Analysis of the Harmonic Producer Circuits

By

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This study introduces a mathematical method for the analysis of the harmonic producer circuits with a ferromagnetic saturable core coil, taking the effect of magnetic hysteresis into consideration.

This method is based on the theorems of the Periodically Interrupted Electric Circuits of the Third Genus, and on the application of the Digital Computer (KDC-1).

First we describe in general how to apply this analytical method to the basic circuit of harmonic producers to clarify its behaviour as accurately as possible, and next to the harmonic producer circuit containing an additional capacitor.

One numerical example is presented to show the performance of the harmonic producer circuit in this paper.

1. Introduction

Circuits producing a number of harmonics by means of saturable core coils have been used for the common supply of carrier currents to multi-channel carrier telephone systems. The quality and stability of the carrier current may affect the reliability of each channel of telephone systems, therefore it is important to analyze and to design the circuit generating a group of harmonics as well as possible.

The study on the hamonic producer circuits was done by various workers, such as E. Peterson, J. M. Manley and L. R. Wrathall[1] who, in 1937, investigated the operation of the harmonic generating circuit only considering the discharge pulse from the transient analytical point of view. In 1949, Fourier analysis was worked by T. Kurokawa[2] taking both the discharge and the charge pulse into consideration, and in 1953, S. Hayashi and K. Nishihara[3] applied at first time the theorems of the Periodically Interrupted Electric Circuits to this problem. Recently, in 1961, an investigation for the same systems based on the analytical method of non-linear problems established by L. A. Pipes was attempted by A. Kyogoku, Y. Oohashi and K. Iishi[4].

These investigations, however, could be considered to be established in

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parctice basing on some assumptions or other, because the analysis of the harmonic producer circuits with a non-linear element such as a saturable core coil might be the most difficult problem from the theoretical point of view.

Therefore the unknown problems exist to be clarified in the harmonic producer circuits. In this paper in order to make clear one of the unknown problems we attempt the analysis of the circuits taking the effect of magnetic hysteresis of the saturable core coil into consideration[5].

2. Harmonic Producer Circuits

The circuit shown in Fig. 1 is the basic circuit of harmonic producers using a saturable core coil. Another circuit containing a capacitance C_1 connected in

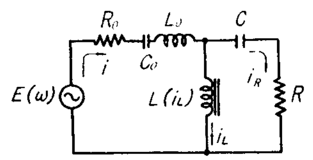

Fig. 1. Harmonic producer circuit with a ferromagnetic core coil.

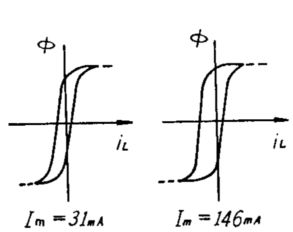

Fig. 2. Observed characteristics of the coil.

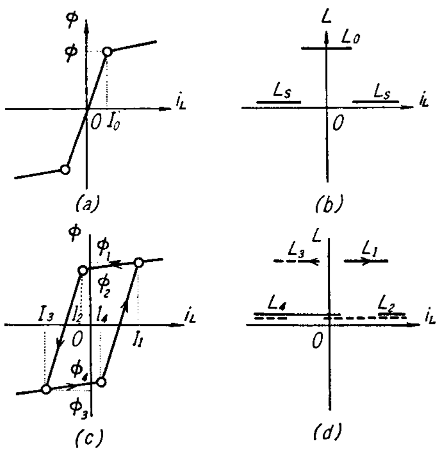

Fig. 3. Approximate characteristics of the coil.

parallel with the coil will be described in section 6.

When the current i due to the input source $E(\omega)$ with a fundamental frequency ω flows in R_0, C_0, L_0 and L circuit, the distorted current i_R including the harmonic components is generated in the output circuit of L, C and R by the change of L due to the instantaneous change of the current i.

Now we assume that the input source with a fundamental frequency ω is the current source of sinusoidal wave form and the characteristic of the ferromagnetic core coils is approximately composed of four rectilinear segments as shown in Fig. 3 (c) and of discrete values of L as shown in Fig. 3 (d).

The previous investigations have done to assume the characteristic of the saturable core coils such as in Fi. 3 (a) composed of three rectilinear segments without regard to the magnetic hysteresis. As the results presented in this paper will show later, it is not suitable to consider that the magnetic hysteresis is not taken into consideration, since it will not deform the shape remarkably but only shifts the phase of repeated pulses.

3. Analysis of the Basic Circuit

3.1. Circuit Conditions

The basic circuit of harmonic producers can be rewritten as shown in Fig. 4.

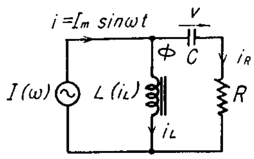

Fig. 4. The basic circuit of a harmonic producer.

The following circuit conditions are assumed:

(a) Input current has a sinusoidal wave form

$$i = I_m \sin \omega t\,.$$

(b) The total magnetic flux-linkage ϕ of the coil is related to the exciting current i_L by four rectilinear segments as shown in Fig. 3 (c).

Accordingly four kinds of the circuit state as follows can be presumed in the steady state per one cycle.

1. The 1-st circuit mode (in the permeable state of the coil):

$$\phi = \phi_4 + L_1(i_L - I_4)\,, \quad I_4 \leq i_L \leq I_1\,.$$

2. The 2-nd circuit mode (in the positive saturated state of the coil):

$$\phi = \phi_1 + L_2(i_L - I_1)\,, \quad i_L \geq I_2\,.$$

3. The 3-rd circuit mode (in the permeable state of the coil):

$$\phi = \phi_2 + L_3(i_L - I_2)\,, \quad I_3 \leq i_L \leq I_2\,.$$

4. The 4-th circuit mode (in the negative saturated state of the coil):

$$\phi = \phi_3 + L_4(i_L - I_3)\,, \quad i_L \leq I_4\,.$$

The transient solutions in each rectilinear region are to be found as the elements of certain matrices on applying the Laplace transformation to the basic differential equations expressed in matrix notation.

Supposing that the phenomena in equation ultimately converge to periodic ones, the resulting solutions on adjacent regions are to be equated to each other at the intersection of the rectilinear characteristics in order to determine the instant of transition.

3.2. Transient Solutions

3.2.1. Transient solutions in the 1-st circuit mode.— The circuit equations are written in the forms

$$\left.\begin{aligned}&\frac{d\phi}{dt}=v+Ri_R\,,\quad i_R=i-i_L\,,\\&C\frac{dv}{dt}=i_R\,,\quad i=I_m\sin(\omega t+\theta_1)\,,\quad i_L=\frac{\phi-\phi_4}{L_1}+I_4\,,\end{aligned}\right\}\tag{3.1}$$

where θ_1 represents the initial phase angle of the input current, at which the permeable state starts, and the initial conditions are supposed to be

$$\phi=\phi_4 \quad \text{at} \quad t=0$$

and

$$\phi=\phi_1 \quad \text{at} \quad t=t_1\,,$$

t_1 being the duration of the state.

Putting

$$\left.\begin{aligned}&y_1(\tau)=\frac{\omega^2C}{I_m}\phi(\tau)\,,\quad z_1(\tau)=\frac{\omega C}{I_m}v(\tau)\,,\quad \tau=\omega t\,,\quad K_1=\frac{I_4}{I_m}\,,\\&y_{\bar{1}}^{0}=\frac{\omega^2C}{I_m}\phi_4\,,\quad \omega CR=k\,,\quad \delta_1=\frac{R}{\omega L_1}\,,\quad \gamma_1=\frac{1}{\omega^2CL}=\frac{\delta_1}{k}\,,\end{aligned}\right\}\tag{3.2}$$

where $y_1(\tau)$ and $z_1(\tau)$ with the subscript "1" belong to the 1-st circuit mode.

Eq. (3.1) reduces to the following dimensionless ones:

$$\left.\begin{aligned}&\left(\frac{d}{d\tau}+\delta_1\right)\{y_1(\tau)-y_{\bar{1}}^{0}\}-z_1(\tau)=k\{\sin(\tau+\theta_1)-K_1\}\,,\\&\gamma_1\{y_1(\tau)-y_{\bar{1}}^{0}\}+\frac{d}{d\tau}z_1(\tau)=\sin(\tau+\theta_1)-K_1\,,\\&i_R/I_m=\sin(\tau+\theta_1)-\gamma_1\{y_1(\tau)-y_{\bar{1}}^{0}\}-K_1\,.\end{aligned}\right\}\tag{3.3}$$

Denoting the operational function corresponding to $y_1(\tau)$ and $z_1(\tau)$ by $Y_1(p)$ and $Z_1(p)$ respectively, in view of Eq. (3.3) we have

$$\begin{pmatrix}p+\delta_1, & -1\\ \gamma_1, & p\end{pmatrix}\begin{pmatrix}Y_1(p)\\ Z_1(p)\end{pmatrix}=\begin{pmatrix}k\\1\end{pmatrix}\frac{\cos\theta_1+p\sin\theta_1}{p^2+1}p+p\begin{pmatrix}y_{\bar{1}}^{0}\\ z_{\bar{1}}^{0}\end{pmatrix}-\begin{pmatrix}k\\1\end{pmatrix}K_1\,.\tag{3.4}$$

Accordingly

$$\begin{aligned}\begin{pmatrix}Y_1(p)\\ Z_1(p)\end{pmatrix}&=\begin{pmatrix}p, & 1\\ -\gamma_1, & p+\delta_1\end{pmatrix}\begin{pmatrix}k\\1\end{pmatrix}\frac{(\cos\theta_1+p\sin\theta_1)p}{(p^2+\delta_1p+\gamma_1)(p^2+1)}+\frac{p}{p^2+\delta_1p+\gamma_1}\begin{pmatrix}p, & 1\\ -\gamma_1, & p+\delta_1\end{pmatrix}\\&\quad\times\begin{pmatrix}y_{\bar{1}}^{0}\\ z_{\bar{1}}^{0}\end{pmatrix}-\frac{K_1}{p^2+\delta_1p+\gamma_1}\begin{pmatrix}p, & 1\\ -\gamma, & p+\delta_1\end{pmatrix}\begin{pmatrix}k\\1\end{pmatrix}.\end{aligned}\tag{3.5}$$

Putting

$$\left.\begin{aligned}&f_{rn}(\tau)=\frac{1}{2\pi j}\int_B\frac{p^n}{(p^2+1)(p^2+\delta_rp+\gamma_r)}\cdot\frac{\varepsilon^{p\tau}}{p}dp\,,\\&g_{rn}(\tau)=\frac{1}{2\pi j}\int_B\frac{p^n}{p^2+\delta_rp+\gamma_r}\cdot\frac{\varepsilon^{p\tau}}{p}dp\,.\end{aligned}\right\}\tag{3.6}$$

$$(r=1,\ 2,\ 3,\ 4,\ )$$

Eq. (3.3) are solved. The result is

$$\begin{pmatrix} y_1(\tau)-y_1^{-0} \\ z_1(\tau) \end{pmatrix} = \begin{pmatrix} F_{11}(\tau), & F_{12}(\tau) \\ f_{12}(\tau), & f_{13}(\tau) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} g_{11}(\tau), & -G_{11}(\tau) \\ G_{12}(\tau), & -g_{11}(\tau) \end{pmatrix} \begin{pmatrix} z_1^{-0} \\ K_1 \end{pmatrix}, \tag{3.7}$$

where

$$\left.\begin{aligned} F_{11}(\tau) &= kf_{12}(\tau)+f_{11}(\tau), \quad G_{11}(\tau) = kg_{11}(\tau)+g_{10}(\tau), \\ F_{12}(\tau) &= kf_{13}(\tau)+f_{12}(\tau), \quad G_{12}(\tau) = g_{12}(\tau)+\delta_1 g_{11}(\tau) \end{aligned}\right\} \tag{3.8}$$

and

$$\left.\begin{aligned} g_{10}(\tau) &= \frac{1}{\gamma_1}\left\{1-\varepsilon^{-\alpha_1\tau}\left(\cos\beta_1\tau+\frac{\alpha_1}{\beta_1}\sin\beta_1\tau\right)\right\}, \\ g_{11}(\tau) &= \frac{1}{\beta_1}\varepsilon^{-\alpha_1\tau}\sin\beta_1\tau, \\ g_{12}(\tau) &= \varepsilon^{-\alpha_1\tau}\left(\cos\beta_1\tau-\frac{\alpha_1}{\beta_1}\sin\beta_1\tau\right), \\ f_{11}(\tau) &= \frac{1}{\Delta}\left\{a_1\sin\tau-2\alpha_1\cos\tau+\varepsilon^{-\alpha_1\tau}\left(2\alpha_1\cos\beta_1\tau+\frac{1+\alpha_1^2-\beta_1^2}{\beta_1}\sin\beta_1\tau\right)\right\}, \\ f_{12}(\tau) &= \frac{1}{\Delta}\{a_1\cos\tau+\delta_1\sin\tau-\varepsilon^{-\alpha_1\tau}(a_1\cos\beta_1\tau+a_2\sin\beta_1\tau)\}, \\ f_{13}(\tau) &= \frac{1}{\Delta}\{\delta_1\cos\tau-a_1\sin\tau+\varepsilon^{-\alpha_1\tau}(-\delta_1\cos\beta_1\tau+a_3\sin\beta_1\tau)\}, \end{aligned}\right\} \tag{3.9}$$

where

$$\alpha_1 = \frac{\delta_1}{2}, \quad \beta_1 = \sqrt{\gamma_1-\alpha_1^2}, \quad a_1 = \gamma_1 - 1, \quad a_2 = \frac{\alpha_1}{\beta_1}(\alpha_1^2+\beta_1^2+1),$$

$$a_3 = \frac{1}{\beta_1}\{(\alpha_1^2+\beta_1^2)^2+\alpha_1^2-\beta_1^2\}, \quad \Delta = (\alpha_1^2+\beta_1^2)^2+2(\alpha_1^2-\beta_1^2)+1.$$

Here the first subsripts of each quantities in Eqs. (3.8) and (3.9) belong to each circuit mode.

3.2.2. Transient solutions in the 2-nd circuit mode.— When the coil is saturated by an exciting current flowing in the positive direction, the circuit equations are written in the forms

$$\left.\begin{aligned} &\frac{d\phi}{dt} = v+Ri_R, \quad i_R = i-i_L, \\ &C\frac{dv}{dt} = i_R, \quad i = I_m\sin(\omega t+\theta_2), \quad i_L = \frac{\phi-\phi_1}{L_2}+I_1, \end{aligned}\right\} \tag{3.10}$$

where θ_2 represents the initial phase angle of the input current, at which the positive saturated state starts, and the conditions are imposed to be

$$\phi = \phi_1 \quad \text{at} \quad t = 0$$

and

$$\phi = \phi_2 \quad \text{at} \quad t = t_2,$$

t_2 being the duration of the state. Using a similar symbols such as in (3.2), Eq. (3.10) may be written in the forms

$$\left.\begin{aligned}&\left(\frac{d}{d\tau}+\delta_2\right)\{y_2(\tau)-y_{\bar{2}}^{0}\}-z_2(\tau)=k\{\sin(\tau+\theta_2)-K_2\},\\&\gamma_2\{y_2(\tau)-y_{\bar{2}}^{0}\}+\frac{d}{d\tau}z_2(\tau)=\sin(\tau+\theta_2)-K_2,\\&i_R/I_m=\sin(\tau+\theta_2)-\gamma_2\{y(\tau)-y_{\bar{2}}^{0}\}-K_2,\end{aligned}\right\}\qquad(3.11)$$

where

$$\delta_2=R/\omega L_2,\quad \gamma_2=\frac{1}{\omega^2 CL_2},\quad K_2=I_1/I_m,\quad y_{\bar{2}}^{0}=\frac{\omega^2 C}{I_m}\phi_1.$$

The solutions are obtained in a similar way as before, that is,

$$\begin{pmatrix}y_2(\tau)-y_{\bar{2}}^{0}\\ z_2(\tau)\end{pmatrix}=\begin{pmatrix}F_{21}(\tau), & F_{22}(\tau)\\ f_{22}(\tau), & f_{23}(\tau)\end{pmatrix}\begin{pmatrix}\cos\theta_2\\ \sin\theta_2\end{pmatrix}+\begin{pmatrix}g_{21}(\tau), & -G_{21}(\tau)\\ G_{22}(\tau), & -g_{21}(\tau)\end{pmatrix}\begin{pmatrix}z_{\bar{2}}^{0}\\ K_2\end{pmatrix},\qquad(3.12)$$

where the $F_2(\tau)$'s, $f_2(\tau)$'s, $g_2(\tau)$'s and $G_2(\tau)$'s are determined on substituting γ_2, α_2, $j\beta_2$, $\cos\theta_2\tau$, and $\frac{1}{\beta_2}\sinh\beta_2\tau$ for γ_1, α_1, β_1, $\cos\beta_1\tau$ and $\frac{1}{\beta_1}\sin\beta_1\tau$ in the expressions of $F_1(\tau)$'s, $f_1(\tau)$'s, $g_1(\tau)$'s and $G_1(\tau)$'s in Eqs. (3.8) and (3.9) respectively, while $\alpha_2=\delta_2/2$ and $\beta_2=\sqrt{\alpha_2^2-\gamma_2}$.

3.2.3. Transient solutions in the 3-rd mode.— The circuit equations are written in the forms

$$\left.\begin{aligned}&\frac{d\phi}{dt}=v+Ri_R,\quad i_R=i-i_L,\\&C\frac{dv}{dt}=i_R,\quad i=I_m\sin(\omega t+\theta_3),\quad i_L=\frac{\phi-\phi_2}{L_3}+I_2,\end{aligned}\right\}\qquad(3.13)$$

where θ_3 represents the initial phase angle of the input current, at which the permeable state begins, and the conditions have to be

$$\phi=\phi_2\quad\text{at}\quad t=0$$

and

$$\phi=\phi_3\quad\text{at}\quad t=t_3,$$

t_3 being the duration of the state.

According to a similar procedure, Eq. (3.13) may be written in the forms

$$\left.\begin{aligned}&\left(\frac{d}{d\tau}+\delta_3\right)\{y_3(\tau)-y_{\bar{3}}^{0}\}-z_3(\tau)=k\{\sin(\tau+\theta_3)-K_3\},\\&\gamma_3\{y_3(\tau)-y_{\bar{3}}^{0}\}+\frac{d}{d\tau}z_3(\tau)=\sin(\tau+\theta_3)-K_3,\\&i_R/I_m=\sin(\tau+\theta_3)-\gamma_3\{y(\tau)-y_{\bar{3}}^{0}\}-K_3,\end{aligned}\right\}\qquad(3.14)$$

where

$$\delta_3=\frac{R}{\omega L_3},\quad \gamma_3=\frac{1}{\omega^2 CL_3},\quad K_3=\frac{I_2}{I_m},\quad y_{\bar{3}}^{0}=\frac{\omega^2 C}{I_m}\phi_2.$$

The solutions are obtained by a similar procedure, that is,

$$\begin{pmatrix} y_3(\tau)-y_3^{-0} \\ z_3(\tau) \end{pmatrix} = \begin{pmatrix} F_{31}(\tau), & F_{32}(\tau) \\ f_{32}(\tau), & f_{33}(\tau) \end{pmatrix} \begin{pmatrix} \cos\theta_3 \\ \sin\theta_3 \end{pmatrix} + \begin{pmatrix} g_{31}(\tau), & -G_{31}(\tau) \\ G_{32}(\tau), & -g_{31}(\tau) \end{pmatrix} \begin{pmatrix} z_3^{-0} \\ K_3 \end{pmatrix}, \tag{3.15}$$

where the elements of the matrices on the right-hand sides, which are denoted by Eqs. (3.6) and (3.8), can be obtained by substituting γ_3, α_3 and β_3 for γ_1, α_1 and β_1 in the expressions of Eqs. (3.8) and (3.9) respectively, while $\alpha_3=\delta_3/2$ and $\beta_3=\sqrt{\gamma_3-\alpha_3^2}$.

3.2.4. Transient solutions in the 4-th circuit mode.— The circuit equations are written in the forms

$$\left.\begin{aligned} &\frac{d\phi}{dt} = v+Ri_R, \quad i_R = i-i_L, \\ &C\frac{dv}{dt} = i_R, \quad i = I_m \sin(\omega t+\theta_4), \quad i_L = \frac{\phi-\phi_3}{L_4}+I_3, \end{aligned}\right\} \tag{3.16}$$

where θ_4 represented the initial phase angle of the input current, at which the negative saturated state starts, and the conditions have to be

$$\phi = \phi_3 \quad \text{at} \quad t=0$$

and

$$\phi = \phi_4 \quad \text{at} \quad t=t_4,$$

t_4 being the duration of the state.

Eq. (3.16) may be written in the forms as before, that is,

$$\left.\begin{aligned} &\left(\frac{d}{d\tau}+\delta_4\right)\{y_4(\tau)-y_4^{-0}\}-z_4(\tau) = k\{\sin(\tau+\theta_4)-K_4\}, \\ &\gamma_4\{y_4(\tau)-y_4^{-0}\}+\frac{d}{d\tau}z_4(\tau) = \sin(\tau+\theta_4)-K_4, \\ &i_R/I_m = \sin(\tau+\theta_4)-\gamma_4\{y_4(\tau)-y_4^{-0}\}-K_4, \end{aligned}\right\} \tag{3.17}$$

where

$$\delta_4 = \frac{R}{\omega L_4}, \quad \gamma_4 = \frac{1}{\omega^2 C L_4}, \quad K_4 = \frac{I_3}{I_m}, \quad y_4^{-0} = \frac{\omega^2 C}{I_m}\phi_3.$$

Accordingly the solutions may be obtained as follows

$$\begin{pmatrix} y_4(\tau)-y_4^{-0} \\ z_4(\tau) \end{pmatrix} = \begin{pmatrix} F_{41}(\tau), & F_{42}(\tau) \\ f_{42}(\tau), & f_{43}(\tau) \end{pmatrix} \begin{pmatrix} \cos\theta_4 \\ \sin\theta_4 \end{pmatrix} + \begin{pmatrix} g_{41}(\tau), & -G_{41}(\tau) \\ G_{42}(\tau), & -g_{41}(\tau) \end{pmatrix} \begin{pmatrix} z_4^{-0} \\ K_4 \end{pmatrix}, \tag{3.18}$$

where the elements of the matrices on the right-hand sides, which are denoted by Eqs. (3.6) and (3.8), are determined on substituting γ_4, α_4 and β_4 for γ_2, α_2 and β_2 in the expressions of the solutions in the 2-nd circuit mode, while $\alpha_4=\delta_4/2$ and $\beta_4=\sqrt{\alpha_4^2-\gamma_4}$.

4. Steady State Solutions considering the Over-all Characteristics

Next we consider the steady state solutions of the circuit where the phenomena in complete cycle behave as follows.

In the preceding sections we obtained generally the solutions in the case of the unsymmetrical characteristic of the coil. Thus in the unsymmetrical case the conditions for periodic solutions must be imposed to be

$$\left.\begin{aligned}&\tau_1+\tau_2+\tau_3+\tau_4=2\pi\,,\\&y_1(\tau_1)=y_2^{-0}\,,\quad y_2(\tau_2)=y_3^{-0}\,,\quad y_3(\tau_3)=y_4^{-0}\,,\quad y_4(\tau_4)=y_1^{-0}\,,\\&z_1(\tau_1)=z_2^{-0},\quad z_2(\tau_2)=z_3^{-0},\quad z_3(\tau_3)=z_4^{-0},\quad z_4(\tau_4)=z_1^{-0}.\end{aligned}\right\} \tag{4.1}$$

On the other hand, the conditions in the case of the symmetrical characteristic of the coil, (i.e. $L_1=L_3$ and $L_2=L_4$) that is,

$$\left.\begin{aligned}&\tau_1+\tau_2=\pi\,,\quad z_1(\tau_1)=z_2^{-0}\,,\quad z_2(\tau_2)=-z_1^{-0}\,,\\&y_1(\tau_1)=-y_3(\tau_3)=y_2^{-0}\,,\quad y_2(\tau_2)=-y_4(\tau_4)=-y_1^{-0}\,.\end{aligned}\right\} \tag{4.2}$$

These relations of Eq. (4.2) can be induced by the fact as follows.

During the first interval of τ_1 seconds, the coil is kept in unsaturated condition and then is saturated in the positive direction during the subsequent interval of τ_2 seconds, while in the next half cycle, the currents, the magnetic flux and voltage in the circuit become equal to magnitude, but opposite in sign.

Here we will discuss the solutions in the case of the symmetrical characteristic of the coil. The initial values of ϕ, i_L and v in the primary unsaturated state and in the subsequent positive saturated state are equal to the final values of those in the negative saturated state and in the primary unsaturated state respectively, and it is intended to study the steady state phenomena in any one of the two half cycles, which are equal in magnitude, but opposite in sign to each other.

Thus the steady state solutions are given ultimately by the elements of

$$\begin{pmatrix}y_1(\tau)+y_2(\tau_2)\\ z_1(\tau)\end{pmatrix}=\begin{pmatrix}F_{11}(\tau), & F_{12}(\tau)\\ f_{12}(\tau), & f_{13}(\tau)\end{pmatrix}\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}-\begin{pmatrix}g_{11}(\tau), & G_{11}(\tau)\\ G_{12}(\tau), & g_{11}(\tau)\end{pmatrix}\begin{pmatrix}z_2(\tau_2)\\ K_1\end{pmatrix},\quad 0\leq\tau\leq\tau_1 \tag{4.3}$$

and

$$\begin{pmatrix}y_2(\tau)-y_1(\tau_1)\\ z_2(\tau)\end{pmatrix}=\begin{pmatrix}F_{21}(\tau), & F_{22}(\tau)\\ f_{22}(\tau) & f_{23}(\tau)\end{pmatrix}\begin{pmatrix}\cos(\tau_1+\theta_1)\\ \sin(\tau_1+\theta_1)\end{pmatrix}+\begin{pmatrix}g_{21}(\tau), & -G_{21}(\tau)\\ G_{22}(\tau), & -g_{21}(\tau)\end{pmatrix}\begin{pmatrix}z_1(\tau_1)\\ K_2\end{pmatrix} \tag{4.4}$$

$$0\leq\tau\leq\tau_2\,.$$

τ_1, τ_2, θ_1, $z_1(\tau_1)$ and $z_2(\tau_2)$ in the above equations are so far unknown, and we have to find their values so that these equations must satisfy the relations in Eq. (4.2) for the given values of ϕ_1 and ϕ_2 or $y_1(\tau_1)$ and $y_2(\tau_2)$.

Putting $\tau=\tau_1$ in Eq. (4.3) and $\tau=\tau_2$ in Eq. (4.4), we can obtain the following equations respectively which will determine the unknown equantities.

$$\begin{pmatrix}1, & 1\\ 0, & 0\end{pmatrix}\begin{pmatrix}y_1(\tau_1)\\ y_2(\tau_2)\end{pmatrix}+\begin{pmatrix}0, & g_{11}(\tau_1)\\ 1, & G_{12}(\tau_1)\end{pmatrix}\begin{pmatrix}z_1(\tau_1)\\ z_2(\tau_2)\end{pmatrix}=\begin{pmatrix}F_{11}(\tau), & F_{12}(\tau_1)\\ f_{12}(\tau_1), & f_{13}(\tau)\end{pmatrix}\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}-\begin{pmatrix}G_{11}(\tau_1)\\ g_{11}(\tau_1)\end{pmatrix}K_1 \tag{4.5}$$

and

$$\begin{pmatrix}-1, & 1\\ 0, & 0\end{pmatrix}\begin{pmatrix}y_1(\tau_1)\\ y_2(\tau_2)\end{pmatrix}+\begin{pmatrix}-g_{21}(\tau_2), & 0\\ -G_{22}(\tau_2), & 1\end{pmatrix}\begin{pmatrix}z_1(\tau_1)\\ z_2(\tau_2)\end{pmatrix}=\begin{pmatrix}E_{21}(\tau_2), & E_{22}(\tau_2)\\ E_{23}(\tau_2), & E_{24}(\tau_2)\end{pmatrix}\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}-\begin{pmatrix}G_{21}(\tau_2)\\ g_{21}(\tau_2)\end{pmatrix}K_2, \tag{4.6}$$

where

$$\begin{pmatrix}E_{21}(\tau_2), & E_{22}(\tau_2)\\ E_{23}(\tau_2), & E_{24}(\tau_2)\end{pmatrix}=\begin{pmatrix}F_{21}(\tau_2), & F_{22}(\tau_2)\\ f_{22}(\tau_2), & f_{23}(\tau_2)\end{pmatrix}\begin{pmatrix}\cos\tau_1, & -\sin\tau_1\\ \sin\tau_1, & \cos\tau_1\end{pmatrix}. \tag{4.7}$$

Eliminating $z_1(\tau_1)$ and $z_2(\tau_2)$ from Eqs. (4.5) and (4.6), first by Eq. (4.5) we have

$$\begin{pmatrix}z_1(\tau_1)\\ z_2(\tau_2)\end{pmatrix}=\begin{pmatrix}0, & g_{11}(\tau_1)\\ 1, & G_{12}(\tau_1)\end{pmatrix}^{-1}\left\{\begin{pmatrix}F_{11}(\tau_1), & F_{12}(\tau_1)\\ f_{12}(\tau_1), & f_{13}(\tau_1)\end{pmatrix}\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}-\begin{pmatrix}G_{11}(\tau_1)\\ g_{11}(\tau_1)\end{pmatrix}K_1-\begin{pmatrix}1, & 1\\ 0, & 0\end{pmatrix}\begin{pmatrix}y_1(\tau_1)\\ y_2(\tau_2)\end{pmatrix}\right\} \tag{4.8}$$

and then substituting Eq. (4.8) into Eq. (4.6), we have

$$\begin{pmatrix}h_{11}(\tau_1), & h_{12}(\tau_1)\\ h_{21}(\tau_1), & h_{22}(\tau_1)\end{pmatrix}\begin{pmatrix}y_1(\tau_1)\\ y_2(\tau_2)\end{pmatrix}+\begin{pmatrix}u_{11}(\tau_1), & u_{12}(\tau_1)\\ u_{21}(\tau_1), & u_{22}(\tau_1)\end{pmatrix}\begin{pmatrix}K_2\\ K_1\end{pmatrix}=\begin{pmatrix}d_{11}(\tau_1), & d_{12}(\tau_1)\\ d_{21}(\tau_1), & d_{22}(\tau_1)\end{pmatrix}\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\end{pmatrix}, \tag{4.9}$$

where

$$\left.\begin{aligned}
\begin{pmatrix}h_{11}(\tau_1), & h_{12}(\tau_1)\\ h_{21}(\tau_1), & h_{22}(\tau_1)\end{pmatrix}&=\begin{pmatrix}-1, & 1\\ 0, & 0\end{pmatrix}-\begin{pmatrix}-g_{21}(\tau_2), & 0\\ -G_{22}(\tau_2), & 1\end{pmatrix}\begin{pmatrix}0, & g_{11}(\tau_1)\\ 1, & G_{12}(\tau_1)\end{pmatrix}^{-1}\begin{pmatrix}1, & 1\\ 0, & 0\end{pmatrix},\\
\begin{pmatrix}u_{11}(\tau_1), & u_{12}(\tau_1)\\ u_{21}(\tau_1), & u_{22}(\tau_1)\end{pmatrix}&=\begin{pmatrix}G_{21}(\tau_2), & 0\\ g_{21}(\tau_2), & 0\end{pmatrix}-\begin{pmatrix}-g_{21}(\tau_2), & 0\\ -G_{22}(\tau_2), & 1\end{pmatrix}\begin{pmatrix}0, & g_{11}(\tau_1)\\ 1, & G_{12}(\tau_1)\end{pmatrix}^{-1}\begin{pmatrix}0, & G_{11}(\tau_1)\\ 0, & g_{11}(\tau_1)\end{pmatrix},\\
\begin{pmatrix}d_{11}(\tau_1), & d_{12}(\tau_1)\\ d_{21}(\tau_1), & d_{22}(\tau_1)\end{pmatrix}&=\begin{pmatrix}E_{21}(\tau_2), & E_{22}(\tau_2)\\ E_{23}(\tau_2), & E_{24}(\tau_2)\end{pmatrix}-\begin{pmatrix}-g_{21}(\tau_2), & 0\\ -G_{22}(\tau_2), & 1\end{pmatrix}\begin{pmatrix}0, & g_{11}(\tau_1)\\ 1, & G_{12}(\tau_1)\end{pmatrix}^{-1}\begin{pmatrix}F_{11}(\tau_1), & F_{12}(\tau_1)\\ f_{12}(\tau_1), & f_{13}(\tau_1)\end{pmatrix}.
\end{aligned}\right\} \tag{4.10}$$

Now the first equation of Eq. (4.9) is

$$h_{11}(\tau_1)y_1(\tau_1)+h_{12}(\tau_1)y_2(\tau_2)+u_{11}(\tau_1)K_2+u_{12}(\tau)K_1=d_{11}(\tau_1)\cos\theta_1+d_{12}(\tau_1)\sin\theta_1. \tag{4.11}$$

In view of Eq. (4.11) we can obtain the phase angle θ_1 as follows.
Putting

$$\left.\begin{aligned}
&\sqrt{d_{11}^2(\tau_1)+d_{12}^2(\tau_1)}=d_0(\tau_1),\\
&\theta_0=\tan^{-1}\frac{d_{12}(\tau_1)}{d_{11}(\tau_1)},
\end{aligned}\right\} \tag{4.12}$$

then, Eq. (4.11) may be written in the forms

$$d_0\cos(\theta_1-\theta_0)=f_n\{y_1(\tau_1),\ y_2(\tau_2),\ \tau_1\}. \tag{4.13}$$

Consequently we have

$$\theta_1 = \theta_0 + \cos^{-1}\frac{1}{d_0} f_n\{y_1(\tau_1),\ y_2(\tau_2),\ \tau_1\}, \tag{4.14}$$

where

$$f_n\{y_1(\tau),\ y_2(\tau_2),\ \tau_1\} = h_{11}(\tau_1)y_1(\tau_1) + h_{12}(\tau_1)y_2(\tau_2) + u_{11}(\tau_1)K_2 + u_{12}(\tau_1)K_1\,. \tag{4.15}$$

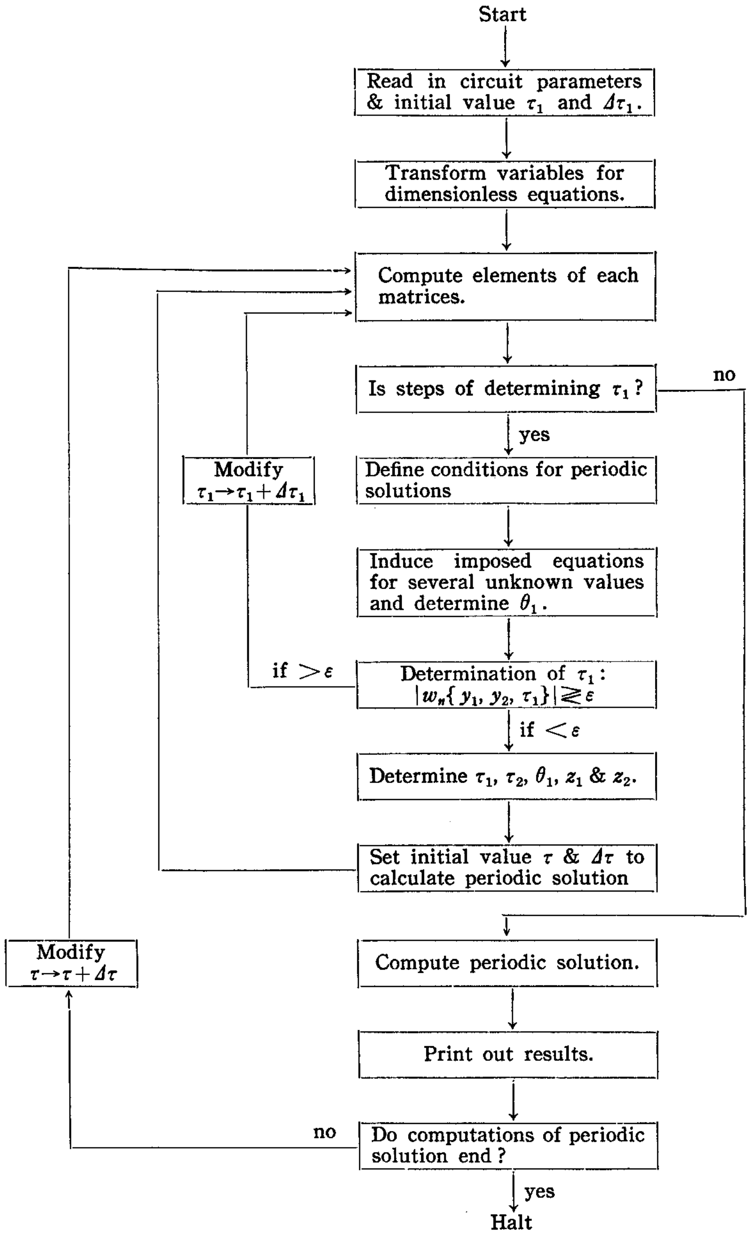

Fig. 5. Digital Computer flowchart for the steady state solutions.

Next to obtain τ_1, substituting θ_1 of Eq. (4.14) into the second equation of Eq. (4.9), i.e.

$$h_{21}(\tau_1)y_1(\tau_1)+h_{22}(\tau_1)y_2(\tau_2)+u_{21}(\tau_1)K_2+u_{22}(\tau_1)K_1 = d_{21}(\tau_1)\cos\theta_1+d_{22}(\tau_1)\sin\theta_1\,, \qquad (4.16)$$

we have the following transcendental equation with regard to the given values of $y_1(\tau_1)$, $y_2(\tau_2)$ and to the unknown values τ_1, and then τ_1 can be determined by solving this equation with respect to the function τ_1.

$$w_n\{y_1(\tau_1),\ y_2(\tau_2),\ \tau_1\} = 0\,, \qquad (4.17)$$

where

$$w_n\{y_1(\tau_1),\ y_2(\tau_2),\ \tau_1\} = h_{21}(\tau_1)y_1(\tau_1)+h_{22}(\tau_1)y_2(\tau_2)+u_{21}(\tau_1)K_2\,. \qquad (4.18)$$

Since θ_1, τ_1 and τ_2 have been found, $z_1(\tau_1)$ and $z_2(\tau_2)$ can be easily found by substituting θ_1, τ_1 and τ_2 into Eq. (4.8).

5. Determination of τ_1 by Digital Computer

It is the most difficult problem to determine τ_1 in the preceding procedure and to obtain the steady state solution.

Since τ_1 cannot be written in the explicit forms because of the complicated transcendental equation, it is comparatively easy to solve this equation by making use of the digital computer for some numerical examples.

Here we can determine τ_1 and the periodic solutions by means of the Digital Computer (KDC-1) according to the procedures as shown in Fig. 5, where approximately the determination τ_1 is confined to solve the following equation

$$|w_n\{y_1(\tau_1),\ y_2(\tau_2),\ \tau_1\}| < \varepsilon\,, \qquad (5.1)$$

ε being a given infinitesimal value.

6. Analysis of the Producer Circuit with a Shunt Condenser

In this section we study the circuit shown in Fig. 6 involving an additional capacitor C_1, which will intensify the generation of several desired components of harmonic currents. This capacitor may, of course, include the effective stray capacitor of the coil. The characteristic of the coil, i.e. the symmetrical characteristic, and other circuit modes are assumed to be just the same as those discussed in the preceding sections.

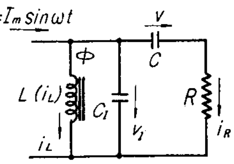

Fig. 6. Harmonic producer circuit involving an additional capacitor.

6.1. Transient Solutions in the 1-st Circuit Mode

The circuit equations are

$$\left.\begin{aligned}&\frac{d\phi}{dt}=v_1\,,\quad v_1=v+Ri_R\,,\\&C\frac{dv}{dt}=i_R\,,\quad i_L=\frac{\phi-\phi_4}{L_1}+I_4\,,\\&i_L+C_1\frac{dv_1}{dt}+C\frac{dv}{dt}=I_m\sin(\omega t+\theta_1)\,,\end{aligned}\right\}\tag{6.1}$$

where θ_1 represents the initial phase angle of the input current, at which the state starts, and the conditions are supposed to be

$$\phi=\phi_4\quad\text{at}\quad t=0$$

and

$$\phi=\phi_1\quad\text{at}\quad t=t_1\,,$$

t_1 being the duration of the state.

Putting

$$\left.\begin{aligned}&y_1(\tau)=\frac{\omega^2C\phi}{I_m},\quad x_1(\tau)=\frac{\omega C_1v_1}{I_m},\quad z_1(\tau)=\frac{\omega Cv}{I_m},\ \tau=\omega t\,,\\&\delta=\frac{1}{\omega CR},\quad \mu=\frac{C}{C_1},\quad \delta_1=\frac{1}{\omega RC_1},\end{aligned}\right\}\tag{6.2}$$

where $y_1(\tau)$, $x_1(\tau)$ and $z_1(\tau)$ with the subscript "1" belong to the 1-st circuit mode.

The following dimensionless equations are obtained from Eq. (6.1), that is,

$$\left.\begin{aligned}&\frac{d}{d\tau}x_1(\tau)+\frac{d}{d\tau}z_1(\tau)+\gamma_1y_1(\tau)-\gamma_1y_1^{-0}+K_1=\sin(\tau+\theta_1)\,,\\&\frac{d}{d\tau}y_1(\tau)-\mu x_1(\tau)=0\,,\\&\left(\frac{d}{d\tau}+\delta\right)z_1(\tau)-\delta_1x_1(\tau)=0\,,\end{aligned}\right\}\tag{6.3}$$

where

$$\gamma_1=\frac{1}{\omega^2CL_1},\quad K_1=\frac{I_4}{I_m},\quad y_1^{-0}=\frac{\omega^2C\phi_4}{I_m}.$$

The corresponding operational equations which determine the operational functions $X_1(p)$, $Y_1(p)$ and $Z_1(p)$ corresponding to $x_1(\tau)$, $y_1(\tau)$ and $z_1(\tau)$ are written in the matrix form

$$\begin{pmatrix}p, & \gamma_1, & p\\-\mu, & p, & 0\\-\delta, & 0, & p+\delta\end{pmatrix}\begin{pmatrix}X_1(p)\\Y_1(p)\\Z_1(p)\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\frac{\cos\theta_1+p\sin\theta_1}{p^2+1}p+p\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1^{-0}\\y_1^{-0}\\z_1^{-0}\end{pmatrix}+\begin{pmatrix}-K_1\\0\\0\end{pmatrix}.\tag{6.4}$$

Hence we have

$$\begin{aligned}\begin{pmatrix}X_1(p)\\Y_1(p)\\Z_1(p)\end{pmatrix}&=\frac{p\cos\theta_1+p^2\sin\theta_1}{(p^2+1)(p^3+p^2\delta_0+p\gamma_{10}+a_1)}\begin{pmatrix}p(p+\delta)\\\mu(p+\delta)\\\delta_1p\end{pmatrix}+\frac{p}{p^3+p^2\delta_0+p\gamma_{10}+a_1}\\&\times\begin{pmatrix}p(p+\delta), & -\gamma_1(p+\delta), & \delta p\\\mu(b+\delta), & p(p+\delta_0), & \delta_1\\\delta_1p, & -a_1, & p^2+\delta_1p+\gamma_{10}\end{pmatrix}\begin{pmatrix}x_1^{-0}\\y_1^{-0}\\z_1^{-0}\end{pmatrix}+\frac{-K_1}{p^3+p^2\delta_0+p\gamma_{10}+a_1}\begin{pmatrix}p(p+\delta)\\\mu(p+\delta)\\\delta_1p\end{pmatrix},\end{aligned}\tag{6.5}$$

where

$$\delta_0 = \delta + \delta_1, \quad \gamma_{10} = \mu\gamma_1, \quad a_1 = \gamma_1\delta_1.$$

The solution of Eq. (6.3) become

$$\begin{pmatrix} x_1(\tau) \\ y_1(\tau)-y_1^{-0} \\ z_1(\tau) \end{pmatrix} = \begin{pmatrix} F_{111}(\tau), & F_{112}(\tau) \\ F_{121}(\tau), & F_{122}(\tau) \\ F_{131}(\tau), & F_{132}(\tau) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} \chi_{111}(\tau), & \chi_{112}(\tau), & \chi_{113}(\tau) \\ \chi_{121}(\tau), & \chi_{122}(\tau), & \chi_{123}(\tau) \\ \chi_{131}(\tau), & \chi_{132}(\tau), & \chi_{133}(\tau) \end{pmatrix} \begin{pmatrix} x_1^{-0} \\ 0 \\ z_1^{-0} \end{pmatrix} - K_1 \begin{pmatrix} \chi_{11}(\tau) \\ \chi_{12}(\tau) \\ \chi_{13}(\tau) \end{pmatrix}, \tag{6.6}$$

where $F(\tau)$'s and $\chi(\tau)$'s are tabulated in Table 1, provided that $f_{rn}(\tau)$ and $g_{rn}(\tau)$ be given by

$$\left.\begin{aligned} f_{rn}(\tau) &= \frac{1}{2\pi j}\int_B \frac{p^n}{(p^2+1)(p^3+p^2\delta_0+p\gamma_{r0}+a_r)}\cdot\frac{\varepsilon^{p\tau}}{p}\,dp, \\ g_{rn}(\tau) &= \frac{1}{2\pi j}\int_B \frac{p^n}{p^3+p^2\delta_0+p\gamma_{r0}+a_r}\cdot\frac{\varepsilon^{p\tau}}{p}\,dp. \end{aligned}\right\} \tag{6.7}$$

$$(r = 1, 2, 3, 4)$$

Table 1.

$F_{111}(\tau)$	$f_{12}(\tau)+\delta f_{12}(\tau)$	$\chi_{111}(\tau)$	$g_{13}(\tau)+\delta g_{12}(\tau)$
$F_{112}(\tau)$	$f_{14}(\tau)+\delta f_{13}(\tau)$	$\chi_{112}(\tau)$	$-\gamma_1\{g_{12}(\tau)+\delta g_{11}(\tau)\}$
$F_{121}(\tau)$	$\mu\{f_{12}(\tau)+\delta f_{11}(\tau)\}$	$\chi_{113}(\tau)$	$\delta g_{12}(\tau)$
$F_{122}(\tau)$	$\mu\{f_{13}(\tau)+\delta f_{12}(\tau)\}$	$\chi_{121}(\tau)$	$\mu\{g_{12}(\tau)+\delta g_{11}(\tau)\}$
$F_{131}(\tau)$	$\delta_1 f_{12}(\tau)$	$\chi_{122}(\tau)$	$g_{13}(\tau)+\delta_0 g_{12}(\tau)$
$F_{132}(\tau)$	$\delta_1 f_{13}(\tau)$	$\chi_{123}(\tau)$	$\delta_1 g_{11}(\tau)$
$\chi_{11}(\tau)$	$g_{12}(\tau)+\delta g_{11}(\tau)$	$\chi_{131}(\tau)$	$\delta_1 g_{12}(\tau)$
$\chi_{12}(\tau)$	$\mu\{g_{11}(\tau)+\delta g_{10}(\tau)\}$	$\chi_{132}(\tau)$	$-a_1 g_{11}(\tau)$
$\chi_{13}(\tau)$	$\delta_1 g_{11}(\tau)$	$\chi_{133}(\tau)$	$g_{13}(\tau)+\delta_1 g_{12}(\tau)+\gamma_{10} g_{11}(\tau)$

6.2. Transient Solutions in the 2-nd Circuit Mode

The circuit equations are

$$\left.\begin{aligned} &\frac{d\phi}{dt} = v_1, \quad v_1 = v + Ri_R, \\ &C\frac{dv}{dt} = i_R, \quad i_L = \frac{\phi-\phi_1}{L_2}+I_1, \\ &i_L + C_1\frac{dv_1}{dt} + C\frac{dv}{dt} = I_m\sin(\omega t+\theta_2), \end{aligned}\right\} \tag{6.8}$$

where θ_2 represents the initial phase angle of the input current, at which the positive saturated state begins, and the conditions are supposed to be

$$\phi = \phi_1 \quad \text{at} \quad t = 0$$

and

$$\phi = \phi_2 \quad \text{at} \quad t = t_2,$$

t_2 being the duration of the state.

Using a similar symbol such as in (6.2), Eq. (6.8) may be written in the dimensionless forms

$$\left.\begin{aligned} &\frac{d}{d\tau}x_2(\tau)+\frac{d}{d\tau}z_2(\tau)+\gamma_2\{y_2(\tau)-y_2^{-0}\}+K_2=\sin(\tau+\theta_2), \\ &\frac{d}{d\tau}\{y_2(\tau)-y_2^{-0}\}-\mu x_2(\tau)=0, \\ &\left(\frac{d}{d\tau}+\delta\right)z_2(\tau)-\delta_1 x_2(\tau)=0, \end{aligned}\right\} \tag{6.9}$$

where

$$\gamma_2=\frac{1}{\omega^2 CL_2},\quad K_2=\frac{I_1}{I_m},\quad y_2^{-0}=\frac{\omega^2 C\phi}{I_m},\quad \gamma_{20}=\mu\gamma_2,\quad a_2=\gamma_2\delta_1.$$

The solutions are obtained in a similar may as before, that is,

$$\begin{pmatrix} x_2(\tau) \\ y_2(\tau)-y_2^{-0} \\ z_2(\tau) \end{pmatrix} = \begin{pmatrix} F_{211}(\tau), & F_{212}(\tau) \\ F_{221}(\tau), & F_{222}(\tau) \\ F_{231}(\tau), & F_{232}(\tau) \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} + \begin{pmatrix} \chi_{211}(\tau), & \chi_{212}(\tau), & \chi_{213}(\tau) \\ \chi_{221}(\tau), & \chi_{222}(\tau), & \chi_{223}(\tau) \\ \chi_{231}(\tau), & \chi_{232}(\tau), & \chi_{233}(\tau) \end{pmatrix}\begin{pmatrix} x_2^{-0} \\ 0 \\ z_2^{-0} \end{pmatrix} - K_2\begin{pmatrix} \chi_{21}(\tau) \\ \chi_{22}(\tau) \\ \chi_{23}(\tau) \end{pmatrix}, \tag{6.10}$$

where $F(\tau)$'s and $\chi(\tau)$'s are tabulated in Table 2.

Table 2.

$F_{211}(\tau)$	$f_{23}(\tau)+\delta f_{22}(\tau)$	$\chi_{211}(\tau)$	$g_{23}(\tau)+\delta g_{22}(\tau)$
$F_{212}(\tau)$	$f_{24}(\tau)+\delta f_{23}(\tau)$	$\chi_{212}(\tau)$	$-\gamma_2\{g_{22}(\tau)+\delta g_{21}(\tau)\}$
$F_{221}(\tau)$	$\mu\{f_{22}(\tau)+\delta f_{21}(\tau)\}$	$\chi_{213}(\tau)$	$\delta g_{22}(\tau)$
$F_{222}(\tau)$	$\mu\{f_{23}(\tau)+\delta f_{22}(\tau)\}$	$\chi_{221}(\tau)$	$\mu\{g_{22}(\tau)+\delta g_{21}(\tau)\}$
$F_{231}(\tau)$	$\delta_1 f_{22}(\tau)$	$\chi_{222}(\tau)$	$g_{23}(\tau)+\delta_0 g_{22}(\tau)$
$F_{232}(\tau)$	$\delta_1 f_{23}(\tau)$	$\chi_{223}(\tau)$	$\delta_1 g_{21}(\tau)$
$\chi_{21}(\tau)$	$g_{22}(\tau)+\delta f_{21}(\tau)$	$\chi_{231}(\tau)$	$\delta_1 g_{22}(\tau)$
$\chi_{22}(\tau)$	$\mu\{g_{21}(\tau)+\delta g_{20}(\tau)\}$	$\chi_{232}(\tau)$	$-a_2 g_{21}(\tau)$
$\chi_{23}(\tau)$	$\delta_1 g_{21}(\tau)$	$\chi_{233}(\tau)$	$g_{23}(\tau)+\delta_1 g_{22}(\tau)+\gamma_{20} g_{21}(\tau)$

Although we can easily obtain the solutions in the 3-rd and 4-th circuit mode by a similar produre, it is unnecessary to get one for considering the steady state of the circuit with the symmetrical characteristic of the coil.

7. Steady State Solutions

Since the duration of the steady state is 2π, and at the transition points of the characteristic curve, the transient solutions in adjacent region should be equated to each other, we have

$$\left.\begin{aligned} &\tau_1+\tau_2=\pi\,, \quad z_1(\tau_1)=z_2^{-0}\,, \quad z_2(\tau_2)=-z_1^{-0}\,, \\ &x_1(\tau_1)=x_2^{-0}\,, \quad x_2(\tau)=-x_1^{-0}\,, \quad y_1(\tau_1)=y_2^{-0}\,, \quad y_2(\tau_2)=-y_1^{-0}\,. \end{aligned}\right\} \tag{7.1}$$

Substituting these relations into Eqs. (6.6) and (6.10) yields

$$\begin{pmatrix} x_1(\tau) \\ y_1(\tau)-y_1^{-0} \\ z_1(\tau) \end{pmatrix} = \begin{pmatrix} F_{111}(\tau), & F_{112}(\tau) \\ F_{121}(\tau), & F_{122}(\tau) \\ F_{131}(\tau), & F_{132}(\tau) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - \begin{pmatrix} \chi_{111}(\tau), & \chi_{113}(\tau) \\ \chi_{121}(\tau), & \chi_{123}(\tau) \\ \chi_{131}(\tau), & \chi_{133}(\tau) \end{pmatrix} \begin{pmatrix} x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} - K_1 \begin{pmatrix} \chi_{11}(\tau) \\ \chi_{12}(\tau) \\ \chi_{13}(\tau) \end{pmatrix} \tag{7.2}$$
$$0 \leq \tau \leq \tau_1$$

and

$$\begin{pmatrix} x_2(\tau) \\ y_2(\tau)-y_2^{-0} \\ z_2(\tau) \end{pmatrix} = \begin{pmatrix} F_{211}(\tau), & F_{212}(\tau) \\ F_{221}(\tau), & F_{222}(\tau) \\ F_{231}(\tau), & F_{232}(\tau) \end{pmatrix} \begin{pmatrix} \cos(\theta_1+\tau_1) \\ \sin(\theta_1+\tau_1) \end{pmatrix} + \begin{pmatrix} \chi_{211}(\tau), & \chi_{213}(\tau) \\ \chi_{221}(\tau), & \chi_{223}(\tau) \\ \chi_{231}(\tau), & \chi_{233}(\tau) \end{pmatrix} \begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \end{pmatrix} - K_2 \begin{pmatrix} \chi_{21}(\tau) \\ \chi_{22}(\tau) \\ \chi_{23}(\tau) \end{pmatrix} \tag{7.3}$$
$$0 \leq \tau \leq \tau_2$$

Next to determine τ_1. θ_1, $z_1(\tau_1)$, $z_2(\tau_2)$, $x_1(\tau_1)$ and $x_2(\tau_2)$, putting $\tau=\tau_1$ in Eq. (7.2) and $\tau=\tau_2$ in Eq. (7.3), we can obtain the following equations respectively,

$$\begin{pmatrix} 1, & 0, & \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ 0, & 0, & \chi_{121}(\tau_1), & \chi_{123}(\tau_1) \\ 0, & 1, & \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{pmatrix} \begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \\ x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} + \begin{pmatrix} 0, & 0 \\ 1, & 1 \\ 0, & 0 \end{pmatrix} \begin{pmatrix} y_2(\tau_2) \\ y_1(\tau_1) \end{pmatrix} = \begin{pmatrix} F_{111}(\tau_1), & F_{112}(\tau_1) \\ F_{121}(\tau_1), & F_{122}(\tau_1) \\ F_{131}(\tau_1), & F_{132}(\tau_1) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - K_1 \begin{pmatrix} \chi_{11}(\tau_1) \\ \chi_{12}(\tau_1) \\ \chi_{13}(\tau_1) \end{pmatrix}, \tag{7.4}$$

$$\begin{pmatrix} -\chi_{211}(\tau_2), & -\chi_{213}(\tau_2), & 1, & 0 \\ -\chi_{221}(\tau_2), & -\chi_{223}(\tau_2), & 0, & 0 \\ -\chi_{231}(\tau_2), & -\chi_{233}(\tau_2), & 0, & 1 \end{pmatrix} \begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \\ x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} + \begin{pmatrix} 0, & 0 \\ 1, & -1 \\ 0, & 0 \end{pmatrix} \begin{pmatrix} y_2(\tau_2) \\ y_1(\tau_1) \end{pmatrix} = \begin{pmatrix} E_{211}(\tau_2), & E_{212}(\tau_2) \\ E_{221}(\tau_2), & E_{222}(\tau_2) \\ E_{231}(\tau_2), & E_{232}(\tau_2) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$$
$$-K_2 \begin{pmatrix} \chi_{21}(\tau_2) \\ \chi_{22}(\tau_2) \\ \chi_{23}(\tau_2) \end{pmatrix}, \tag{7.5}$$

where

$$\begin{pmatrix} E_{211}(\tau_2), & E_{212}(\tau_2) \\ E_{221}(\tau_2), & E_{222}(\tau_2) \\ E_{231}(\tau_2), & E_{232}(\tau_2) \end{pmatrix} = \begin{pmatrix} F_{211}(\tau_2), & F_{212}(\tau_2) \\ F_{221}(\tau_2), & F_{222}(\tau_2) \\ F_{231}(\tau_2), & F_{232}(\tau_2) \end{pmatrix} \begin{pmatrix} \cos\tau_1, & -\sin\tau_1 \\ \sin\tau_1, & \cos\tau_1 \end{pmatrix}. \tag{7.6}$$

The first and third equations of Eq. (7.4) are

$$\begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \end{pmatrix} + \begin{pmatrix} \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{pmatrix} \begin{pmatrix} x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} = \begin{pmatrix} F_{111}(\tau_1), & F_{112}(\tau_1) \\ F_{131}(\tau_1), & F_{132}(\tau_1) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - K_1 \begin{pmatrix} \chi_{11}(\tau_1) \\ \chi_{13}(\tau_1) \end{pmatrix} \tag{7.7}$$

and the first and third equations of Eq. (7.5) are

$$-\begin{pmatrix} \chi_{211}(\tau_2), & \chi_{213}(\tau_2) \\ \chi_{231}(\tau_2), & \chi_{233}(\tau_2) \end{pmatrix} \begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \end{pmatrix} + \begin{pmatrix} x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} = \begin{pmatrix} E_{211}(\tau_2), & E_{212}(\tau_2) \\ E_{231}(\tau_2), & E_{232}(\tau_2) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - K_2 \begin{pmatrix} \chi_{21}(\tau_2) \\ \chi_{23}(\tau_2) \end{pmatrix}. \tag{7.8}$$

Accordingly

$$\begin{pmatrix} x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} = \begin{pmatrix} \chi_{211}(\tau_2), & \chi_{213}(\tau_2) \\ \chi_{231}(\tau_2), & \chi_{233}(\tau_2) \end{pmatrix} \begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \end{pmatrix} + \begin{pmatrix} E_{211}(\tau_2), & E_{212}(\tau_2) \\ E_{231}(\tau_2), & E_{232}(\tau_2) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - K_2 \begin{pmatrix} \chi_{21}(\tau_2) \\ \chi_{23}(\tau_2) \end{pmatrix}. \tag{7.9}$$

Substituting Eq. (7.9) into Eq. (7.7) yields

$$\begin{aligned} \begin{pmatrix} k_{11}(\tau_1), & k_{12}(\tau_1) \\ k_{13}(\tau_1), & k_{14}(\tau_1) \end{pmatrix} \begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \end{pmatrix} &= \begin{pmatrix} c_{11}(\tau_1), & c_{12}(\tau_1) \\ c_{13}(\tau_1), & c_{14}(\tau_1) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - K_1 \begin{pmatrix} \chi_{11}(\tau_1) \\ \chi_{13}(\tau_1) \end{pmatrix} \\ &\quad + K_2 \begin{pmatrix} \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{pmatrix} \begin{pmatrix} \chi_{21}(\tau_2) \\ \chi_{23}(\tau_2) \end{pmatrix}, \end{aligned} \tag{7.10}$$

where

$$\begin{pmatrix} k_{11}(\tau_1), & k_{12}(\tau_1) \\ k_{13}(\tau_1), & k_{14}(\tau_1) \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} + \begin{pmatrix} \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{pmatrix} \begin{pmatrix} \chi_{211}(\tau_2), & \chi_{213}(\tau_2) \\ \chi_{231}(\tau_2), & \chi_{233}(\tau_2) \end{pmatrix} \tag{7.11}$$

and

$$\begin{pmatrix} c_{11}(\tau_1), & c_{12}(\tau_1) \\ c_{13}(\tau_1), & c_{14}(\tau_1) \end{pmatrix} = \begin{pmatrix} F_{111}(\tau_1), & F_{112}(\tau_1) \\ F_{131}(\tau_1), & F_{132}(\tau_1) \end{pmatrix} - \begin{pmatrix} \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{pmatrix} \begin{pmatrix} E_{211}(\tau_2), & E_{212}(\tau_2) \\ E_{231}(\tau_2), & E_{232}(\tau_2) \end{pmatrix}. \tag{7.12}$$

Eq. (7.10) gives the following equations for obtaining the unknown values $x_1(\tau_1)$ and $z_1(\tau_1)$,

$$\begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \end{pmatrix} = \begin{pmatrix} x_{11}(\tau_1), & x_{12}(\tau_1) \\ z_{11}(\tau_1), & z_{12}(\tau_1) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} q_{11}(\tau_1) \\ q_{12}(\tau_1) \end{pmatrix}, \tag{7.13}$$

where

$$\begin{pmatrix} x_{11}(\tau_1), & x_{12}(\tau_1) \\ z_{11}(\tau_1), & z_{12}(\tau_1) \end{pmatrix} = \begin{pmatrix} k_{11}(\tau_1), & k_{12}(\tau_1) \\ k_{13}(\tau_1), & k_{14}(\tau_1) \end{pmatrix}^{-1} \begin{pmatrix} c_{11}(\tau_1), & c_{12}(\tau_1) \\ c_{13}(\tau_1), & c_{14}(\tau_1) \end{pmatrix} \tag{7.14}$$

and

$$\begin{pmatrix} q_{11}(\tau_1) \\ q_{12}(\tau_1) \end{pmatrix} = \begin{pmatrix} k_{11}(\tau_1), & k_{12}(\tau_1) \\ k_{13}(\tau_1), & k_{14}(\tau_1) \end{pmatrix}^{-1} \left\{ -K_1 \begin{pmatrix} \chi_{11}(\tau_1) \\ \chi_{13}(\tau_1) \end{pmatrix} + K_2 \begin{pmatrix} \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{pmatrix} \begin{pmatrix} \chi_{21}(\tau_2) \\ \chi_{23}(\tau_2) \end{pmatrix} \right\}, \tag{7.15}$$

From Eq. (7.7), we have

$$\begin{pmatrix} x_1(\tau_1) \\ z_1(\tau_1) \end{pmatrix} = - \begin{pmatrix} \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{pmatrix} \begin{pmatrix} x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} + \begin{pmatrix} F_{111}(\tau_1), & F_{112}(\tau_1) \\ F_{131}(\tau_1), & F_{132}(\tau_1) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - K_1 \begin{pmatrix} \chi_{11}(\tau_1) \\ \chi_{13}(\tau_1) \end{pmatrix}. \tag{7.16}$$

Substituting Eq. (7.16) into Eq. (7.8) gives

$$\begin{aligned} \begin{pmatrix} k_{21}(\tau_2), & k_{22}(\tau_2) \\ k_{23}(\tau_2), & k_{24}(\tau_2) \end{pmatrix} \begin{pmatrix} x_2(\tau_2) \\ z_2(\tau_2) \end{pmatrix} &= \begin{pmatrix} c_{21}(\tau_2), & c_{22}(\tau_2) \\ c_{23}(\tau_2), & c_{24}(\tau_2) \end{pmatrix} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} - K_2 \begin{pmatrix} \chi_{21}(\tau_2) \\ \chi_{23}(\tau_2) \end{pmatrix} \\ &\quad - K_1 \begin{pmatrix} \chi_{211}(\tau_2), & \chi_{213}(\tau_2) \\ \chi_{231}(\tau_2), & \chi_{233}(\tau_2) \end{pmatrix} \begin{pmatrix} \chi_{11}(\tau_1) \\ \chi_{13}(\tau_1) \end{pmatrix}, \end{aligned} \tag{7.17}$$

where

$$\begin{bmatrix} k_{21}(\tau_2), & k_{22}(\tau_2) \\ k_{23}(\tau_2), & k_{24}(\tau_2) \end{bmatrix} = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} + \begin{bmatrix} \chi_{211}(\tau_2), & \chi_{213}(\tau_2) \\ \chi_{231}(\tau_2), & \chi_{233}(\tau_2) \end{bmatrix} \begin{bmatrix} \chi_{111}(\tau_1), & \chi_{113}(\tau_1) \\ \chi_{131}(\tau_1), & \chi_{133}(\tau_1) \end{bmatrix} \tag{7.18}$$

and

$$\begin{bmatrix} c_{21}(\tau_2), & c_{22}(\tau_2) \\ c_{23}(\tau_2), & c_{24}(\tau_2) \end{bmatrix} = \begin{bmatrix} \chi_{211}(\tau_2), & \chi_{213}(\tau_2) \\ \chi_{231}(\tau_2), & \chi_{233}(\tau_2) \end{bmatrix} \begin{bmatrix} F_{111}(\tau_1), & F_{112}(\tau_1) \\ F_{131}(\tau_1), & F_{132}(\tau_1) \end{bmatrix} + \begin{bmatrix} E_{211}(\tau_2), & E_{212}(\tau_2) \\ E_{231}(\tau_2), & E_{232}(\tau_2) \end{bmatrix}. \tag{7.19}$$

Similarly Eq. (7.17) gives the following equations for obtaining the unknown values $x_2(\tau_2)$ and $z_2(\tau_2)$,

$$\begin{bmatrix} x_2(\tau_2) \\ z_2(\tau_2) \end{bmatrix} = \begin{bmatrix} x_{21}(\tau_2), & x_{22}(\tau_2) \\ z_{21}(\tau_2), & z_{22}(\tau_2) \end{bmatrix} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix} + \begin{bmatrix} q_{21}(\tau_2) \\ q_{22}(\tau_2) \end{bmatrix}, \tag{7.20}$$

where

$$\begin{bmatrix} x_{21}(\tau_2), & x_{22}(\tau_2) \\ z_{21}(\tau_2), & z_{22}(\tau_2) \end{bmatrix} = \begin{bmatrix} k_{21}(\tau_2), & k_{22}(\tau_2) \\ k_{23}(\tau_2), & k_{24}(\tau_2) \end{bmatrix}^{-1} \begin{bmatrix} c_{21}(\tau_2), & c_{22}(\tau_2) \\ c_{23}(\tau_2), & c_{24}(\tau_2) \end{bmatrix} \tag{7.21}$$

and

$$\begin{bmatrix} q_{21}(\tau_2) \\ q_{22}(\tau_2) \end{bmatrix} = \begin{bmatrix} k_{21}(\tau_2), & k_{22}(\tau_2) \\ k_{23}(\tau_2), & k_{24}(\tau_2) \end{bmatrix}^{-1} \left\{ -K_2 \begin{bmatrix} \chi_{21}(\tau_2) \\ \chi_{23}(\tau_2) \end{bmatrix} - K_1 \begin{bmatrix} \chi_{211}(\tau_2), & \chi_{213}(\tau_2) \\ \chi_{231}(\tau_2), & \chi_{233}(\tau_2) \end{bmatrix} \begin{bmatrix} \chi_{11}(\tau_1) \\ \chi_{13}(\tau_1) \end{bmatrix} \right\}. \tag{7.22}$$

Next the second equation of Eq. (7.5) is

$$-\chi_{221}(\tau_2)x_1(\tau_1) - \chi_{223}(\tau_2)z_1(\tau_1) + y_2(\tau_2) - y_1(\tau_1) = E_{221}(\tau_2)\cos\theta_1 + E_{222}(\tau_2)\sin\theta_1 - K_2\chi_{22}(\tau_2). \tag{7.23}$$

Substituting of Eq. (7.13) into Eq. (7 23) yields

$$y_{10}(\tau_1) = F_{1c}(\tau_1)\cos\theta_1 + F_{1s}(\tau_1)\sin\theta_1, \tag{7.24}$$

where

$$\left.\begin{aligned} y_{10}(\tau_1) &= K_2\chi_{22}(\tau_2) - \chi_{221}(\tau_2)q_{11}(\tau_1) - \chi_{223}(\tau_2)q_{12}(\tau_1) + y_2(\tau_2) - y_1(\tau_1), \\ F_{1c}(\tau_1) &= E_{221}(\tau_2) + \chi_{221}(\tau_2)x_{11}(\tau_1) + \chi_{223}(\tau_2)z_{11}(\tau_1), \\ F_{1s}(\tau_1) &= E_{222}(\tau_2) + \chi_{221}(\tau_2)x_{12}(\tau_1) + \chi_{223}(\tau_2)z_{12}(\tau_1). \end{aligned}\right\}. \tag{7.25}$$

By this equation, θ_1 can be obtained as follows

$$\theta_1 = \theta_0 + \cos^{-1}\frac{y_{10}(\tau_1)}{F_{10}(\tau_1)}, \tag{7.26}$$

where

$$\theta_0 = \tan^{-1}\frac{F_{1s}(\tau_1)}{F_{1c}(\tau_1)}, \quad F_{10}(\tau_1) = \sqrt{F_{1c}^2(\tau_1) + F_{1s}^2(\tau_1)}. \tag{7.27}$$

The second equation of Eq. (7.4) is

$$\chi_{121}(\tau_1)x_2(\tau_2) + \chi_{123}(\tau_1)z_2(\tau_2) + y_2(\tau_2) + y_1(\tau_1) = F_{121}(\tau_1)\cos\theta_1 + F_{122}(\tau_1)\sin\theta_1 - K_1\chi_{12}(\tau_1). \tag{7.28}$$

Substitution of Eq. (7.20) in Eq. (7.28) yields

$$y_{20}(\tau_1) = F_{2c}(\tau_1)\cos\theta_1 + F_{2s}\sin(\tau_1)\theta_1, \tag{7.29}$$

where

$$\left.\begin{aligned} y_{20}(\tau_1) &= K_1\chi_{12}(\tau_1)+\chi_{121}(\tau_1)q_{21}(\tau_2)+\chi_{123}(\tau_1)q_{22}(\tau_2)+y_2(\tau_2)+y_1(\tau_1)\,, \\ F_{2c}(\tau_1) &= F_{121}(\tau_1)-\chi_{121}(\tau_1)x_{21}(\tau_2)-\chi_{123}(\tau_1)z_{21}(\tau_2)\,, \\ F_{2s}(\tau_1) &= F_{122}(\tau_1)-\chi_{121}(\tau_1)x_{22}(\tau_2)-\chi_{123}(\tau_1)z_{22}(\tau_2) \end{aligned}\right\}. \tag{7.30}$$

Consequently, substituting of Eqs. (7.14), (7.15), (7.21), (7.22) and (7.26) into Eq. (7.29), the transcendental equation can be obtained as follows, which containing unexplicitly the unknown function τ_1,

$$w_n'\{y_1(\tau_1),\ y_2(\tau_2),\ \tau_1\} = 0\,, \tag{7.31}$$

then we can determine τ_1 by solving this equation.

8. $f_{rn}(\tau)$ and $g_{rn}(\tau)$

(a) When the equation $p^3+\delta_0 p^2+p\gamma_{10}+a_1=0$ has three distinct roots.
Putting

$$p^3+\delta_0 p^2+p\gamma_{10}+a_1 = (p+a_1)(p+a_2)(p+a_3)\,, \tag{8.1}$$

$f_{rn}(\tau)$'s and $g_{rn}(\tau)$'s are tabulated in Tables 3, 4 and 5, when $r=1$,

Table 3.

$g_{10}(\tau)$	$\frac{1}{a_1}+\frac{1}{a_1A_1}\varepsilon^{-a_1\tau}+\frac{1}{a_2A_2}\varepsilon^{-a_2\tau}+\frac{1}{a_3A_3}\varepsilon^{-a_3\tau}$
$g_{11}(\tau)$	$-\frac{1}{A_1}\varepsilon^{-a_1\tau}-\frac{1}{A_2}\varepsilon^{-a_2\tau}-\frac{1}{A_3}\varepsilon^{-a_3\tau}$
$g_{12}(\tau)$	$\frac{a_1}{A_1}\varepsilon^{-a_1\tau}+\frac{a_2}{A_2}\varepsilon^{-a_2\tau}+\frac{a_3}{A_3}\varepsilon^{-a_3\tau}$
$g_{13}(\tau)$	$-\frac{d_1^2}{A_1}\varepsilon^{-a_1\tau}-\frac{d_2^2}{A_2}\varepsilon^{-a_2\tau}-\frac{a_3^2}{A_3}\varepsilon^{-a_3\tau}$

Table 4.

$f_{11}(\tau)$	$b_0\cos\tau+b_1\sin\tau-\frac{1}{B_1}\varepsilon^{-a_1\tau}-\frac{1}{B_2}\varepsilon^{-a_2\tau}-\frac{1}{B_3}\varepsilon^{-a_3\tau}$
$f_{12}(\tau)$	$b_0\sin\tau+b_1\cos\tau+\frac{a_1}{B_1}\varepsilon^{-a_1\tau}+\frac{a_2}{B_2}\varepsilon^{-a_2\tau}+\frac{a_3}{B_3}\varepsilon^{-a_3\tau}$
$f_{13}(\tau)$	$-b_0\cos\tau-b_1\sin\tau-\frac{a_1^2}{B_1}\varepsilon^{-a_1\tau}-\frac{a_2^2}{B_2}\varepsilon^{-a_2\tau}-\frac{a_3^2}{B_3}\varepsilon^{-a_3\tau}$
$f_{14}(\tau)$	$b_0\sin\tau-b_1\cos\tau+\frac{a_1^3}{B_1}\varepsilon^{-a_1\tau}+\frac{a_2^3}{B_2}\varepsilon^{-a_2\tau}+\frac{a_3^3}{B_3}\varepsilon^{-a_3\tau}$

Table 5.

b_1	$\frac{a_1-\delta_0}{(a_1^2+1)(a_2^2+1)(a_3^2+1)}$	b_0	$\frac{1-\gamma_0}{(a_1^2+1)(a_2^2+1)(a_3^2+1)}$
B_1	$(1+a_1^2)A_1$	A_1	$(a_1-a_2)(a_3-a_1)$
B_2	$(1+a_2^2)A_2$	A_2	$(a_2-a_3)(a_1-a_2)$
B_3	$(1+a_3^2)A_3$	A_3	$(a_3-a_1)(a_2-a_3)$

The values of $f_{2n}(\tau)$'s and $g_{2n}(\tau)$'s are obtained by replacing γ_{10} and a_1 by γ_{20} and a_2 in the expressions of $f_{1n}(\tau)$'s and $g_{1n}(\tau)$'s.

(b) When the equation $p^3+\delta_0 p^2+\gamma_{10}p+a_1=0$ has one root and a pair of conjugate roots.

Putting

$$p^3+\delta_0 p^2+\gamma_{10}p+a_1 = (p+\alpha_1)(p+\alpha+j\beta)(p+\alpha-j\beta), \tag{8.2}$$

we have Tables 6, 7 and 8.

Table 6.

$g_{10}(\tau)$	$\frac{1}{a_1}-\frac{e_0}{a_1}\varepsilon^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(e_1\cos\beta-e_2\sin\beta\tau)$
$g_{11}(\tau)$	$e_0\varepsilon^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(-e_3\cos\beta\tau+e_4\sin\beta\tau)$
$g_{12}(\tau)$	$-\alpha_0 e_0\varepsilon^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(e_5\cos\beta\tau-e_6\sin\beta\tau)$
$g_{13}(\tau)$	$\alpha_1^2 e_0\varepsilon^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(-e_7\cos\beta\tau+e_8\sin\beta\tau)$

Table 7.

$f_{11}(\tau)$	$b_0\cos\tau+b_1\sin\tau+d_0\varepsilon^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(-d_1\cos\beta\tau+d_2\sin\beta\tau)$
$f_{12}(\tau)$	$b_1\cos\tau-b_0\sin\tau-\alpha_1 d_0\varepsilon^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(d_3\cos\beta\tau-d_4\sin\beta\tau)$
$f_{13}(\tau)$	$-b_0\cos\tau-b_1\sin\tau+\alpha_1^2 d_0\varepsilon^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(-d_5\cos\beta\tau+d_6\sin\beta\tau)$
$f_{14}(\tau)$	$-b_1\cos\tau+b_0\sin\tau-\alpha_1^3 d_0{}^{-\alpha_1\tau}+\varepsilon^{-\alpha\tau}(d_7\cos\beta\tau-d_8\sin\beta\tau)$

Table 8.

e_0	$\frac{1}{(\alpha_1-\alpha)^2+\beta^2}$	d_0	$\frac{e_0}{\alpha_1^2+1}$	d_1	$\frac{e_0}{\Delta}(3\alpha^2-\beta^2+1-2\alpha_1\alpha)$
e_2	$\frac{\alpha_1}{\alpha_1\beta}\{\alpha(\alpha_1-\alpha)+\beta^2\}e_0$	e_1	$\frac{e_0}{\alpha_1}-\frac{1}{a_1}$	d_2	$\frac{e_0}{\beta\Delta}\{(\alpha_1-\alpha)(\alpha^2-\beta^2+1)+2\alpha\beta^2\}$
e_3	e_0	e_5	$\alpha_1 e_0$	d_3	$\alpha d_1+\beta d_2$
e_6	$\frac{e_0}{\beta}\{\alpha(\alpha_1-\alpha)-\beta^2\}$	e_4	$e_0\cdot\frac{\alpha_1-\alpha}{\beta}$	d_4	$\alpha d_2-\beta d_1$
				d_5	$\alpha d_3+\beta d_4$
e_8	$\alpha e_6-\beta e_5$	e_7	$-e_0(\alpha-\beta)^2$	d_6	$\alpha d_4-\beta d_3$
b_0	$\frac{1-\gamma_{10}}{(\alpha_1^2+1)\cdot\Delta}$	b_1	$\frac{\alpha_1-\delta_0}{(\alpha_1^2+1)\cdot\Delta}$	d_7	$\alpha d_5+\beta d_6$
Δ	$(\alpha^2-\beta^2+1)^2+4\alpha^2\beta^2$	a_1	$\alpha_1(\alpha^2+\beta^2)$	d_8	$\alpha d_6-\beta d_5$

As in the previous case, the values $f_{2n}(\tau)$'s and $g_{2n}(\tau)$'s are obtained on replacing γ_{10} and a_1 by γ_{20} and a_2 in the expressions of $f_{1n}(\tau)$'s and $g_{1n}(\tau)$'s respectively.

9. Numerical Example

In the present section a numerical example of the basic circuit is illustrated, and the results of this example were obtained by making use of the Digital Computer (KDC-1). Here, for the sake of comparing with those of Reference (3) we assume that the circuit constants are the same ones in Reference (3) as follows:

$$L_1 = L_3 = 1.17\,H, \quad L_2 = L_4 = 1.7\,mH,$$

$$C = 0.002\,\mu F, \quad R = 2000\,\Omega, \quad f = \frac{\omega}{2\pi} = 1000\,c/s.$$

In the numerical calculation of determing τ_1, we take

$$\Delta\tau_1 = 0.001, \quad \varepsilon = 0.0001.$$

The results are shown in Figs. 7, 8, 9, 10 and 11.

Fig. 7 shows the determined values τ_1 with respect to a measure of the hysteresis of the coil, i.e.

$$\Delta y\,\% = \frac{y_1(\tau_1) - y_2(\tau_2)}{y_1(\tau_1)} \times 100\%$$

for some parameters K', i.e. $K' = 1/K_2 = I_m/I_1$.

From this result, we can find that the pulse width τ_1 increases in proportion to increasing of the measure of the hysteresis, and that the values τ_1 are equal to those of Reference (3) when $\Delta y\% = 0$, without the hysteresis.

Fig. 8 shows the values of θ_1 which also increase in accordance with increasing $\Delta y\%$. The peak value of $E_L(t)$ is shown in Figs. 9 and 10, from which, the effect of the hysteresis of the coil makes the peak voltake decrease. Fig. 11

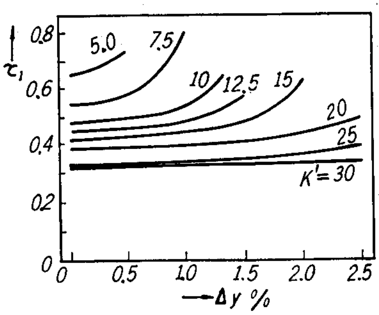

Fig. 7. Durations of the 1-st circuit mode (Widths of pulse voltages across the coil).

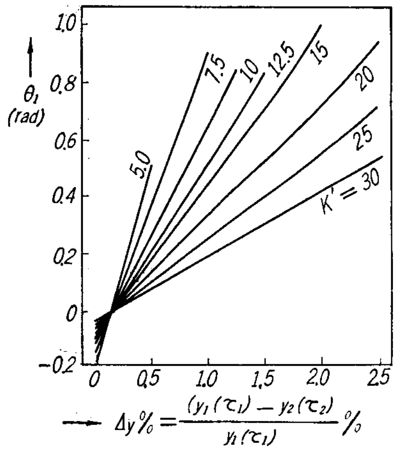

Fig. 8. Phase angles θ_1.

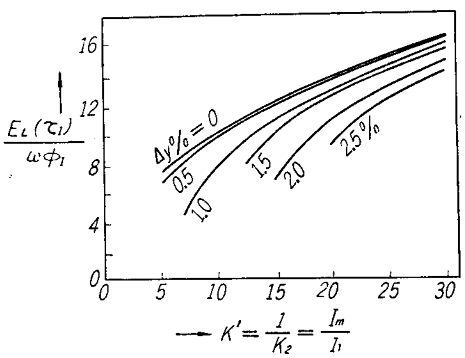

Fig. 9. Peak pulse voltage across the coil.

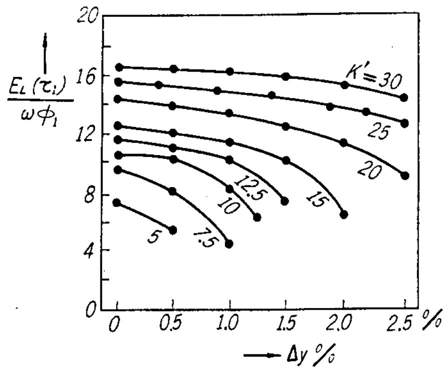

Fig. 10. Peak pulse voltage across the coil.

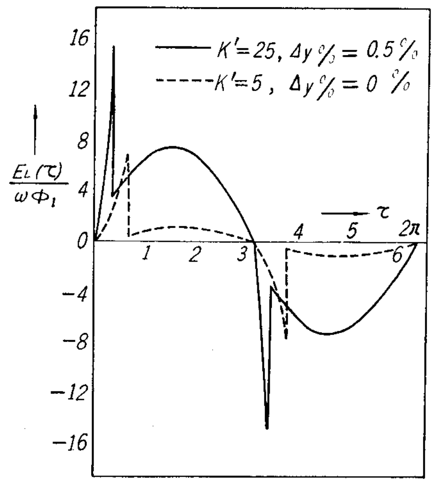

Fig. 11. Wave forms of voltage across the coil.

shows the wave forms of the steady state voltage across the coil, in this case we can find that the wave form was remarkably distorted by the effect of the hysteresis, comparing with the case without the hysteresis.

10. Conclusion

In this paper, the study on the harmonic producer circuits is presented by the use of the analytical method based on the theorems of the Periodically Interrupted Electric Circuits of the Third Genus.

Taking the effect of magnetic hysteresis of the coil into consideration, we can make clear the behaviour of the producer circuits as accurately as possible, and show the numerical results of one example.

It is evident that this analytical method is useful for the analysis of the non-linear problems such as those containing the hysteresis or other non-linearity in order to obtain the steady state solutions, but it is difficult to determine the periods which decided by the natural phenomena in the considered problems, in that case, by means of the digital computer we can easily determine these values for the case of some numerical examples.

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