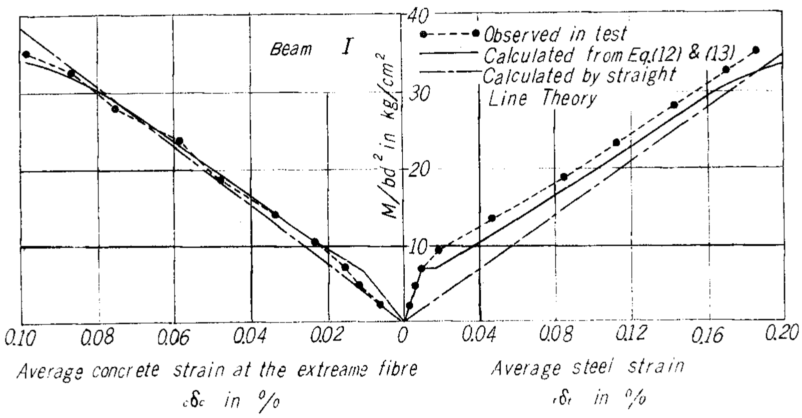


Fig. 11. Comparison between calculated and observed average strains.

5. Spacing and Opening Width of Crack.

5.1. Theoretical equations.

The spacing and the opening width of crack are also affected by the bond between the tensile reinforcement and the concrete around it. Applying the assumption of effective tensile reinforcement to the analysis, the relation between the average spacing of crack and the average bond stress can be written by following simple form.

$$e_{av} = \frac{2\,{}_sA_t({}_s\sigma_{t-\max} - {}_s\sigma_{t-\min})}{\tau_{av}\cdot\phi} = \frac{D({}_s\sigma_{t-\max} - {}_s\sigma_{t-\min})}{2\cdot\tau_{av}} \tag{16}$$

Substituting Eq. (10) into Eq. (16),

$$e_{av} = \frac{k_2}{2}\cdot\frac{D}{p_r}\cdot\frac{{}_c\sigma_{tB}}{\tau_{av}} = \frac{1}{2k_1}k_1k_2\frac{D}{p_r}\cdot\frac{{}_c\sigma_{tB}}{\tau_{av}} \tag{17}$$

Also, the average width of crack can be obtained from

$$w_{av} = e_{av}({}_r\delta_t - {}_c\delta_t) \tag{18}$$

From Eq. (12), we obtain

$${}_r\delta_t = \frac{{}_r\sigma_t - k_1k_2\,{}_c\sigma_{tB}/p_r}{E_s}$$

and

$${}_c\delta_t = \frac{k_1k_2\,{}_c\sigma_{tB}}{E_c}$$

Thus, Eq. (18) becomes

$$w_{av} = e_{av}\left(\frac{{}_r\sigma_t - k_1k_2\,{}_c\sigma_{tB}/p_r}{E_s} - \frac{k_1k_2\,{}_c\sigma_{tB}}{E_c}\right) \tag{19}$$

The second term in the bracket of Eq. (19) is so small as to be negligible in comparison with the first one. Then, neglecting the second term in the bracket, Eq. (19) becomes

$$w_{av} = \frac{e_{av}}{E_s}\left({}_r\sigma_t - \frac{k_1k_2\,{}_c\sigma_{tB}}{p_r}\right) \tag{20}$$

For the rectangular beam section shown in Fig. 3, p_r is given by

$$p_r = \frac{{}_sA_t}{{}_cA_t} = p\cdot\frac{d}{2(h-d)} \tag{21}$$

Substituting Eq. (21) into Eq. (20),

$$w_{av} = \frac{k_1k_2}{k_1E_s}\cdot\frac{{}_c\sigma_{tB}}{\tau_{av}}\left(\frac{h-d}{d}\right)\frac{D}{p}\left[{}_r\sigma_t - \frac{2k_1k_2\,{}_c\sigma_{tB}}{p}\left(\frac{h-d}{d}\right)\right] \tag{22}$$

Eq. (22) is quite similar to the experimental equation obtained by A. Clark,[7] that is,

$$w_{av} = 2.27\times10^{-8}\left(\frac{h-d}{d}\right)\frac{D}{p}\left[f_s-56.6\left(\frac{1}{p}+\frac{E_s}{E_r}\right)\right] \text{ (in inches)} \qquad (23).$$

where D is the diameter of reinforcement in inches and f_s the steel stress in psi. calculated from the ordinary straight-line theory, which can be assumed to be approximately equal to the stress ${}_r\sigma_t$.

5. 2. Comparison between theory and experiments.

The theoretical equations discussed above were compared with the beam test results described in Section 4. 2. For calculating the spacing and opening width of crack by Eqs. (17) and (22), the values of k_1, k_1k_2 and ${}_c\sigma_{tB}/\tau_{av}$ are necessary. The value of k_1k_2 can be obtained from tensile test results on the effective tensile reinforcement itself, which is given by Eq. (15) in this study. However, the remainings can not be obtained directly from the test results. When the load is applied to the reinforced concrete beam, a few flexural cracks take place initially in the pure flexural span and thereafter several cracks follow them with increase of applied load. After the tensile stress in the reinforcement reaches a critical value, any additional cracks do not occur even if the tensile stress in the reinforcement becomes larger. Of course, the same phenomena are also recognized in the axially loading tests on the effective tensile reinforcement. At such critical stage, the coefficient k_1 can be assumed to 0.5. If assumed as

$$k_1 = 0.5 \qquad (24),$$

the corresponding value of ${}_c\sigma_{tB}/\tau_{av}$ can be reduced from the tests on the effective tensile reinforcement itself. In the tests on effective tensile reinforcement described in Section 4. 1, the specimens were regarded as reaching such critical stage when the average steel strain of tensile reinforcement ${}_r\delta_t$ became 0.1 to 0.15%, which corresponds to the tensile stress of 2,000 to 3,000 kg/cm² in tensile reinforcement. And the corresponding values of ${}_c\sigma_{tB}/\tau_{av}$ obtained are as follows:

$$\left.\begin{array}{ll}\text{For the deformed bar} & \dfrac{{}_c\sigma_{tB}}{\tau_{av}} = 1.0 \\ \text{For the plain bar} & \dfrac{{}_c\sigma_{tB}}{\tau_{av}} = 1.2\end{array}\right\} \qquad (25).$$

Using Eqs. (24) and (25) as well as Eq. (15), the crack spacing of the beam at the critical stage can be presumed from Eq. (17). In the beam test performed in this study, the stresses in the tensile reinforcement at the critical stage were approximately 3,000 kg/cm². The corresponding theoretical crack spacing for eath beam calculated by Eq. (17) is listed in Table 2. In Table 2, the observed crack spacings are also summarized.

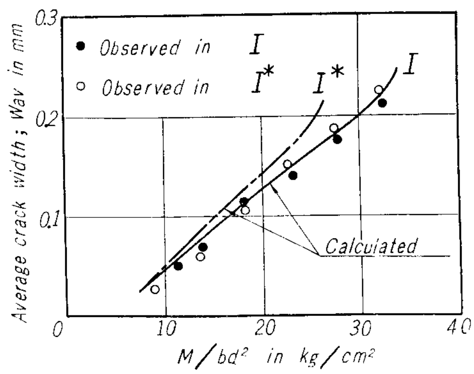
Table 2. Comparison between calculated and observed average crack spacing.

Beam Designation		I	I*	II	III	III*	IV	V	V*
p_r (%)		3.30	3.35	3.30	4.95	5.03	7.13	2.37	2.38
e_{av} (cm)	Caluculated	12.1	14.5	12.1	8.0	9.6	6.7	19.8	23.3
	Observed	12.0	13.5	12.0	11.0	12.0	10.6	21.0	16.6

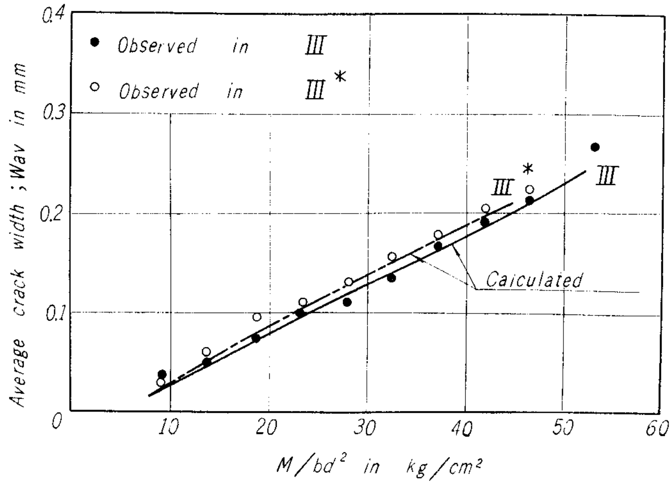
It can be seen from Table 2 that in Beams I, I*, II and V the calculated crack spacings agreed fairly well with those observed and in the other beams deviated considerably. Such considerable differences between the theory and the test results will be caused by the fact that the experimental values of ${}_c\sigma_{tB}/\tau_{av}$ given by Eq. (25) were reduced from the tests on the effective tensile reinforcement having the ratio of reinforcing bar p_r less than 3.5%.

Similarly, the average crack width of beams can be calculated from Eq. (20). Strictly speaking, the theoretical crack spacing obtained from Eq. (17) should be used in the calculation of crack width. However, in this study the experimental values of crack spacings listed in Table 2 were used in the calculation. The reasons are that according to Eq. (20) the width of crack increases in proportion to the value of spacing and a more exact estimation of the crack spacing by Eq. (17) seems to be impossible from limited test results described in this study. The calculated results of the width of crack are shown in Fig. 12 in comparison with the measured values. From Fig. 12 it can be seen that if the crack spacing is given as adequately as obtained by the actual test, the crack width can be calculated correctly from Eq. (20). Anyhow, further investigation on the values of k_1, k_1k_2 and ${}_c\sigma_{tB}/\tau_{av}$ would be

(a)



(b)



(c)

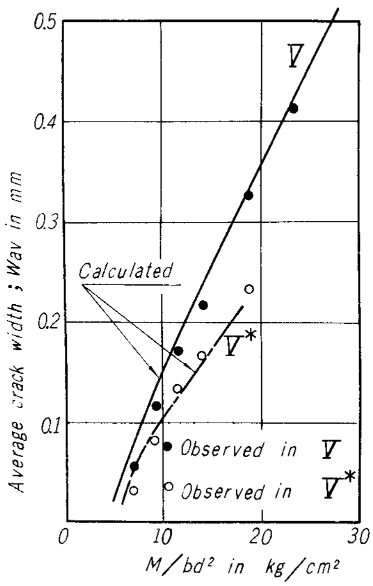


Fig. 12. Comparsion between calculated and observed crack widths $(a) \sim (e)$.

necessary for the exact estimations of the width and the spacing of crack.

Moreover, from Table 2 and Fig. 12 it is evident that only a few differences of the average crack spacing as well as the average width are recognized between the beams reinforced by deformed bars and those by plain bars.

6. Conclusive Remarks.

In this study we attempt to establish the general theory for calculating the flexural rigidity as well as the average spacing and the average width of crack in the reinforced concrete beam by introducing the idea of effective tensile reinforcement. The theoretical equation for the average flexural rigidity of cracked beam derived in this study agreed fairly well with the experiments.

The formula for calculating the average crack width developed in this study was quite similar to that proposed by A. Clark. Also, the comparison between theory and experiments showed that if the average crack spacing was given as adequately as obtained from the experiments the average crack width could be calculated correctly by using the theoretical equation developed in this study.

References

1) Johnson, A., "Deformation of Reinforced Concrete", Publications I.A.B.S.E., Vol. 11, pp. 251–290, 1951.
2) Murashev, V. I., "Investigations into the Rigidity and the Opening-up of Cracks in Reinforced Concrete Bending Members", Building Research and Documentation, Contributions and Discussions at the First CIB Congress, Rotterdam, pp. 96–106, 1961.
3) Eppes, B. G., "Comparison of Measured and Calculated Stiffness for Beams Reinforced in Tension Only", ACI Journal, Vol. 31, No. 4, pp. 313–325, Oct. 1959.
4) Ban, S., "Flexural Rigidity, Crack Spacing and Crack Width of Reinforced Concrete Beams", Transaction of Architectural Institute of Japan, Vol. 48, pp. 21–25, March 1954 (in Japanese).
5) Graf, O., "Die Eigenschaften des Betons", s. 160, Berlin, 1950.
6) Saliger, R., "Der Stahlbetonbau", s. 71, Wien, 1956.
7) Clark, A., "Cracking in Reinforced concrete Flexural Members", ACI Journal, Vol. 27, No. 8, pp. 851–862, April 1956.

Notation

E_c, E_s : Moduli of elasticity of concrete and steel, respectively

E_0 : Initial tangent modulus of elasticity of concrete

I_c, I_s : Moments of inertia of cross sections of concrete and reinforcement, respectively, with respect to center of gravity of reinforced concrete beam section

EI : Flexural rigidity of reinforced concrete beam section

M : Applied bending moment

φ : Rotation of beam section per unit length

${}_c\delta_c$: Average compressive strain of concrete at extreme fibre

${}_c\delta_z$: Average compressive strain of concrete at the distance of z from neutral axis

$_c\delta_t$: Average tensile strain of concrete at the level of reinforcement
$_s\delta_c$: Average strain of compressive reinforcement
$_r\delta_t$: Average strain of tensile reinforcement embedded in concrete (Average strain of effective tencile reinforcement)
$_c\sigma_z$: Average stress of concrete at the distance of z from neutral axis
$_c\sigma_{cB}$, $_c\sigma_{tB}$: Ultimate compressive and tensile strengths of concrete, respectively
$_s\sigma_y$: Yielding stress of reinforcement
$_s\sigma_{t-\max}$, $_s\sigma_{t-\min}$: Maximum and minimum tensile stresses in reinforcement embedded in concrete, respectively
$_s\sigma_{t-av}$: Average tensile stress in reinforcement embedded in concrete
$_r\sigma_t$: Nominal stress in tensile reinforcement embedded in concrete, calculated under the assumption that the whole tensile force is carried by bare tensile reinforcement
τ_{av} : Average bond stress
b, d : Width and effective depth of beam, respectively
h : Total depth of beam
$n=n_1d$: Average depth of compression zone of concrete
$d_c=d_{c1}\cdot d$: Depth of compressive reinforcement measured from extreme fibre in compression
D : Diameter of tensile reinforcing bar
ϕ : Total perimeter of tensile reinforcing bar
e_{av}, w_{av} : Average spacing and average width of crack, respectively
$_cA_t$: Cross sectional area of concrete prism (Cross sectional area of concrete of effective tensile reinforcement)
$_sA_c=p'bd$: Area of compressive reinforcement
$_sA_t=pbd$: Area of tensile reinforcement $(=p_r\cdot{}_cA_t)$