A Study on the Space and Energy Dependent Reactor Kinetics, with Direct Physical Interpretation of the Effective Neutron Lifetime and Criticality Factor

By

Hiroshi NISHIHARA* and Seiji UKAI*

(Received January 31, 1963)

First, the concept of neutron importance is introduced. It is assumed that each of the neutrons produced by fission in a chain reactor possesses an importance proportional to the number of its descendants. Secondly, on the basis of the law of conservation, a transport equation of the neutron importance is derived. Then, the effective neutron lifetime is defined as the mean interval of successive fission events in the course of the importance transport. The consistent definition of the criticality factor is the neutron multiplicity during the effective neutron lifetime so defined.

After defining the basic reactor kinetics parameters, such as the effective neutron lifetime and criticality factor, the persistent time behavior of nuclear chain reactors has been investigated.

The kernel form reactor equation is used because of its physical intelligibility.

The formulas obtained are applicable to any reactor, provided that the neutron flux and its adjoint function is known either analytically or numerically.

I. Introduction

The effective neutron lifetime and criticality factor are of essential importance in the kinetics of nuclear reactors. In spite of their significance, however, they are not very clearly defined, in the sense that their physical entities are not directly reflected in their definitions.

We now intend to define these parameters on the basis of a direct physical interpretation of the time behavior of chain reactors.

The general neutron transport equation in kernel form is[1]

$$\phi(\boldsymbol{P}, t) = \int_{-\infty}^{t} dt' \int d\boldsymbol{P}' H(\boldsymbol{P}', t' \rightarrow \boldsymbol{P}, t) S(\boldsymbol{P}', t') , \qquad (1)$$

where $\boldsymbol{P}$ represents a point $(\boldsymbol{r}, E, \Omega)$ in the neutron phase space. $S(\boldsymbol{P}', t')$ neutrons are supposed to be born in unit volume of the phase space around $\boldsymbol{P}'$ per unit time at t'.

* Department of Nuclear Engineering

The medium is characterized by the general transport kernel $H(\boldsymbol{P}', t \to \boldsymbol{P}, t)$ which gives the neutron flux at time t in unit volume of neutron phase space around $\boldsymbol{P}$ due to a single neutron produced by fission or some other process at point $\boldsymbol{P}'$ at time t'.

In the case of prompt neutron kinetics, where the effect of delayed neutrons is neglected, $S(\boldsymbol{P}', t')$ is given by

$$S(\boldsymbol{P}', t') = \frac{1}{4\pi} f(E') \int dE'' \nu(E'') \Sigma_f(\boldsymbol{r}', E'') \phi_0(\boldsymbol{r}', E'', t') , \tag{2}$$

where $f(E)$ is the normalized spectrum of prompt fission neutrons. The total flux ϕ_0 is the integral of the angular flux ϕ over all directions:

$$\phi_0(\boldsymbol{r}', E'', t') = \int d\Omega'' \phi(\boldsymbol{r}', E'', \Omega'', t') . \tag{3}$$

$\nu(E)$ is the neutron multiplicity which gives the number of neutrons born per fission induced by a neutron of energy E, while $\Sigma_f(\boldsymbol{r}, E)$ is the medium's fission cross section for neutrons of energy E.

In order to eliminate the energy and angular dependences, we multiply the equation (1) through by $\nu(E)\Sigma_f(\boldsymbol{r}, E)$ and integrate over energy E and direction Ω. Then we obtain

$$S(\boldsymbol{r}, t) = \int_{-\infty}^{t} dt' \int d\boldsymbol{r}' G(\boldsymbol{r}', t' \to \boldsymbol{r}, t) S(\boldsymbol{r}', t') , \tag{4}$$

where

$$S(\boldsymbol{r}, t) = \int dE \nu(E) \Sigma_f(\boldsymbol{r}, E) \Phi_0(\boldsymbol{r}, E, t) , \tag{5}$$

and

$$G(\boldsymbol{r}', t' \to \boldsymbol{r}, t) = \frac{1}{4\pi} \int dE \int dE' \int d\Omega \int d\Omega' \nu(E) \Sigma_f(\boldsymbol{r}, E) f(E') H(\boldsymbol{P}', t' \to \boldsymbol{P}, t) . \tag{6}$$

The kernel $G(\boldsymbol{r}', t' \to \boldsymbol{r}, t)$ gives the number of neutrons produced, in unit volume around $\boldsymbol{r}$ in unit time interval at time t, by fissions induced by a single neutron born by fission at $(\boldsymbol{r}', t')$.

II. Fisson Neutron Importance

The fission neutron importance is related to the rate of fission neutron production due to a family of neutrons originated in the past from a single ancestor.

It is assumed that the descendants of a single neutron born at $(\boldsymbol{r}, t)$ by fission will produce in the reactor system a total of $F(\boldsymbol{r}, t; T)$ neutrons by fission in unit time interval at T in the future. The function $F(\boldsymbol{r}, t; T)$ satisfies Lewin's axiom of conservation[2], because the ability of a neutron to cause fission is expected to be transfered to its issue through the transport kernel $G(\boldsymbol{r}', t' \rightarrow \boldsymbol{r}, t)$; that is, we have

$$F(\boldsymbol{r}, t; T) = \int_t^T dt' \int d\boldsymbol{r}' G(\boldsymbol{r}, t \rightarrow \boldsymbol{r}', t') F(\boldsymbol{r}', t'; T), \tag{7}$$

which shows that the quantity represented by the function $F(\boldsymbol{r}, t; T)$ is conserved in the course of chain reactions. This quantity is evidently the adjoint of the neutron transport equation (1).

Now we may consider that each of the neutrons within the reactor system possesses, when born by fission at $(\boldsymbol{r}, t)$, the importance $F(\boldsymbol{r}, t; T)$ in sustaining the chain reaction at T in the future.

We introduce, hereon, the effective fission neutron source $S^*(\boldsymbol{r}, t)$ which is defined as the fission neutron source $S(\boldsymbol{r}, t)$ weighted by the importance $F(\boldsymbol{r}, t; T)$;

$$S^*(\boldsymbol{r}, t) = F(\boldsymbol{r}, t : T) S(\boldsymbol{r}, t) . \tag{8}$$ †

Substituting the neutron transport equation (1) into the above equation we obtain,

$$S^*(\boldsymbol{r}, t) = \int_{-\infty}^{t} dt' \int d\boldsymbol{r}' G^*(\boldsymbol{r}', t' \rightarrow \boldsymbol{r}, t) S^*(\boldsymbol{r}', t') \tag{9}$$

where $G^*(\boldsymbol{r}', t' \rightarrow \boldsymbol{r}, t)$ is a biased kernel whicn is defined by

$$G^*(\boldsymbol{r}', t' \rightarrow \boldsymbol{r}, t) = \frac{F(\boldsymbol{r}, t; T)}{F(\boldsymbol{r}', t'; T)} G(\boldsymbol{r}' t' \rightarrow \boldsymbol{r}, t) . \tag{10}$$

The biased kernel is normalized in the sense that

$$\int_{t'}^{T} dt \int d\boldsymbol{r} G^*(\boldsymbol{r}', t' \rightarrow \boldsymbol{r}, t) = 1 . \tag{11}$$

Since, according to the equation (9), a single effective neutron born at $(\boldsymbol{r}', t')$ is expected to produce effective neutrons of $G^*(\boldsymbol{r}', t' \rightarrow \boldsymbol{r}, t)$, in unit volume around $(\boldsymbol{r}, t)$, then $G^*(\boldsymbol{r}', t' \rightarrow \boldsymbol{r}, t)$ is the probability that a single effective neutron born at $(\boldsymbol{r}', t')$ will die to produce another effective neutron by fission, in unit volume around $\boldsymbol{r}$ and in unit time interval at t.

III. Effective Neutron Lifetime and Criticality Factor

The average time interval between birth and death of an effective neutron born at $(\boldsymbol{r}, t)$ is given by

$$l(\boldsymbol{r}, t) = \int_t^{\infty} dt' \int d\boldsymbol{r}' (t'-t) G^*(\boldsymbol{r}, t \rightarrow \boldsymbol{r}', t') . \tag{12}$$

† $S^*(\boldsymbol{r}, t)$ may be considered as the importance rate at $(\boldsymbol{r}, t)$.

The effective neutron lifetime is, then, defined as the average value of $l(\boldsymbol{r}, t)$ in the reactor system:

$$l(t) = \left[\int d\boldsymbol{r}\, l(\boldsymbol{r}, t) S^*(\boldsymbol{r}, t)\right] \times \left[\int d\boldsymbol{r}\, S^*(\boldsymbol{r}, t)\right]^{-1}, \tag{13}$$

where

$$\int d\boldsymbol{r}\, S^*(\boldsymbol{r}, t)$$

is the integrated importance rate.

A possible definition of the criticality factor is

$$k(t) = \left[\int d\boldsymbol{r} \int_t^\infty dt' \int d\boldsymbol{r}' G(\boldsymbol{r}, t \to \boldsymbol{r}', t') S(\boldsymbol{r}, t)\right] \times \left[\int d\boldsymbol{r}\, S(\boldsymbol{r}, t)\right]^{-1}, \tag{14}$$

which represents the multiplication of fission rate after the average neutron lifetime.

The criticality factor should, however, rather be defined as the effective multiplication factor, which is the neutron multiplicity during the effective neutron lifetime $l(t)$.

During an effective neutron's lifetime, from its birth at $(\boldsymbol{r}, t)$ to its death at $(\boldsymbol{r}', t,)$, the neutron importance at $\boldsymbol{r}'$ is supposed to be changed from $F(\boldsymbol{r}', t)$ into $F(\boldsymbol{r}', t')$. Since the multiplication factor of neutrons is the reciprocal multiplication factor $F(\boldsymbol{r}', t)/F(\boldsymbol{r}', t')$ of their importance, the criticality factor is defined by the equation:

$$k(t) = \left[\int d\boldsymbol{r} \int_t^\infty dt' \int d\boldsymbol{r}' \frac{F(\boldsymbol{r}', t)}{F(\boldsymbol{r}', t')} G^*(\boldsymbol{r}, t \to \boldsymbol{r}', t') S^*(\boldsymbol{r}, t)\right] \times \left[\int d\boldsymbol{r}\, S^*(\boldsymbol{r}, t)\right]^{-1}. \tag{15}$$

IV. Evaluation of the Reactor Kinetics Parameters

In order to evaluate, numerically, the reactor kinetics parameters defined in the preceding sections, some modifications are necessary in their expressions, because the equations (13) and (15) are not directly applicable to numerical work.

As in most reactor theories, we now assume that the kernel $H(\boldsymbol{P}', t' \to \boldsymbol{P}, t)$ is the Green's function for the Boltzmann transport equation:

$$\frac{1}{v} \frac{\partial \phi(\boldsymbol{P}, t)}{\partial t} + Q(\boldsymbol{P}) \phi(\boldsymbol{P}, t) = \frac{1}{4\pi} f(E) S(\boldsymbol{r}, t), \tag{16}$$

where the isotropic emission of fission neutrons is assumed.

The Green's function $H(\boldsymbol{P}', t' \to \boldsymbol{P}, t)$ satisfies the equation

$$\frac{1}{v} \frac{\partial}{\partial t} H(\boldsymbol{P}', t' \to \boldsymbol{P}, t) + Q(\boldsymbol{P}) H(\boldsymbol{P}', t' \to \boldsymbol{P}, t) = \delta(\boldsymbol{P} - \boldsymbol{P}') \delta(t - t'), \tag{17}$$

with the initial condition $H=0$ for $t \leqq t'$ and with appropriate boundary conditions.

It is readily shown that the adjoint equation of the transport equation (16) is given by

$$-\frac{1}{v}\frac{\partial\phi^*(\boldsymbol{P},t)}{\partial t}+Q^*(\boldsymbol{P})\phi^*(\boldsymbol{P},t)=\frac{1}{4\pi}\nu(E)\Sigma_f(\boldsymbol{r},E)\int dE'\int d\Omega' f(E')\phi^*(\boldsymbol{r},E',\Omega',t)\,, \tag{18}$$

where $\phi^*(\boldsymbol{P},t)$ is the adjoint function of the neutron flux $\phi(\boldsymbol{P},t)$, while Q^* is the adjoint operator of Q.

By reciprocity, the Green's function $H(\boldsymbol{P},t\to\boldsymbol{P}',t')$ satisfies the following equation:

$$-\frac{1}{v}\frac{\partial}{\partial t}H(\boldsymbol{P},t\to\boldsymbol{P}',t')+Q^*(\boldsymbol{P})H(\boldsymbol{P},t\to\boldsymbol{P}',t')=\delta(\boldsymbol{P}-\boldsymbol{P}')\delta(t-t')\,, \tag{19}$$

which shows that $H(\boldsymbol{P},t\to\boldsymbol{P}',t')$ is the Green's function for the adjoint equation (18).

It can easily be shown that the function

$$F(\boldsymbol{r},t)=\int d\Omega\int dEf(E)\phi^*(\boldsymbol{r},E,\Omega,t) \tag{20}$$

is well defined mathematically within the reactor system and is adaptable to all physical and mathematical conditions imposed on the neutron importance function.

It is assumed hereafter that the operator Q, subsequently Q^*, is time-invariant, so that the transport kernels $H(\boldsymbol{P}',t'\to\boldsymbol{P},t)$ and $G(\boldsymbol{r}',t'\to\boldsymbol{r},t)$ do not depend explicitly on absolute time but on the transport delay $\tau=t'-t$.

Then dependence of the reactor variables on time can be separated as follows:

$$\begin{aligned}\phi(\boldsymbol{P},t)&=\phi_\lambda(\boldsymbol{P})e^{\lambda t}\,,\quad \phi^*(\boldsymbol{P},t)=\phi_\lambda^*(\boldsymbol{P})e^{-\lambda t}\,,\\ S(\boldsymbol{r},t)&=S_\lambda(\boldsymbol{r})e^{\lambda t}\,,\quad F(\boldsymbol{r},t)=F_\lambda(\boldsymbol{r})e^{-\lambda t}\,.\end{aligned} \tag{21}$$

Substituting the above expressions into the equations (16), (18), (4), and (7), we have

$$\left.\begin{aligned}&\frac{\lambda}{v}\phi_\lambda(\boldsymbol{P})+Q(\boldsymbol{P})\phi_\lambda(\boldsymbol{P})=\frac{1}{4\pi}f(E)S_\lambda(\boldsymbol{r})\,,\\ &\frac{\lambda}{v}\phi_\lambda^*(\boldsymbol{P})+Q^*(\boldsymbol{P})\phi_\lambda^*(\boldsymbol{P})=\frac{1}{4\pi}\nu(E)\Sigma_f(\boldsymbol{r},E)F_\lambda(\boldsymbol{r})\,,\end{aligned}\right\} \tag{22}$$

where

$$S_\lambda(\boldsymbol{r})=\int d\boldsymbol{r}'G_\lambda(\boldsymbol{r}'\to\boldsymbol{r})S_\lambda(\boldsymbol{r}')\,,\quad F_\lambda(\boldsymbol{r})=\int d\boldsymbol{r}'G_\lambda(\boldsymbol{r}\to\boldsymbol{r}')F(\boldsymbol{r}')\,, \tag{23}$$

ϕ_λ and ϕ_λ^* being the eigenfunctions associated with the eigenvalue λ.

In this case, the effective neutron lifetime and criticality factor defined in the previous section become time independent and are given by

$$l_\lambda = \left[\int dr \int dr' F_\lambda(\boldsymbol{r}) S_\lambda(\boldsymbol{r}') \int_0^\infty \tau e^{-\lambda\tau} G(\boldsymbol{r}' \to \boldsymbol{r}, \tau)\, d\tau\right] \times \left[\int dr\, S_\lambda(\boldsymbol{r}) F_\lambda(\boldsymbol{r})\right]^{-1}, \quad (24)$$

$$k_\lambda = \left[\int dr \int dr' F_\lambda(\boldsymbol{r}) S_\lambda(\boldsymbol{r}') \int_0^\infty G(\boldsymbol{r}' \to \boldsymbol{r}, \tau)\, d\tau\right] \times \left[\int dr\, S_\lambda(\boldsymbol{r}) F_\lambda(\boldsymbol{r})\right]^{-1}. \quad (25)$$

The following equations can be derived immediately from the equation (22):

$$\frac{1}{v}\frac{\partial}{\partial\tau}[\tau H(\tau)] + Q[\tau H(\tau)] = \frac{1}{v} H(\tau)\,, \quad (26)$$

$$\frac{1}{v}\frac{\partial}{\partial\tau}[e^{-\lambda\tau} H(\tau)] + Q[e^{-\lambda\tau} H(\tau)] = \delta(\boldsymbol{P} - \boldsymbol{P}') - \frac{\lambda}{v} e^{-\lambda\tau} H(\tau)\,, \quad (27)$$

where $H(\tau)$ is written for the kernel $H(\boldsymbol{P}' \to \boldsymbol{P}, \tau)$.

Since $H(\tau)$ is the Green's function for the equations (26) and (27), then $\tau H(\tau)$ and $e^{-\lambda\tau} H(\tau)$ can be expressed in the kernel form:

$$\tau H(\tau) = \int_0^\tau d\tau' \int d\boldsymbol{P}'' H(\boldsymbol{P}'' \to \boldsymbol{P}, \tau') \frac{1}{v''} H(\boldsymbol{P}' \to \boldsymbol{P}'', \tau - \tau')\,, \quad (28)$$

$$e^{-\lambda\tau} H(\tau) = H(\tau) - \lambda \int_0^\tau d\tau' \int d\boldsymbol{P}'' H(\boldsymbol{P}'' \to \boldsymbol{P}, \tau') \frac{1}{v''} H(\boldsymbol{P}' \to \boldsymbol{P}'', \tau - \tau') e^{-\lambda(\tau - \tau')}. \quad (29)$$

Being multiplied by $e^{-\lambda\tau}$ and integrated over τ, the equation (28) yields

$$\int_0^\infty \tau e^{-\lambda\tau} H(\tau)\, d\tau = \int d\boldsymbol{P}'' H_\lambda(\boldsymbol{P}'' \to \boldsymbol{P}) \frac{1}{v''} H_\lambda(\boldsymbol{P}' \to P'')\,, \quad (30)$$

where

$$H_\lambda(\boldsymbol{P}' \to \boldsymbol{P}) = \int_0^\infty e^{-\lambda\tau} H(\boldsymbol{P}' \to \boldsymbol{P}, \tau)\, d\tau\,.$$

Meanwhile, on integrating the equation (29) over τ, we obtain

$$H_\lambda = \int_0^\infty e^{-\lambda\tau} H(\tau)\, d\tau = H_0 - \lambda \int d\boldsymbol{P}'' H_0(\boldsymbol{P}'' \to \boldsymbol{P}) \frac{1}{v''} H_\lambda(\boldsymbol{P}' \to \boldsymbol{P}'')\,, \quad (31)$$

where the kernel H_0 is defined by the integral of H

$$H_0(\boldsymbol{P}' \to \boldsymbol{P}) = \int_0^\infty d\tau\, H(\boldsymbol{P}' \to \boldsymbol{P}, \tau)\,. \quad (32)$$

Since, in most cases, we are concerned with the largest eigenvalue, then the kernel appearing in the second term on the right-hand side of the equation (31) can be replaced by H. This approximation is equivalent to the neglecting of terms of the order λ^2.

Under this assumption, the following relation can easily be proved:

$$\lambda = (k_\lambda - 1)/l_\lambda\,. \quad (33)$$

In the case where the reactor is not very far from criticality, the calculus of perturbations can be used in evaluating the reactor kinetics parameters.

Let δG_0 be a small variation in the kernel G_0. Then in the first order approximation we have†

$$k_\lambda - 1 = \left[\int d\boldsymbol{r}\int d\boldsymbol{r}'\delta G_0(\boldsymbol{r}'\to\boldsymbol{r})F_c(\boldsymbol{r})S_c(\boldsymbol{r}')\right]\times\left[\int d\boldsymbol{r}F_c(\boldsymbol{r})S_c(\boldsymbol{r})\right]^{-1}. \quad (34)$$

The variation $\delta G_0(\boldsymbol{r}'\to\boldsymbol{r})$ can be divided into two parts:

$$\delta G_0(\boldsymbol{r}'\to\boldsymbol{r}) = \frac{1}{4\pi}\int dE\int dE'\int d\Omega\int d\Omega'\{\delta[\nu\Sigma_f(\boldsymbol{r}, E)]H_{0c}(\boldsymbol{P}'\to\boldsymbol{P})f(E') + [\nu(E)\Sigma_f(\boldsymbol{r}, E)]_c\delta H_0(\boldsymbol{P}'\to\boldsymbol{P})f(E')\}, \quad (35)$$

where $\delta H_0(\boldsymbol{P}'\to\boldsymbol{P})$ approximately satisfies the equation which follows:

$$Q_c(\boldsymbol{P})\,\delta H_0(\boldsymbol{P}'\to\boldsymbol{P}) = -\delta Q(\boldsymbol{P})H_{0c}(\boldsymbol{P}'\to\boldsymbol{P}). \quad (36)$$

Since we have, in the critical state of the reactor,

$$Q_c(\boldsymbol{P})H_{0c}(\boldsymbol{P}'\to\boldsymbol{P}) = \delta(\boldsymbol{P}-\boldsymbol{P}'), \quad (37)$$

then δH_0 can be expressed in the kernel form:

$$\delta H_0(\boldsymbol{P}'\to\boldsymbol{P}) = -\int d\boldsymbol{P}''H_{0c}(\boldsymbol{P}''\to\boldsymbol{P})\,\delta Q(\boldsymbol{P}'')\,H_{0c}(\boldsymbol{P}'\to\boldsymbol{P}''). \quad (38)$$

Substitution of (38) into (35) and subsequently (34) yields

$$k_\lambda - 1 = \left[\frac{1}{4\pi}\int d\boldsymbol{r}\int dE\int dE'\int d\Omega\int d\Omega' f(E)\,\phi_c^*(\boldsymbol{r}, E, \Omega)\,\delta[\nu(E)\Sigma_f(\boldsymbol{r}, E)]\phi_c(\boldsymbol{r}, E', \Omega') - \int d\boldsymbol{P}\phi_c^*(\boldsymbol{P})\delta Q(\boldsymbol{P})\,\phi(\boldsymbol{P})\right]\times\left[\int d\boldsymbol{r}S_c(\boldsymbol{r})F_c(\boldsymbol{r})\right]^{-1}. \quad (39)$$

According to the equation (30), the exact expression for the effective neutron lifetime is

$$l_\lambda = \left[\int d\boldsymbol{P}\phi_\lambda^*(\boldsymbol{P})\frac{1}{v}\phi_\lambda(\boldsymbol{P})\right]\times\left[\int d\boldsymbol{r}S_\lambda(\boldsymbol{r})\,F_\lambda(\boldsymbol{r})\right]^{-1}. \quad (40)$$

In the first order approximation we have

$$l_\lambda = \left[\int d\boldsymbol{P}\phi_c^*(\boldsymbol{P})\frac{1}{v}\phi_c(\boldsymbol{P})\right]\times\left[\int d\boldsymbol{r}S_c(\boldsymbol{r})\,F_c(\boldsymbol{r})\right]^{-1}. \quad (41)$$

V. Inhour Equation

Here we will discuss the case where delayed neutrons are taken into accout, in another approach.

First we assume that the delayed neutron precursors do not move in space.

† Subscript c denotes the critical state.

Then the fission neutron source with delayed neutrons can be written as

$$\frac{1}{4\pi}\{f_p(E)[1-\beta(\boldsymbol{r})]S(\boldsymbol{r},t)+\sum_i \lambda_i C_i(\boldsymbol{r},t)\,\delta(E-E_i)\} \tag{42}$$

where $f_p(E)$ is the normalized prompt fission neutron spectrum, and $\beta(\boldsymbol{r})$ is the delayed neutron fraction at $\boldsymbol{r}$, while $C_i(\boldsymbol{r},t)$ is the density of the precursors of i-th species from which delayed neutrons of energy E_i are emitted. $C_i(\boldsymbol{r},t)$ obeys the equation;

$$\frac{\partial}{\partial t}C_i(\boldsymbol{r},t) = \beta_i(\boldsymbol{r})S(\boldsymbol{r},t)-\lambda_i C_i(\boldsymbol{r},t)\,. \tag{43}$$

We further assume that $\phi(\boldsymbol{r},t)$ and $S(\boldsymbol{r},t)$ vary in time in the way described by the equation (21). Then the equation (43) can easily be solved, giving:

$$C_i(\boldsymbol{r},t) = \frac{\beta_i(\boldsymbol{r})}{\lambda+\lambda_i}S_\lambda(\boldsymbol{r})e^{\lambda t}\,. \tag{44}$$

Substitution of the equations (42) and (44) into (1) yields

$$\phi_\lambda(\boldsymbol{P}) = \frac{1}{4\pi}\int d\boldsymbol{P}'\,H_\lambda(\boldsymbol{P}'\to\boldsymbol{P})\,F(\boldsymbol{r}',E',\lambda)\,S_\lambda(\boldsymbol{r}') \tag{45}$$

where

$$F(\boldsymbol{r},E,\lambda) = F_0(\boldsymbol{r},E)-\sum_i\frac{\lambda}{\lambda+\lambda_i}\beta_i(\boldsymbol{r})\,\delta(E-E_i') \tag{46}$$

and

$$F_0(\boldsymbol{r},E) = f_p(E)\{1-\beta(\boldsymbol{r})\}+\sum_i\beta_i(\boldsymbol{r})\,\delta(E-E_i)\,. \tag{47}$$

Then the equation (45) is rewritten, using the equation (31);

$$\begin{aligned}\phi_\lambda(\boldsymbol{P}) &= \frac{1}{4\pi}\int d\boldsymbol{P}'H_0(\boldsymbol{P}'\to\boldsymbol{P})F(\boldsymbol{r}',E',\lambda)\,S_\lambda(\boldsymbol{r})\\ &-\lambda\int d\boldsymbol{P}'\int d\boldsymbol{P}''H_0(\boldsymbol{P}''\to\boldsymbol{P})\frac{1}{v''}H_\lambda(\boldsymbol{P}'\to\boldsymbol{P}'')F(\boldsymbol{r}',E',\lambda)\,S_\lambda(\boldsymbol{r}')\,.\end{aligned} \tag{48}$$

Multiplying this equation by $F_\lambda(\boldsymbol{r})\,\nu\Sigma_f(\boldsymbol{r},E)$ and integrating over all variables, we obtain the following equation, in the first order approximation with respect to δ,

$$\begin{aligned}\lambda\int d\boldsymbol{P}\phi_c^*(\boldsymbol{P})\frac{1}{v}\phi_c(\boldsymbol{P}) &= \frac{1}{4\pi}\int d\boldsymbol{r}\int dE\int dE'\int d\Omega\int d\Omega'\,\phi_c^*(\boldsymbol{r},E,\Omega)\,\delta[\nu(E')\Sigma_f(\boldsymbol{r},E')]\\ &\times F_0(\boldsymbol{r},E')\,\phi_c(\boldsymbol{r},E',\Omega')-\int d\boldsymbol{P}\phi_c^*(\boldsymbol{P})\,\delta Q(\boldsymbol{P})\,\phi_c(\boldsymbol{P})\\ &-\frac{1}{4\pi}\sum_i\frac{\lambda}{\lambda+\lambda_i}\int d\boldsymbol{r}\int dE\int dE'\int d\Omega\int d\Omega'\phi_c(\boldsymbol{r},E,\Omega)\beta_i(\boldsymbol{r})\delta(E-E_i)\nu\Sigma_{fc}(\boldsymbol{r}',E',\Omega')\,,\end{aligned} \tag{49}$$

where it is assumed that the terms involving λ^2/v can be neglected and that

$F_0(\boldsymbol{r}, E)$ can be replaced by $f_p(E)$ when involved in the first order term of the flux.

The equation (49) can be reduced into a simpler form

$$\lambda l = (k-1) - \sum_i \frac{\lambda}{\lambda+\lambda_i}\beta_{i\,\mathrm{eff}}, \tag{50}$$

where the reactor kinetics parameters are defined as follows:

$$l = \frac{1}{W}\int d\boldsymbol{P}\phi_c^*(\boldsymbol{P})\frac{1}{v}\phi_c(\boldsymbol{P}), \quad W = \int d\boldsymbol{r} F_c(\boldsymbol{r}) S_c(\boldsymbol{r}), \tag{51}$$

$$k-1 = \frac{1}{W}\Big[\frac{1}{4\pi}\int d\boldsymbol{r}\int dE\int dE'\int d\Omega\int d\Omega'\,\phi_c^*(\boldsymbol{r}, E, \Omega)\delta[\nu(E')\Sigma_f(\boldsymbol{r}, E')]F_0(\boldsymbol{r}, E')\phi_c(\boldsymbol{r}, E', \Omega') - \int d\boldsymbol{P}\phi_c^*(\boldsymbol{P})\delta Q(\boldsymbol{P})\phi_c(\boldsymbol{P}_0)\Big], \tag{52}$$

$$\beta_{i\,\mathrm{eff}} = \frac{1}{W}\cdot\frac{1}{4\pi}\int d\boldsymbol{r}\int dE\int dE'\int d\Omega\int d\Omega'[\phi_c^*(\boldsymbol{r}, E, \Omega)\beta_i(\boldsymbol{r})\delta(E-E_i)\nu(E')\Sigma_f(\boldsymbol{r}, E') \times\phi_c(\boldsymbol{r}, E', \Omega')]. \tag{53}$$

The equation (50) is well known as the inhour equation. The definition (53) of the effective delayed neutron fraction seems quite natural.

It should be noted that the procedure developed in this section, as well as the results obtained as the consequence, is valid only for the largest eigenvalue of the fundamental mode of the flux distribution.

Acknowledgement

The authors wish to express their appreciation to Messrs. M. Ohta and K. Uematsu for their discussions.

References

1) A. M. Weinberg: Proceedings of Symposia in Applied Mathematics, XI, Nuclear Reactor Theory, Amer. Math. Soc., 1961.
2) J. Lewins: Nucl. Science and Engineering, 7, 268–274, 1960.