An Analysis of Non-Linear Sampled-Data Feedback Control Systems

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By

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Higher order sampled-data feedback systems which contain a saturating element or a backlash element are investigated in this paper.

This study introduces a new approach to the analysis of non-linear sampled-data control systems. At first the authors describe a new analytlcal method for such systems using the theorem of Periodically Interrupted Electric Circuits and how to apply the Digital Computer (KDC-1) to this theorem. The method presented here can be applied to any higher order systems with any non-linear elements by making use of the digtal computer simulation of the above theorem and the non-linear element. Some illustrative examples are given to clarify the method involved. One example of the third order sampled-data feedback system wlth a saturating element is investigated in case of initial conditions being given without external forces, a unit step function and sine wave inputs being given. The examples show that in the case of a step input as well as initial conditions existing, slight variations of initial values result in difierent modes of periodic oscillations, while in the case of a sinusoidal input, a slight modification of non-linear characteristics results in forced oscillations in one case and in sub-harmynic oscillations in another.

Two illustrative examples of the second order system with a backlash element are considered in the case where the linear system is followed by the backlash or follows the backlash.

Some results obtained by numerical computations are presented to show the performance of the system dynamics on the basis of the new analytical method presented here.

1. Introduction

To feedback control systems belongs a sampled-data feedback system for which the input signal is represented by samples at regular intervals of time as a discontinious waveform, for example, the control system with contactor relay mechanism, the radar system and the digital computer control system. There are two types in the sampled-data feedback systems, one of which is a

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linear sampled-data feedback system containing linear elements and another is a non-linear sampled-data feedback system containing non-linear elements.

The theory of the former has progressed and been well constructed by means of the techniques of the z transform and the modified z transform. However the theory of the latter containing non-linear elements, for example, a saturating element, a backlash element, and a hysteresis or deadzone element, has hardly been established because of difficulties of analyzing the complicated phenomena arising between the discontinious signal and nonlinearity. However C.K. Chow[1] has recently studied about the self-sustained oscillations in relay servomechanisms with sampling by using the describing function method. R.E. Kalman[2], on the other hand, has investigated the initial condition problems for the same system by the phase plane method and has obtained unexpected results from the theoretical point of view.

The approach to the analysis of the transient behavior of the low order relay sampled-data feedback system has been introduced in the past by F.J. Mullin, E.I. Jury[3], K. Izawa and L.E. Weaver[4] by use of the phase plane method. Especially F.J. Mullin[5] has studied the stability and the compensation of the saturating sampled-data feedback systems and S. Kodama[6], from a different point of view, investigated the stability of a non-linear sampled-data feedback system and he gave the counter theory for the stability to F.J. Mullin's study. These investigations based on various assumptions of non-linear elements as well as the dynamic chracteristic of the feedback systems have not been established generally for the non-linear sampled-data feedback systems.

In this paper we introduce a new approach to the analysis of the higher order sampled-data feedback systems containing non-linear elements. This analytical technique for the transient and steady state behavior of the systems is based on the theorem of Periodically Interrupted Electrical Circuits[7] applying the digital computer[8][9]. Although only saturating and backlash elements are here taken into account as non-linear elements, the analytical method presented here may serve as a useful tool in the system with other non-linear elements.

2. Application of the Theorem of Periodically Interrupted Electric Circuits

The system under consideration is shown in Fig. 1. The non-linearity of the memory or the zero-memory type follows a zero-order hold network.

The controlled system is considered

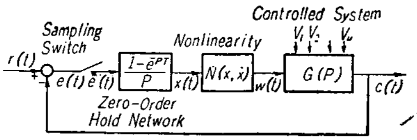

Fig. 1 Nonlinear sampled-data control system

generally as an m-th order linear element and subject to the disturbances $v(t)$'s which are applied to points of the cascaded output elements.

When the input to the nonlinearity is $x(t)$, the output $w(t)$ of the nonlinearity is written as

$$\begin{aligned} w(t) &= N\{x(t),\ \dot{x}(t)\} \\ &= w_n \qquad\qquad t = \overline{n-1}T+0 \end{aligned} \tag{1}$$

where $N(x, \dot{x})$ is the non-linear function representing the non-linearity, T is the sampling period, $\dot{x}(t)=\dfrac{d}{dt}x(t)$ and $n=1, 2, \cdots$. This output $w(t)$ becomes constant during one sampling period as shown in Fig. 2.

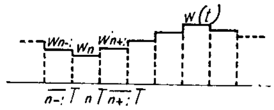

Fig. 2 Output of the nonlinearity

The system equation can be set up, in general, in the matrix form as follows during some (n) sampling period

$$\begin{pmatrix} D+a_{11} & \lambda_{12} \cdots\cdots & \lambda_{1m} \\ \lambda_{21} & D+a_{22} & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \lambda_{m-1, m} \\ \lambda_{m1}\cdots\lambda_{m, m-1} & & D+a_{mm} \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_{m-1}(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} w_n \\ v_1(t)+v_{1n} \\ v_2(t)+v_{2n} \\ \vdots \\ v_u(t)+v_{un} \end{pmatrix} \quad 0 \leqq t \leqq T \tag{2}$$

where

$$D = \frac{d}{dt},\ v_{un} = v_u(t)],\quad t = \overline{n-1}T \qquad \lambda_{12}, \cdots, \lambda_{m, m-1}, = 0 \text{ or } \pm 1$$

and $a_{11}, \cdots, a_{mm}$ are constants determined by the controlled system, $c_1(t), \cdots, c_{m-1}(t)$ are outputs at points of the cascaded first order output elements of the controlled system.

Now we rewrite Eq. (2) in the abbreviated form as

$$[Z(D)][g_n(t)] = [y_n(t)] \qquad 0 \leqq t \leqq T \tag{3}$$

and Eq. (3) can be replaced by the operational form (the second kind Laplace transform)

$$[Z(p)][g_n(p)] = [y_n(p)] + p[L][g_{n-1}(T)] \tag{4}$$

where $[L]$ is a constant determined by $[Z(D)]$ and generally this matrix is a unit matrix.

Premultiplying both sides of Eq. (4) by $[Z(p)]^{-1}$, the following equation can be obtained

$$[g_n(p)] = [Z(p)]^{-1}[y_n(p)] + p[Z(p)]^{-1}[L][g_{n-1}(T)] \tag{5}$$

Hence the corresponding time function to Eq. (5) can be derived by taking the inverse transform of Eq. (5) directly

$$[g_n(t)] = [\varphi_n(t)]+[\chi(t)][g_{n-1}(T)] \qquad 0\leqq t\leqq T \tag{6}$$

where

$$[\varphi_n(t)] = \mathfrak{H}^*[Z(p)]^{-1}[y_n(p)]$$
$$[\chi(t)] = \mathfrak{H}p[Z(p)]^{-1}[L]$$

This result is the solution of the system Eq. (2) under consideration during n-th sampling period and corresponds to the solution obtaind by the modified Z trasform used in the linear sampled-data feedback system.

Now the initial matrix $[g_{n-1}(T)]$ of Eq. (6) is evaluated by the following reccurrence formulae in matrix notation as

$$\begin{aligned}[g_{n-1}(T)] &= [\varphi_{n-1}(T)]+[\chi(T)][g_{n-2}(T)] \\ &= [\varphi_{n-1}(T)]+[\chi(T)][\varphi_{n-2}(T)]+[\chi(T)]^2[\varphi_{n-3}(T)]+\cdots\cdots \\ &\qquad +[\chi(T)]^{n-2}[\varphi_1(T)]+[\chi(T)]^{n-1}[g_0^{-0}]\end{aligned} \tag{7}$$

then, substituting Eq. (7) into Eq. (6) results in the solution during any sampling interval.

If the disturbances $v_1(t), \cdots, v_u(t)$ in Fig. 1 were not applied, then $[\varphi_n(t)]$ in Eq. (6) would be replaced and deduced easily in the form of the product of two matrices, that is, $[y_n(t)]$ becomes equal to $[y_n]$ composed of the constant elements being indifferent to time t during one sampling interval, therefore in such a case Eq. (6) can be written as

$$[g_n(t)] = [\varphi(t)][y_n]+[\chi(t)][g_{n-1}(T)] \qquad 0\leqq t\leqq T \tag{8}$$

where

$$\left.\begin{aligned}[\varphi(t)] &= \mathfrak{H}[Z(p)]^{-1} \\ [y_n] &= [N\{[R_n]+[T]'g[_{n-1}(T)]\}]\end{aligned}\right\} \tag{9}$$

In Eq. (9), $[R_n]$ is determined by the $r(t)$ input to the feedback system and includes $r(\overline{n-1}T)$ component, $[(T)]'$ is the so-called transfer matrix whose components of 0 or ± 1. In this case the initial matrix $[g_{n-1}(T)]$ can be written as

$$\begin{aligned}[g_{n-1}(T)] &= [\varphi(T)][y_{n-1}]+[\chi(T)][g_{n-2}(T)] \\ &= [\varphi(T)][y_{n-1}]+[\chi(T)][\varphi(T)][y_{n-2}]+[\chi(T)]^2[\varphi(T)][y_{n-3}]+\cdots\cdots \\ &\qquad +[\chi(T)]^{n-2}[\varphi(T)][y_1]+[\chi(T)]^{n-1}[g_0^{-0}]\end{aligned} \tag{10}$$

Next we shall induce the condition of the self-sustained oscillations. In the case where the $r(t)$ input to the system exists, it is possible to obtain its conditions, though we now consider the case where the $r(t)$ input is not given.

If the self-sustained oscillations of period $rT(r=1, 2, \cdots)$ are to exist in the system when $r(t)=0$, we can put, from Eq. (9)

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* $\mathfrak{H}\{f(p)\} = \dfrac{1}{2\pi j}\lim_{\beta\to\infty}\displaystyle\int_{r-j\beta}^{r+j\beta}\frac{f(p)}{p}e^{pt}dp$

$$[N\{[T]'[g_{n-1}(T)]\}] = [\varepsilon_{n-1}] \tag{11}$$

and substituting Eq. (11) into Eq. (8) yields the following matrix according to the condition that Eq. (8) should have the same value at the r-th sampling instant as the value at the previous r-th sampling instant, that is,

$$\begin{aligned}[B_\mu] = &\{[\varphi(T)][\varepsilon_{\mu-1}]+[\chi(T)]\}\{[\varphi(T)][\varepsilon_{\mu-2}]+[\chi(T)]\}\cdots \\ &\{[\varphi(T)][\varepsilon_1]+[\chi(T)]\}\{[\varphi(T)][\varepsilon_r]+[\chi(T)]\}\cdots\{[\varphi(T)][\varepsilon_n]+[\chi(T)]\}\end{aligned} \tag{12}$$

where

$$\mu = 1, 2, \cdots, r.$$

Hence we can obtain the β's latent roots of the matrix $[B_\mu]$ by solving the characteristic equation of the form

$$\delta\{\beta[U]-[B_\mu]\} = 0 \tag{13}$$

Then we know that the necessary and sufficient condition* where the system would generate the self-sustained oscillations is that the absolute values of all the latent roots $\beta_1, \beta_2, \cdots, \beta_m$ should be equal to unity respectively. If there is only one latent root which is more than unity, the system is unstable and if all the latent roots are less than unity, the system is stable because the solution becomes infinitesimal as time goes.

3. Simulation by the Digital Computer and Numerical Examples

In practice it is difficult to clarify theoreticaly the dynamic characteristics of the system according to the above theorem, but it is easily possible to place the characteristics more clearly in sight by using the digital computer to simutate this analytical method applied to some examples.

Now consider the case where the controlled system is the 3rd order system without disturbances and the non-linearity is a saturaing element shown in Fig. 3.

Eq. (8) represents the solution during n-th sampling interval when the saturating element is given

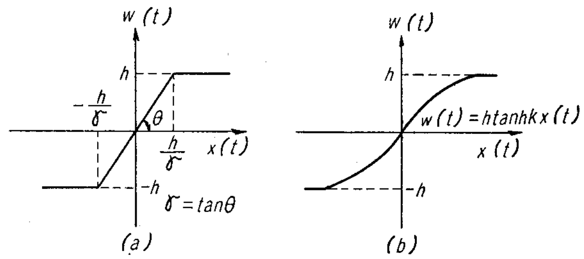


Fig. 3 Nonlinarity (saturating element)

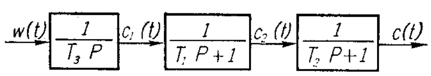


Fig. 4 Controlled system $G(p)$

* When r is a rational number, similarly we consider the self-sustained oscillation, therefore the condition of the self-sustained oscillatlons could be established in general.

by Fig. 3(a), where $[y_n]$ of Eq. (9) can be determined by the following three regions denoted such that

(1), if the linear region exists;

$$|x_n| \equiv |\{r(t)-c(t)\}| \leq h/r \qquad t=\overline{n-1}T+0$$

then Eq. (9) becomes

$$\begin{aligned}[y_n] &= r\{[r(\overline{n-1}T)]+[T]'[g_{n-1}(T)]\} \\ &= \begin{pmatrix} rx_n \\ 0 \\ 0 \end{pmatrix},\end{aligned} \tag{14}$$

(2), if the positive saturate region exists;

$$x_n > \frac{h}{r}$$

then Eq. (9) becomes

$$[y_n] = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}, \tag{15}$$

(3), and if the negative saturating region exists;

$$x_n < \frac{h}{r}$$

then Eq. (9) becomes

$$[y_n] = \begin{pmatrix} -h \\ 0 \\ 0 \end{pmatrix}. \tag{16}$$

In the case of this example, the digital computer flowchart is shown in Fig. 5.

When the saturating element whose characteristic is represented by the following Eq. (17) is given in Fig. 3(b), $[y_n]$ can be more easily determined rather than in the above case.

$$w(t) = h \tanh kx(t) \tag{17}$$

Fig. 6 and Fig. 7 show the numerical results in the case of these examples. Fig. 6(a) shows the response to a unit step function input in which the different modes of periodic oscillation occurring depend on a slight variations of initial values.

Other different modes of periodic oscillation depending on other initial conditions in the same system was generated.

Fig. 6(b) shows the response of the system with various saturating elements to a unit step function input in the case where the γ of the non-linearity is only changed at same initial conditions.

The modes (2), (3) and (4) in Fig. 6(b) illustrate the periodic oscillation, therefore we know that it is possible for one value of the γ of the non-linearity that the periodic oscillation should exist in the system, but for another value

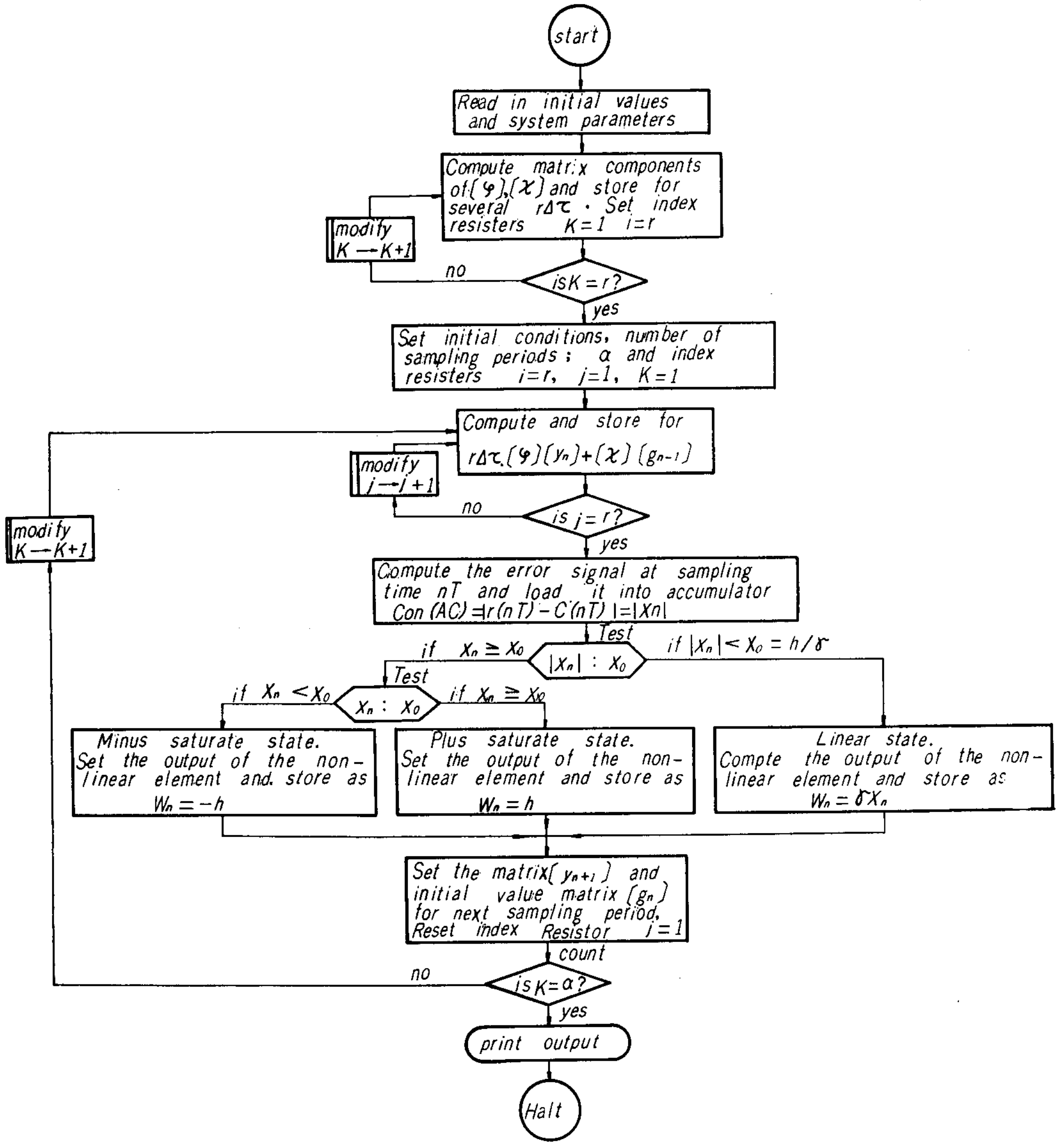
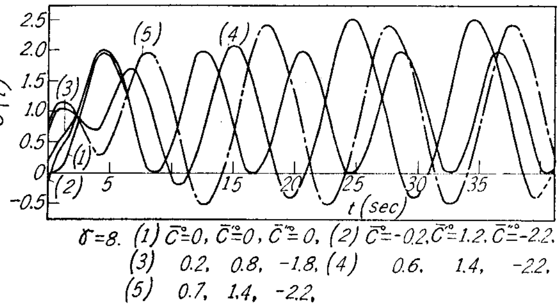
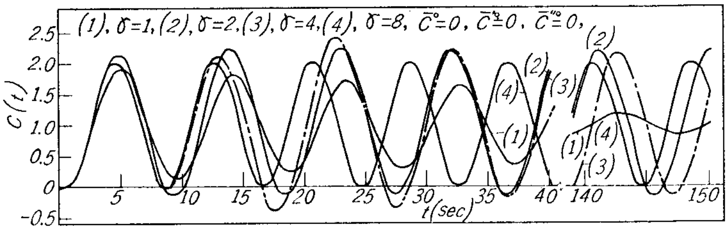


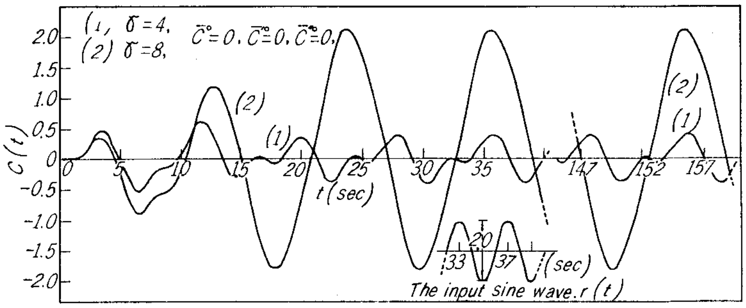
Fig. 5 Digital computer flowchart for the higher order sampled-data control system with a saturating element



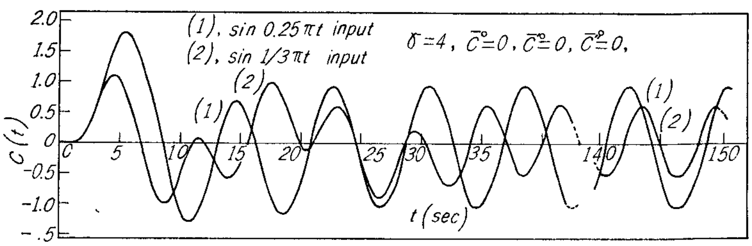
(a) Different modes of periodic oscillations in response to a unit step function input



(b) The time response of the system with various saturating elements to a unit step function input

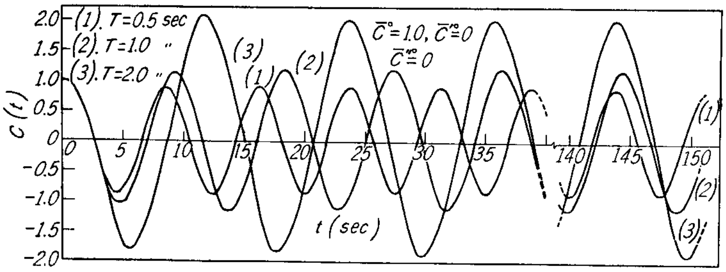


(c) Forced oscillations in the system with a saturating element for $r(t)=\sin 0.5\pi t$

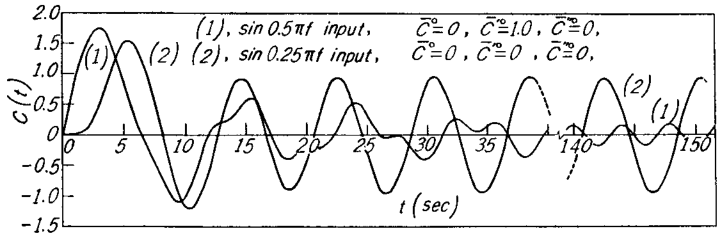


(d) The transient response of the system with a saturating element to sinusoidal inputs

Fig. 6 Response of the system with a saturating element, $h=1.0$, $T=1.0$ (sec) and $G(p)=\dfrac{1}{p(p+1)^2}$ to a unit step function and sinusoidal inputs

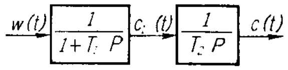


(a) The response of the system with a saturating element ($w(t)=\tanh 3x(t)$) to $r(t)=0$

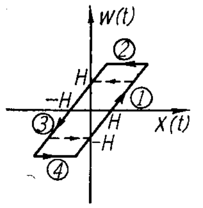


(b) The response of the system with a saturating element ($w(t)=\tanh 2x(t)$) and $T=1.0$ to sinusoidal inputs

Fig. 7 Response of the system with a saturating element described $w(t)=h \tanh kx(t)$ and $G(p)=\dfrac{1}{p(p+1)^2}$ to sine wave inputs and no input but only initial values



(a) Controlled system $G(p)$



(b) Nonlinearity (backlash)

Fig. 8 Controlled system and backlash element

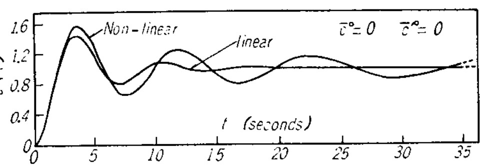
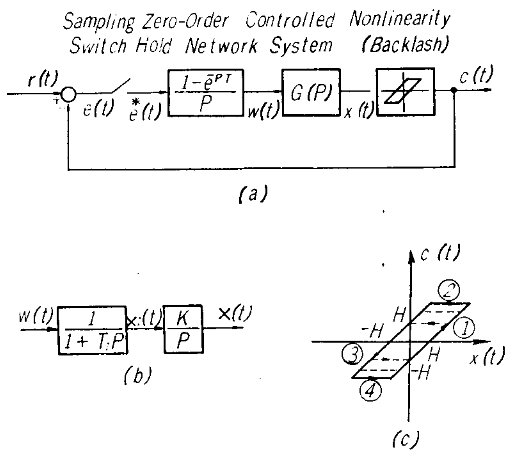


Fig. 9 The response of the second order system of $G(p)=\dfrac{1}{p(p+1)}$, $T=1.0$ with a backlash element $H=0.1$ and without (linear) to a unit step function input



(a) Sampled-data sarvo with backlash element
(b) Controlled system, $G(p)$
(c) Nonlinearity (backlash)

Fig. 10 Non-linear sampled data control system

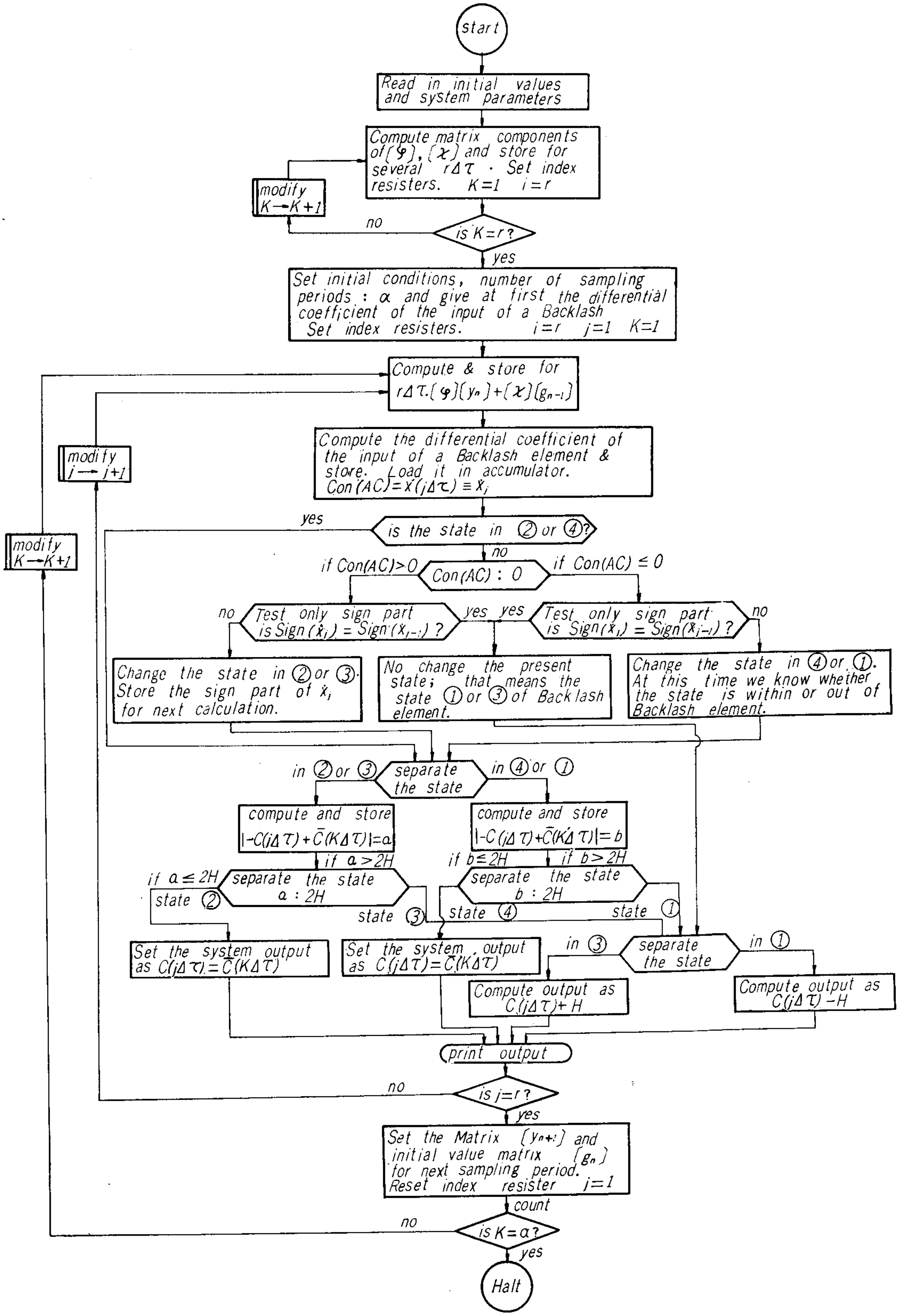


Fig. 11 Digital computer flowchart for the second order sampled-data control system with a backlash element

it is impossible.

Fig. 6 (c) and (d) show the response of the system to a sine wave input. The forced oscillations shown in Fig. 6 (c) are the subharmonic oscillation of order 1/2 illustrated by curve 1 and order 1/3 by curve 2, however in the case shown in Fig. 6 (d) the subharmonic oscillation don't occur under the same condition as in Fig. 6 (c).

Fig. 7(a) shows the response of the system to no inputs only due to the given initial value for changing the sampling period and Fig. 7(b) indicates the response of the same system to a sine wave input under the same conditions in Fig. 7(a).

Next we show the numerical examples of the system with a backlash element as the non-linearity shown in Fig. 8 (b).

At first we consider that the controlled system is the 2nd order system without disturbances and a backlash element follows the zero-order hold network as shown in Fig. 1.

In such a case, $[y_n]$ of Eq. (9) is determined by the following separete regions, thus

$$[y_n] = \begin{pmatrix} w(\overline{n-1}T+0) \\ 0 \end{pmatrix}$$

here $w(t)$ is found to be

① $w(t) = x(t) - H,\ (x'(t) = 0)$

② $w(t) = \{x(t) - H\} \qquad t = \overline{n-2}T + 0$ (within backlash)

③ $w(t) = x(t) + H,\ (x'(t) < 0)$

④ $w(t) = \{x(t) + H\} \qquad t = \overline{n-2}T + 0$ (within backlash)

where

H: half of the backlash width, $t = \overline{n-1}T + 0$ (at n-th sampling instant).

Fig. 9 shws the response of the system containing a backlash element to a unit step function input which is compared with the response of the linear system.

When the backlash element follows the controlled system as shown in Fig. 10, one numerical results in such a case is illustrated in Fig. 12 which gives the response of the system to a unit step

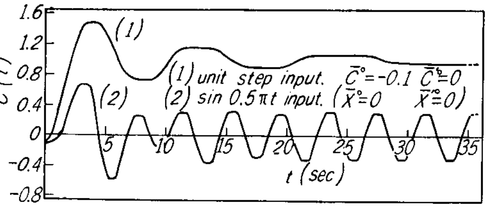


Fig. 12 The transient response of the system of Fig. 10 with $T=1.0$, $T_1=1.0$, $K=1.0$ and $H=0.1$ to a unit step function and sine wave input

function input and to a sine wave input.

The digital computer flowchart for numerical calculation of the feedback system shown in Fig. 10 is illustrated in Fig. 11.

4. Conclusion

An attempt has been made to analyze the higher order sampled-data feedback systems with non-linear element by appling the theorem of Periodically Interrupted Electric Circuits.

The application of this analytical method to some non-linear sampled-data system can be useful in practice only by making use of the digital computer simulation.

In this paper we consider the 3rd order sampled-data feedback system with a saturating element and the 2nd order sampled-data feedback system with a backlash, however it may be possible to analyzed the higher order sampled-dat feedback system with other non-linear elements by means of the analytical method presented here.

It is evident that this approach will be useful to clarify the performance of the sampled-data feedback system with non-linear elements, which has hardly been analyzed by other methods.

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