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On the Coupled Free Vibrations of a Suspension Bridge – (I)

By

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In Part I, a set of fundamental equations of motions of a suspension bridge is derived and classified into two classes of modes, the first of which can be termed as the deflectional modes, while the second can be termed as the torsional modes. Analytical solutions for the deflectional modes of free vibrations are discussed and determined by employing the Ritz method with and without a fix point at midspan of stiffening floor. Detailed informations for the torsional modes and some basic applications of the theory developed here will be described in Part II.

1. Introduction

This paper presents general dynamic characteristics of a suspension bridge in the vertical direction as well as in the horizontal direction including the torsional rotation of a stiffening floor system. Stress is placed on derivation of fundamental equations of motions and classification of free vibrational modes from a general point of view. Because of complexity in structural conjunctions and mutual reactions of individual members, it is expected that the fundamental equations are eventually of the coupled non-linear form of expressions, for which to obtain the exact solutions may be scarcely possible. Our consideration, thus, at the present state of investigation, is limited to linear elastic responses of a suspension bridge assumed to consist merely of extensional cables,

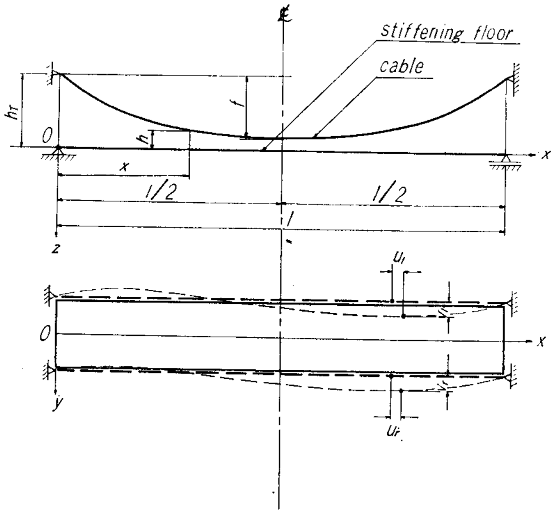

Fig. 1.

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stiffening floor and hangers distributed uniformly along the spanwise direction. (Fig. 1) Fundamental differential equations are obtained with the aid of the variational principle, which results in a somewhat involved form of expressions. Employing the Ritz method for this eigenvalue problem of free vibrations, the solutions are sought as an infinite series of sinusoidal functions for the case without constraint at the midspan and a modified form of the same functions with constraint. As well-known, a suspension bridge is rather different from other types of structures used in civil engineering on an account of its flexibility and our present investigation aims to clarify combined responses of cables and stiffening floor.

2. Formulation of the problem

In this paragraph we shall consider the mathematical formulation of free vibrations of a suspension bridge as shown in Fig. 1. Three components of displacements u_0, x_0, w_0, which are longitudinal (horizontal), lateral (horizontal), and lateral (vertical), suffice to define the deformed state of floor system. The same notations are employed for displacements of cables in horizontal and vertical directions and l, r are attached to them in order to distinguish the configurations of the two cables (Fig. 2). In addition to these the angle of rotation θ with respect to the center line, which varies with the coordinate x only, is considered so that torsional motions of the floor can be described simultaneously with horizontal and vertical modes of deformations. By virtue of the assumption that initial plane remains plane after deformation, the displacements at a typical point of the stiffening floor are expressed as follows:

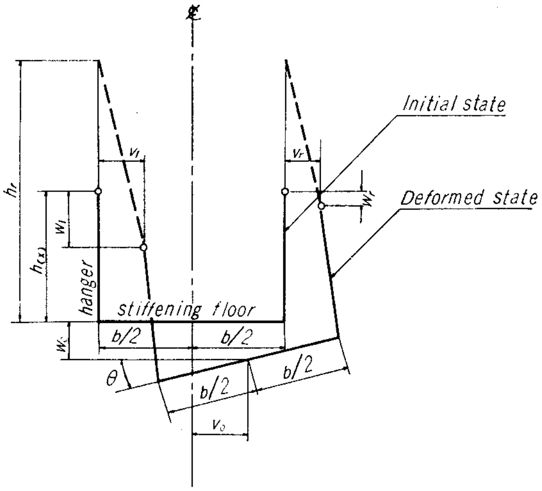

Fig. 2.

$$u(x, y, z) = u_0(x) - \frac{\partial w_0}{\partial x} z - \frac{\partial v_0}{\partial x} y \quad \text{for } x\text{-direction}$$

$$w(x, y, z) = v_0(x) - \theta(x) z \quad \text{for } y\text{-direction} \qquad (1)$$

$$w(x, y, z) = w_0(x) + \theta(x) y \quad \text{for } z\text{-direction}$$

Since the strain energy density for an isotropic homogeneous material is written as

$$\Phi_1 = \frac{1}{2}(\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij})\varepsilon_{ij} \tag{2}$$

λ, μ: Lame's constants

the strain energy of the floor system becomes

$$\int_0^l\int_{-b/2}^{b/2}\int_{-d/2}^{d/2}\Phi_1(x,y,z)dzdydx = \frac{1}{2}\int_0^l\Big[(\lambda+2\mu)\Big\{I_y\Big(\frac{\partial^2 w_0}{\partial x^2}\Big)^2+I_z\Big(\frac{\partial^2 v_0}{\partial x^2}\Big)^2 + \mu(I_y+I_z)\Big(\frac{\partial\theta}{\partial x}\Big)^2\Big]dx \tag{3}$$

where l, b, d denote span length, width and depth of floor and I_y, I_z are the moments of inertia with respect to y and z axes respectively.

For evaluation of strain energy of floor it is suggested in general to consider separately the bending deformation of stiffening web plates attached to the floor slab, forming an H-shaped cross section as a result. As long as the deflectional modes defined later are concerned, the integral (3) remains of the same form, but for the torsional modes, the bending resistance of the web-plates contributes to an increase of trosional rigidity of the floor. The latter consideration, to include the bending rigidity of the web in torsional deformation, results in the fourth order differential equation of θ with respect to x, while the former in the second order differential equation.

The strain energy of cables can be given in terms of three components of displacements at a typical cross-section. Let E_c and A_c be the Young's modulus and the sectional area of a cable, respectively, and elongation of cable signified by ds is written for element dx as

$$\begin{aligned} ds &= \{(h'-w')^2+(1+u')^2+(v')^2\}^{1/2}dx - \{1+h'^2\}^{1/2}dx \\ &= (1+h'^2)^{1/2}\Big\{\frac{u'-h'w'}{1+h'^2}+\frac{w'^2+2h'w'u'+h'^2u'^2}{2(1+h'^2)^2}+\frac{v'^2}{2(1+h'^2)}\Big\}dx \end{aligned} \tag{4}$$

where high order terms are eliminated and furthermore no effect due to Poisson's ratio is involved. By use of this one may have the strain energy of a cable as follows

$$\frac{1}{2}\int_0^l E_cA_c\Big\{(1+h'^2)^{-1/2}\frac{ds}{dx}\Big\}^2dx+\int_0^l H_w(1+h'^2)^{1/2}ds \tag{5}$$

The first term signifies the strain energy due to elongation of cable and the second term the work done by the inital cable stress due to the dead weight. Substitution of eq. (4) into eq. (5) yields to

$$\frac{1}{2}\int_0^l E_cA_c\left\{\left(\frac{u_l'-h'w_l'}{1+h'^2}\right)^2+\left(\frac{u_r'-h'w_r'}{1+h'^2}\right)^2\right\}dx$$
$$+\int_0^l H_w\left\{u_l'-h'w_l'+\frac{1}{2}\frac{w_l'^2+2h'w_l'u_l'+h'^2u_l'^2}{1+h'^2}+\frac{1}{2}v_l'^2\right\}dx$$
$$+\int_0^l H_w\left\{u_r'-h'w_r'+\frac{1}{2}\frac{w_r'^2+2h'w_r'u_r'+h'^2u_r'^2}{1+h'^2}+\frac{1}{2}v_r'^2\right\}dx \tag{6}$$

The kinetic energies of floor system and cables are easily obtained, as follows:

$$T_f=\frac{1}{2}\int_0^l\int_{-b/2}^{b/2}\int_{-d/2}^{d/2}\rho_f(\dot{u}^2+\dot{v}^2+\dot{w}^2)dzdydx$$
$$=\frac{1}{2}\int_0^l\frac{w_f}{gA}\left\{I_y\left(\frac{\partial^2w_0}{\partial x\partial t}\right)^2+I_z\left(\frac{\partial^2v_0}{\partial x\partial t}\right)^2+(I_y+I_z)\left(\frac{\partial\theta}{\partial t}\right)^2+A\left(\frac{\partial v_0}{\partial t}\right)^2+A\left(\frac{\partial w_0}{\partial t}\right)^2\right\}dx \tag{7}$$

for the sitiffening floor system and

$$T_c=\frac{1}{2}\int_0^l\frac{w_c}{g}(\dot{u}^2+\dot{v}^2+\dot{w}^2)(1+h'^2)^{1/2}dx$$
$$=\frac{1}{2}\int_0^l\frac{w_c}{g}(1+h'^2)^{1/2}(\dot{u}_l^2+\dot{v}_l^2+\dot{w}_l^2+\dot{u}_r^2+\dot{v}_r^2+\dot{w}_r^2)dx \tag{8}$$

for cables which are assumed to possess the same mechanical properties for both left and right cables with respect to the section perpendicular to the center line of bridge and, if different, one may easily find the corresponding expressions in the exactly same fashion.

In addition to the above mentioned, the change of potential energy due to vertical displacement of the stiffening floor by gravitation should be taken into an account since the workdone by initial cable stresses are included in the strain energy of cables. The structure with own weight W_f works by an amount equal to

$$W=\int_0^l\{w_f w_0+w_c\sqrt{1+h'^2}(w_l+w_r)\}dx \tag{9}$$

Thus the total energy expressions associated with stiffening girders and two cables plus two constraint conditions can be reduced to the fundamental equations of motion with the aid of the variational principle[1] which can be written as

$$\delta[T-U+W-\lambda_1f_1-\lambda_2f_2]=0 \tag{10}$$

where

$$T=\frac{1}{2}\int_0^l\frac{w_f}{gA}\left\{I_y\left(\frac{\partial^2w_0}{\partial x\partial t}\right)^2+I_z\left(\frac{\partial^2v_0}{\partial x\partial t}\right)^2+(I_y+I_z)\left(\frac{\partial\theta}{\partial t}\right)^2+A\left(\frac{\partial v_0}{\partial t}\right)^2+A\left(\frac{\partial w_0}{\partial t}\right)^2\right\}dx$$
$$+\frac{1}{2}\int_0^l\frac{w_c}{g}(1+h'^2)^{1/2}\left\{\left(\frac{\partial u_l}{\partial t}\right)^2+\left(\frac{\partial v_l}{\partial t}\right)^2+\left(\frac{\partial w_l}{\partial t}\right)^2+\left(\frac{\partial u_r}{\partial t}\right)^2+\left(\frac{\partial v_r}{\partial t}\right)^2+\left(\frac{\partial w_r}{\partial t}\right)\right\}dx$$

$$U = \frac{1}{2}\int_0^l\left[(\lambda+2\mu)\left\{I_y\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2+I_z\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2\right\}+\mu(I_y+I_z)\left(\frac{\partial\theta}{\partial x}\right)^2\right]dx$$
$$+\frac{1}{2}\int_0^l E_cA_c\left\{\left(\frac{u_l'-h'w_l'}{1+h'^2}\right)^2+\left(\frac{u_r'-h'w_r'}{1+h'^2}\right)^2\right\}dx$$
$$+\int_0^l H_w\left\{u_l'-h'w_l'+\frac{1}{2}\left(\frac{w_l'^2+2h'w_l'u_l'+h'^2u_l'^2}{1+h'^2}\right)+\frac{1}{2}v_l'^2\right\}dx$$
$$+\int_0^l H_w\left\{u_r'-h'w_r'+\frac{1}{2}\left(\frac{w_r'^2+2h'w_r'u_r'+h'^2u_r'^2}{1+h'^2}\right)+\frac{1}{2}v_r'^2\right\}dx$$

and

$$f_1 = \left(\frac{b}{2}\frac{\partial v_0}{\partial x}-u_l\right)^2+(v_0-v_l)^2+\left(h+w_0+\frac{b}{2}\theta-w_l\right)^2-h^2$$
$$f_2 = \left(\frac{b}{2}\frac{\partial v_0}{\partial x}+u_r\right)^2+(v_0-v_r)^2+\left(h+w_0-\frac{b}{2}\theta-w_r\right)^2-h^2$$

λ_1, λ_2: Lagrange's multipliers

assuming inextensible hangers.

Using the ordinary method of calculus of variation, eq. (10) provides a following set of differential equations, viz.,

$$\frac{w_f}{g}\ddot{w}_0-\rho_f\frac{\partial^2}{\partial x\partial t}\left(I_y\frac{\partial^2 w_0}{\partial x\partial t}\right)+\frac{\partial^2}{\partial x^2}\left\{(\lambda+2\mu)I_y\frac{\partial^2 w_0}{\partial x^2}\right\}$$
$$+2\lambda_1\left(h+w_0+\frac{b}{2}\theta-w_l\right)+2\lambda_2\left(h+w_0-\frac{b}{2}\theta-w_r\right) = w_f \quad (11)$$

$$\frac{w_f}{g}\ddot{v}_0-\rho_f\frac{\partial^2}{\partial x\partial t}\left(I_z\frac{\partial^2 v_0}{\partial x\partial t}\right)+\frac{\partial^2}{\partial x^2}\left\{(\lambda+2\mu)I_z\frac{\partial^2 v_0}{\partial x^2}\right\}-\frac{\partial}{\partial x}\left\{b\lambda_1\left(\frac{b}{2}\frac{\partial v_0}{\partial x}-u_l\right)\right\}$$
$$-\frac{\partial}{\partial x}\left\{b\lambda_2\left(\frac{b}{2}\frac{\partial v_0}{\partial x}+u_r\right)\right\}+2\lambda_1(v_0-v_l)+2\lambda_2(v_0-v_r) = 0 \quad (12)$$

$$\frac{w_f}{gA}(I_y+I_z)\ddot{\theta}-\frac{\partial}{\partial x}\left\{\mu(I_y+I_z)\frac{\partial\theta}{\partial x}\right\}$$
$$+b\lambda_1\left(h+w_0+\frac{b}{2}\theta-w_l\right)-b\lambda_2\left(h+w_0-\frac{b}{2}\theta-w_r\right) = 0 \quad (13)$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{u}_l-\frac{\partial}{\partial x}\left\{E_cA_c\left(\frac{u_l'-h'w_l'}{(1+h'^2)^2}\right)\right\}-H_w\frac{\partial}{\partial x}\left\{\frac{h'(w_l'+h'u_l')}{1+h'^2}\right\}$$
$$-2\lambda_1\frac{b}{2}\left(\frac{\partial v_0}{\partial x}-u_l\right) = 0 \quad (14)$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{u}_r-\frac{\partial}{\partial x}\left\{E_cA_c\frac{u_r'-h'w_r'}{(1+h'^2)^2}\right\}-H_w\frac{\partial}{\partial x}\left\{\frac{h'(w_r'+h'u_r')}{1+h'^2}\right\}$$
$$+2\lambda_2\left(\frac{b}{2}\frac{\partial v_0}{\partial x}+u_r\right) = 0 \quad (15)$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{v}_l-H_w\frac{\partial^2 v_l}{\partial x^2}-2\lambda_1(v_0-v_l) = 0 \quad (16)$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{v}_r-H_w\frac{\partial^2 v_r}{\partial x^2}-2\lambda_2(v_0-v_r) = 0 \quad (17)$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{w}_l+\frac{\partial}{\partial x}\left\{E_cA_c\frac{h'(u_l'-h'w_l')}{(1+h'^2)^2}\right\}+H_wh''-H_w\frac{\partial}{\partial x}\left(\frac{w_l'+h'u_l'}{1+h'^2}\right)$$
$$-2\lambda_1\left(h+w_0+\frac{b}{2}\theta-w_l\right)=w_c\sqrt{1+h'^2} \tag{18}$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{w}_r+\frac{\partial}{\partial x}\left\{E_cA_c\frac{h'(u_r'-h'w_r')}{(1+h'^2)^2}\right\}+H_wh''-H_w\frac{\partial}{\partial x}\left(\frac{w_r'+h'u_r'}{1+h'^2}\right)$$
$$-2\lambda_2\left(h+w_0-\frac{b}{2}\theta-w_r\right)=w_c\sqrt{1+h'^2} \tag{19}$$

The above nine equations contain eleven unkown dependent variables, w_0, v_0, θ, u_l, v_l, w_l, u_r, v_r, w_r, λ_1 and λ_2 to be determined by the equations plus two constraint conditions

$$\left(\frac{b}{2}\frac{\partial v_0}{\partial x}-u_l\right)^2+(v_0-v_l)^2+\left(h+w_0+\frac{b}{2}\theta-w_l\right)^2=h^2$$
$$\left(\frac{b}{2}\frac{\partial v_0}{\partial x}+u_r\right)^2+(v_0-v_r)^2+\left(h+w_0-\frac{b}{2}\theta-w_r\right)^2=h^2$$

where λ and μ signify Lame's constants. Physically Lagrange's multiplers λ_1, λ_2 correspond to mutual reactions for cables and floor as the result of the constraint conditions. Eq's (11) through (19) are of the general form of expressions for describing the dynamic behavior of a suspension bridge. For example combining eq's (11), (18) and (19) to eliminate both λ_1 and λ_2, the result can coincide with the form of differential equation derived by F. Eleich and others[2] except for coupling of vertical and horizontal displacements of a cable. In this derivation the increment of the horizontal components of cable stress is equivalently replaced by two components of displacements in the longitudinal and vertical directions[3].

Based on the above fundamental equations, the free vibrations of a suspension bridge are classified into two types of modes, the first of which can be termed as the deflectional modes, while the second of which as the torsional modes. The former modes of vibrations indicate that neither lateral (horizontal) displacement nor rotation of floor with respect to the center line of bridge contributes to the oscillations, which therefore are described two-dimensionally with the relations $u_l=u_r\equiv u$, $w_l=w_r\equiv w$ and $\lambda_1=\lambda_2\equiv\bar{\lambda}$. Thus under the assumption that $\theta=v_0=v_l=v_r=0$ we obtain three out of nine expressions to describe the configuration for this mode:

$$\frac{w_f}{g}\ddot{w}_0-\frac{w_f}{gA}\frac{\partial^2}{\partial x\partial t}\left(I_y\frac{\partial^2 w_0}{\partial x\partial t}\right)+\frac{\partial^2}{\partial x^2}\left\{(\lambda+2\mu)I_y\frac{\partial^2 w_0}{\partial x^2}\right\}+4\bar{\lambda}(h+w_0-w)=w_f \tag{20}$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{u}+\frac{\partial}{\partial x}\left\{E_cA_c\frac{h'w'-u'}{(1+h'^2)^2}\right\}-H_w\frac{\partial}{\partial x}\left\{\frac{h'(w'+h'u')}{1+h'^2}\right\}+2\bar{\lambda}u=0 \tag{21}$$

$$\frac{w_c}{g}(1+h'^2)^{1/2}\ddot{w}-\frac{\partial}{\partial x}\left\{E_cA_c\frac{h'(h'w'-u')}{(1+h'^2)^2}\right\}+H_wh''-H_w\frac{\partial}{\partial x}\left(\frac{w'+h'u'}{1+h'^2}\right)$$
$$-2\bar{\lambda}(h+w_0-w)=w_c\sqrt{1+h'^2} \tag{22}$$

with a constraint condition

$$u^2+(h+w_0-w)^2=h^2 \tag{23}$$

For an other mode of free vibrations one has to consider all expressions, none of which is reduced nor combined together. This indicates obviously complex behaviors of a suspension bridge in the torsional oscillation and it should be noticed that the torsional modes are dependent on the other modes of displacements at this order of approximation. One should thus recognize the coupling of torsional and lateral displacements as a characteristic of free vibrations of suspension bridges. Detailed discussion will be made later regarding this problem.

3. Initial funicular curve of cable

In the preceding paragraph we have obtained a set of equations for free vibrations of suspension bridges, in which Lagrange's multiplers signify the mutual reactions between cables and stiffening girder. Let all displacements vanish identically, then we have

$$4\bar{\lambda}h=w_f,\quad 2H_wh''-4\bar{\lambda}h=2w_c\sqrt{1+h'^2} \tag{24}$$

which determines the initial funicular curve of cable. In eq. (24) w_f denotes the dead weight of unit length of floor system and eq (24) is rewritten as

$$w_f+2w_c(1+h'^2)^{1/2}=2H_wh'' \tag{25}$$

Neglecting higher order term h'^2 in eq. (25), $h(x)$, is approximately expressed by a parabola, while with h'^2 it is defined as a solution of the non-linear equation

$$h''-\frac{w_c}{H_w}(1+h'^2)^{1/2}-\frac{w_f}{2H_w}=0$$

the first integral of which is easily obtained by denoting p as h', namely

$$\frac{dp}{dx}=\frac{w_c}{H_w}(1+p^2)^{1/2}+\frac{w_f}{2H_w}$$
$$x=\int\frac{dp}{\frac{w_f}{2H_w}+\frac{w_c}{H_w}(1+p^2)^{1/2}}=\frac{2H_w}{w_f}\int\frac{d\theta}{\cos\theta(\cos\theta+\alpha)}$$

where $\alpha=\dfrac{2w_c}{w_f},\quad p=\tan\theta$

Then $$x=\frac{2H_w}{w_f}\left\{\frac{1}{a}\ln\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)-\frac{1}{a}\int\frac{d\theta}{a+\cos\theta}\right\}^{4)} \tag{26}$$

$$=\frac{2H_w}{w_f}\left[\frac{1}{a}\ln\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)-\frac{1}{a}\int\frac{2}{(a^2-1)^{1/2}}\tan^{-1}\left\{\frac{(a-1)\tan\theta/2}{(a^2-1)^{1/2}}\right\}\right] \tag{26.1}$$

for $a^2>1$

$$=\frac{2H_w}{w_f}\left[\ln\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)-\tan\frac{\theta}{2}\right] \tag{26.2}$$

for $a^2=1$

$$=\frac{2H_w}{w_f}\left[\frac{1}{a}\ln\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)-\frac{1}{a}\frac{1}{(1-a^2)^{1/2}}\ln\left\{\frac{(1-a)\tan\frac{\theta}{2}+(1-a^2)^{1/2}}{(1-a)\tan\frac{\theta}{2}-(1-a^2)^{1/2}}\right\}\right] \tag{26.3}$$

for $a^2<1$

Eq. (26) is therefore the first integral which forms three different types of expressions as eq's (26.1), (26.2) and (26.3) depending on the ratios of w_f vs w_c. Ignoring the high order term h'^2 it is reduced to the customary parabolic curve of cable which is given as

$$x=\frac{2H_w}{w_f}\int\frac{dp}{1+a}=\frac{2H_w}{w_f}\frac{1}{1+a}\tan\theta \tag{27}$$

From the practical point of view a stiffening truss of a suspension bridge weighes it down more than cables, the initial funicular curve of which can be approximated by a parabola.

4. General remarks on the deflectional modes of free vibrations

As mentioned in 2. the modes of our present consideration are defined by eq's (20) through (23). Let us specify the associate boundary conditions with the above equations as

$$\begin{aligned} &u(0)=u(l)=w(0)=w(l)=0\\ &w_0(0)=w_0(l)=w_0''(0)=w_0''(l)=0 \end{aligned} \tag{28}$$

assuming that the cable is perfectly flexible and the stiffening girder is simply supported at both ends of the span. Classically the fundamental differential equation for this mode of vibrations is written in the form[2)]

$$\frac{d^2}{dx^2}\left(EI_y\frac{d^2w_0}{dx^2}\right)-2H_w\frac{d^2w_0}{dx^2}+\frac{2\bar{h}w_f}{H_w}-\left(\frac{w_f}{g}+\frac{2w_c}{g}\right)\omega^2w_0=0 \tag{29}$$

where $\bar{h}$, ω^2 denotes the increment of horizontal component of cable stress and the circular frequency, respectively, and for sake of simplicity the symbol w_0 is used for the mode of deflection. In terms of eq's (20) and (22) we find the corresponding expression to eq. (29) requiring the $u=0$, $w_0=w$, $(h'\ll 1)$ identically, that is,

$$\frac{d^2}{dx^2}\left(EI_y\frac{d^2w_0}{dx^2}\right)-2H_w\left(\frac{d}{dx}\left(\frac{w'+h'u'}{1+h'^2}\right)-\frac{d}{dx}\left\{2E_cA_c\frac{h'(h'w'-u')}{(1+h'^2)^2}\right\}\right.$$
$$-\omega^2\left\{\frac{w_f}{g}+\frac{2w_c}{g}(1+h'^2)\right\}w_0=0 \tag{30}$$

It should be noticed that the assumption $u=0$, $w_0=w$ does not satisfy eq. (21) and thus eq. (30) is only valid in the sense that it forms an approximation of the deflectional modes.

There appears to be no closed form of solutions in terms of tabulated functions for this coupled free vibrational problem. It is known, however, that the equations with boundary conditions, eq. (28), provide two branches of curves in the spectrum diagram. Physically it follows that one branch approaches to the spectrum corresponding to the vibrations of stiffening girder only and another to the spectrum corresponding to the vibrations of stiffening girder only and another to the spectrum for cables only. Our attention is primarily restricted to the lower branch of natural frequency spectra. For higher frequencies ignorance of shear effect in floor girders may not be justified any longer and more thoroughful consideration will be required.

The deflectional modes are featured at the following structural point; recalling eq. (24) one may notice that the lowest approximation for reactive force of hangers can be given by $\bar{\lambda}=H_wh''/2h$[5]. Substituting this into eq's (20), (21) and (22) we have

$$\left.\begin{aligned}
&\frac{d^2}{dx^2}\left\{(\lambda+2\mu)I_y\frac{d^2w_0}{dx^2}\right\}+2H_w\frac{h''}{h}(w_0-w)-\omega^2\left\{\frac{w_f}{g}w_0-\frac{w_f}{gA}\frac{d}{dx}\left(I_y\frac{dw_0}{dx}\right)\right\}=0\\
&\frac{d}{dx}\left\{E_cA_c\frac{h'w'-u'}{(1+h'^2)^2}\right\}-H_w\frac{d}{dx}\left\{\frac{h'(w'+h'u')}{1+h'^2}\right\}+H_w\frac{h''}{h}u-\omega^2\frac{w_c}{g}(1+h'^2)^{1/2}u=0\\
&-\frac{d}{dx}\left\{E_cA_c\frac{h'(h'w'-u')}{(1+h'^2)^2}\right\}-H_w\frac{d}{dx}\left(\frac{w'+h'u'}{1+h'^2}\right)-H_w\frac{h''}{h}(w_0-w)-\omega^2\frac{w_c}{g}(1+h'^2)^{1/2}w=0
\end{aligned}\right\} \tag{31}$$

for which a singular point is introduced if $h_T=f$ (sag of cable), i.e., $h=0$ at the midspan of bridge. This classifies the modes of the present problem into two classes for which solutions will be described in the following paragraphs. It is worthwhile to notice that center-diagonal stays, commonly adopted to modern types of suspension bridge, constrain horizontal displacement of cable at the midspan and as result augment remarkably the stiffness of a suspension bridge[6]. Mathematically, installment of diagonal stays is conceived as the fundamental equations are associated with singularity at the midspan of the bridge. Thus the problem should be treated separately depending on whether there is a fix point at the center point of the span.

5. Analytical solutions of the deflectional modes without a fix point at midspan

The solutions of three coupled simultaneous differential equations (31) with boundary conditions (28) can be expressed in the form

$$w_0 = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi x}{l}, \qquad w = \sum_{k=1}^{\infty} B_k \sin\frac{k\pi x}{l}, \qquad u = \sum_{k=1}^{\infty} C_k \sin\frac{k\pi x}{l} \tag{32}$$

Restricting our attention to eq's (31), the stationary theorem for our eigenvalue problem is written as

$$\delta\Big[\sum_k \sum_j \{D^{(1)}_{kj}A_kA_j + D^{(2)}_{kj}A_kB_j + D^{(3)}_{kj}B_kB_j + D^{(4)}_{kj}B_kC_j + D^{(5)}_{kj}C_kC_j\}\Big] = 0$$

which leads to

$$\begin{vmatrix} D^{(1)}_{kj}+D^{(1)}_{jk} & D^{(2)}_{mj} & 0 \\ D^{(2)}_{jl} & D^{(3)}_{lm}+D^{(3)}_{ml} & D^{(4)}_{ln} \\ 0 & D^{(4)}_{mr} & D^{(5)}_{ru}+D^{(5)}_{ur} \end{vmatrix} \begin{vmatrix} A_j \\ B_m \\ C_n \end{vmatrix} = 0 \quad \text{for} \left\{ \begin{array}{l} j,\, k = 1, \cdots, j_\lambda, \cdots \\ m,\, l = 1, \cdots, m_\mu, \cdots \\ n,\, r = 1, \cdots, n_\nu, \cdots \end{array} \right. \tag{33}$$

Thus, the characteristic equation becomes

$$\begin{vmatrix} D^{(1)}_{kj}+D^{(1)}_{jk} & D^{(2)}_{mj} & 0 \\ D^{(2)}_{jl} & D^{(3)}_{lm}+D^{(3)}_{m} & D^{(4)}_{lu} \\ 0 & D^{(4)}_{mr} & D^{(5)}_{ru}+D^{(5)}_{ur} \end{vmatrix} = 0 \tag{34}$$

or explicitly

$$\begin{vmatrix} 2D^{(1)}_{11},\ D^{(1)}_{12}+D^{(1)}_{21},\cdots\cdots & D^{(2)}_{11},\ D^{(2)}_{12},\cdots\cdots\cdots & \\ D^{(1)}_{21}+D^{(1)}_{12},\ 2D^{(1)}_{22},\cdots\cdots & D^{(2)}_{21},\ D^{(2)}_{22},\cdots\cdots\cdots & 0 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \\ D^{(2)}_{11},\ D^{(2)}_{21},\cdots\cdots\cdots & 2D^{(3)}_{11},\ D^{(3)}_{12}+D^{(3)}_{21},\cdots\cdots & D^{(4)}_{11},\ D^{(4)}_{12},\cdots\cdots\cdots \\ D^{(2)}_{12},\ D^{(2)}_{22},\cdots\cdots\cdots & D^{(3)}_{21}+D^{(3)}_{12},\ 2D^{(3)}_{22},\cdots\cdots & D^{(4)}_{21},\ D^{(4)}_{22},\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ & D^{(4)}_{11},\ D^{(4)}_{21},\cdots\cdots\cdots & 2D^{(5)}_{11},\ D^{(5)}_{12}+D^{(5)}_{21},\cdots\cdots \\ 0 & D^{(4)}_{12},\ D^{(4)}_{22},\cdots\cdots\cdots & D^{(5)}_{21}+D^{(5)}_{12},\ 2D^{(5)}_{22},\cdots\cdots \\ & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \end{vmatrix} = 0$$

The equation (34) is the basic equation for investigating the dynamic behavior of a suspension bridge and each element forms an infinite which converges the more rapidly the smaller the sag ratios. Explicit expressions for $D^{(k)}_{ij}$ are given in the appendix II. For the simplest case of the fundamental modes numerical results by eq. (34) are compared with those obtained by Bleich (Appendix I). The result indicates that a sufficient amount of stiffness of floor system provides a simple shape of fundamental modes and the lesser stiffness the more the flexibility of the cables effects the vibrational characterictics of a suspension bridge.

6. Analytical solutions of the deflectional modes with a fix point at midspan

In the preceding paragraph we have assumed non-vanishing $h(x)$ in the interval of definition, $0 \leq x \leq l$. However most of long spanned suspension bridges of these days are stiffened by center diagonal stays to increase torsional rigidity and to prevent the structure from catastropic oscillations[6]. The practical attachment of this stay envisages physically the solutions for eq. (31) and for this case one may not adopt the mode functions as described in eq. (32). Since analytical solutions are possibly sought in the same fashion as before, we assume them as follows

$$w_0 = \sum_{i=1}^{\infty} A_i \sin\frac{i\pi x}{l}, \quad w = \sum_{i=1}^{\infty}\left(A_i + B_i \frac{h}{l}\right)\sin\frac{i\pi x}{l}, \quad u = \sum_{i=1}^{\infty} C_i \frac{h}{l}\sin\frac{i\pi x}{l} \tag{35}$$

It is easily known that the deflectional modes under consideration, eq. (35), can be subdivided into *symmetric modes* by taking odd terms for vertical displacements and even terms for horizontal displacements and *anti-symmetric modes* by having even terms for vertical and odd terms for horizontal displacements.

The second expression in eq. (35) is alternatively expressed as

$$w = \sum_{i=1}^{\infty}\left\{A_i + {}_1B_i \frac{h}{l} + {}_2B_i\left(\frac{h}{l}\right)^2\right\}\sin\frac{i\pi x}{l}$$

introducing high order corrections to lateral displacements of cables (Fig. 3).

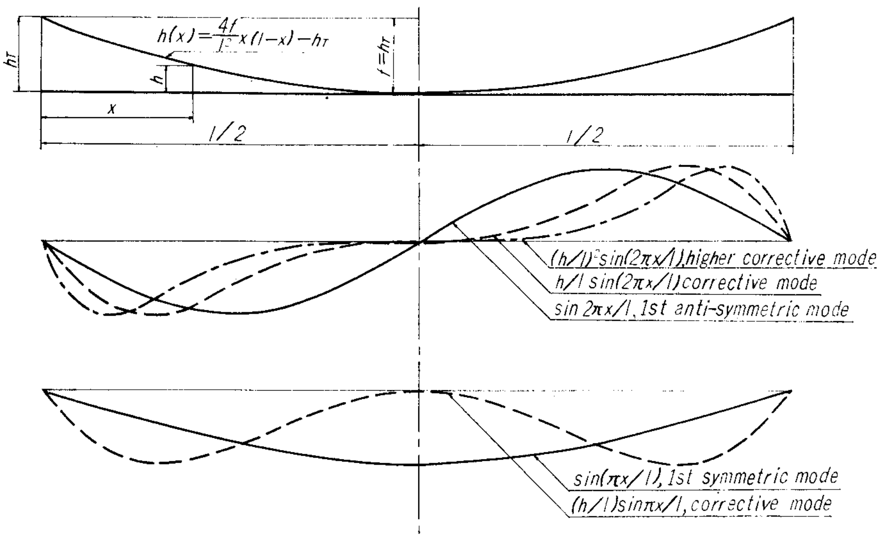

Fig. 3.

For the sake of brevity, we illustrate the symmetric modes only. The anti-symmetric modes can be considered in exactly same way and any expressions discussed hereafter are easily extended to this case. By virtue of eq. (35) the stationary condition associated with eq. (31) is given as

$$\delta I = \delta \Big[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{2i+1}A_{2j+1}D^{(1)}_{2i+1,2j+1} + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} B_{2i+1}B_{2j+1}D^{(2)}_{2i+1,2j+1} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_{2i}C_{2j}D^{(3)}_{2i,2j}$$
$$+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{2i+1}B_{2j+1}D^{(4)}_{2i+1,2j+1} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} B_{2i+1}C_{2j}D^{(5)}_{2i+1,2j} + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} A_{2i+1}C_{2j}D^{(6)}_{2i+1,2j}\Big] = 0 \quad (36)$$

which yields

$$\frac{\partial I}{\partial A_{2i+1}} = \frac{\partial I}{B_{2i+1}} = \frac{\partial I}{\partial C_{2i}} = 0, \qquad \text{and}$$

the characterictic equation

$$\begin{vmatrix} D^{(1)}_{2i+1,2l+1} + D^{(1)}_{2l+1,2i+1} & D^{(4)}_{2i+1,2m+1} & D^{(6)}_{2i+1,2n} \\ D^{(4)}_{2l+1,2j+1} & D^{(2)}_{2j+1,2m+1} + D^{(2)}_{2m+1,2j+1} & D^{(5)}_{2j+1,2n} \\ D^{(6)}_{2l+1,2k} & D^{(5)}_{2m+1,2k} & D^{(3)}_{2n,2k} + D^{(3)}_{2k\cdot 2n} \end{vmatrix} = 0 \qquad (37)$$

or explicity

$$\begin{vmatrix} 2D^{(1)}_{1,1}, \ D^{(1)}_{1,3}+D^{(1)}_{3,1}, \ \cdots & D^{(4)}_{1,1}, \ D^{(4)}_{1,3}, \ \cdots\cdots & D^{(6)}_{1,2}, \ D^{(6)}_{1,4}, \ \cdots\cdots \\ D^{(1)}_{3,1}+D^{(1)}_{1,3}, \ 2D^{(1)}_{3,3}, \ \cdots & D^{(4)}_{3,1}, \ D^{(4)}_{3,3}, \ \cdots\cdots & D^{(6)}_{3,2}, \ D^{(6)}_{3,4}, \ \cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ D^{(4)}_{1,1}, \ D^{(4)}_{3,1}, \ \cdots\cdots & 2D^{(2)}_{1,1}, \ D^{(2)}_{1,3}+D^{(2)}_{3,1}, \ \cdots & D^{(5)}_{1,2}, \ D^{(5)}_{1,4}, \ \cdots\cdots \\ D^{(4)}_{1,3}, \ D^{(4)}_{3,3}, \ \cdots\cdots & D^{(2)}_{3,1}+D^{(2)}_{1,3}, \ 2D^{(2)}_{3,3}, \ \cdots & D^{(5)}_{3,2}, \ D^{(5)}_{3,4}, \ \cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ D^{(6)}_{1,2}, \ D^{(6)}_{3,2}, \ \cdots\cdots & D^{(5)}_{1,2}, \ D^{(5)}_{3,2}, \ \cdots\cdots & 2D^{(3)}_{2,2}, \ D^{(3)}_{2,4}+D^{(3)}_{4,2}, \ \cdots \\ D^{(6)}_{1,4}, \ D^{(6)}_{3,4}, \ \cdots\cdots & D^{(5)}_{1,4}, \ D^{(5)}_{3,4}, \ \cdots\cdots & D^{(3)}_{4,2}+D^{(3)}_{2,4}, \ 2D^{(3)}_{4,4}, \ \cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \end{vmatrix} = 0$$

where elements in eq. (37) are explicitly specified in the appendix III. Eq. (35) are therfore known to suffice to describe general dynamic responses of a suspension bridge with center diagonal stays and can be reduced to the classical solutions for symmetric and anti-symmetric modes by annihilating B_i and C_i. Under this classical assumption, however, the significance of singularity at the central point can not be conceived properly and justification of the fact that the central diagonal stays are pertinent to stiffen the structure can be accomplished by taking into account the higher order approximations.

7. Conclusion

Free vibrations of a suspension bridge are classified into two classes of modes, deflectional and torsional. Analytical solutions for the deflectional modes which converge rapidly for small sag-ratios are presented in the general form for cases with and without a fix point at midspan. The significance for center diagonal stays is thus clarified to correspond to singularity in fundamental differential equations. The dynamic design of such flexible structures as suspension bridges calls for thorough detailed analysis including the higher order terms so that we are accessible to proper justification of any physical phenomena.

8. Acknowldegement

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Appendix I

For the simplest case of free vibrations of a suspenfion bridge the potential energy and the kinetic energy become

$$U_{\max}=\frac{1}{2}\int_0^l\left[EI_y\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2+2E_cA_c\left(\frac{u'-h'w'}{1+h'^2}\right)+4H_w\left(u'+\frac{w'^2+2h'w'u'+h'^2u'^2}{2(1+h'^2)}\right)\right]dx$$

$$T_{\max}=\frac{\omega^2}{2}\int_0^l\left[\frac{w_f}{g}w_0^2+\frac{2w_c}{g}(1+h'^2)^{1/2}(u^2+w^2)+\frac{w_f}{gA}I_y\left(\frac{\partial w_0}{\partial x}\right)^2\right]dx$$

by which the approximate natural frequency of the lowest order is written as

$$\omega_1^2=\frac{\frac{1}{4}\cdot\frac{\pi^4 EI_y}{l^3}+\frac{16f^2\pi^2E_cA_c}{l^3}F_c\left(\frac{f}{l}\right)+\frac{\pi^2H_w}{l}F_H\left(\frac{f}{l}\right)}{\frac{1}{4}\frac{w_fl}{g}+\frac{\pi^2}{4}\frac{w_f}{gA}\frac{I_y}{l}+\frac{w_cl}{g}F_d\left(\frac{f}{l}\right)}$$

where

$$F_c\left(\frac{f}{l}\right)=\int_0^1\frac{(1-2\xi)^2}{\left\{1-\frac{16f^2}{l^2}(1-2\xi)^2\right\}^2}\cos^2\pi\xi d\xi\,,\qquad F_H\left(\frac{f}{l}\right)=\int_0^1\frac{\cos^2\pi\xi}{1+\frac{16f^2}{l^2}(1-2\xi)^2}d\xi$$

$$F_d\left(\frac{f}{l}\right) = \int_0^1 \left(1+\frac{16f^2}{l^2}(1-2\xi)^2\right\}^{1/2} \sin^2 \pi\xi d\xi$$

under the assumption that $w_0 = w = \sin\frac{\pi x}{l}$ and $u=0$

Example: $l=2{,}800$ ft, $f=232$ ft, $w_c=0.647$ kips/ft/cable, $w_f=4{,}406$ kips/ft, $H_w=12{,}040$ kips, $A_c=191.5$ in^2, $E_c=26{,}000$ kps/in^2, $E=29{,}600$ kips/in^2, $I_y=2\times 128{,}400$ ft^2/in^2

then $\kappa=\frac{16f^2}{l^2}=0.109{,}84$ ond $\omega_1^2 \fallingdotseq 1.907$

while $\omega^2=1.965$ by the Ritz method.[7]

Appendix II

$$D_{kj}^{(1)} = EI_y\left(\frac{k\pi}{l}\right)^2\left(\frac{j\pi}{l}\right)^2\frac{l}{2}\delta_{kj}+2H_w\int_0^l\frac{h''}{h}\sin\frac{k\pi x}{l}\sin\frac{j\pi x}{l}dx-\omega^2\frac{w_f l}{2g}\delta_{kj}$$

$$= \left\{EI_y\frac{\pi^4}{2l^3}k^2j^2-\omega^2\frac{w_f l}{2g}\right\}\delta_{kj}+2H_w\frac{8f}{h_T l}\sum_{p=0}^{\infty}\sum_{q=0}^{p}(-1)^{p+q}\left(\frac{4f}{h_T}\right)^p {}_pC_q$$

$$\times\int_0^1 \xi^{p+q}\sin k\pi\xi\sin j\pi\xi d\xi$$

$$D_{kj}^{(2)} = -4H_w\int_0^l\frac{h''}{h}\sin\frac{k\pi x}{l}\sin\frac{j\pi x}{l}dx$$

$$= -\frac{32H_w f}{h_T l}\sum_{p=0}^{\infty}\sum_{q=0}^{p}(-1)^{p+q}\left(\frac{4f}{h_T}\right)^p {}_pC_q\int_0^1\xi^{p+q}\sin k\pi\xi\sin j\pi\xi d\xi$$

$$D_{ki}^{(3)} = 2E_cA_c\frac{\pi^2}{l^2}kj\int_0^l\frac{h'^2}{1+h'^2}\cos\frac{k\pi x}{l}\cos\frac{j\pi x}{l}dx+2H_w\frac{\pi^2}{l^2}kj\int_0^l\frac{1}{1+h'^2}\cos\frac{k\pi x}{l}\cos\frac{j\pi x}{l}dx$$

$$+2H_w\int_0^l\frac{h''}{h}\sin\frac{k\pi x}{l}\sin\frac{j\pi x}{l}dx-\omega^2\frac{2w_c}{g}\int_0^l(1+h'^2)^{1/2}\sin\frac{k\pi x}{l}\sin\frac{j\pi x}{l}dx$$

$$= \frac{2\pi^2E_cA_c}{l}kj\sum_{p=0}^{\infty}\sum_{q=0}^{2p+1}(-1)^{p+q}2^q\left(\frac{4f}{l}\right)^{2(p+1)}{}_{2(p+1)}C_q\int_0^1\xi^q\cos k\pi\xi\cos j\pi\xi d\xi$$

$$+\frac{2\pi^2H_w}{l}kj\sum_{p=0}^{\infty}\sum_{q=0}^{2p}(-1)^{p+q}2^q\left(\frac{4f}{l}\right)^{2p}{}_{2p}C_q\int_0^1\xi^q\cos k\pi\xi\cos j\pi\xi d\xi$$

$$+\frac{16H_wf}{h_Tl}\sum_{p=0}^{\infty}\sum_{q=0}^{p}(-1)^{p+q}\left(\frac{4f}{h_T}\right)^p{}_pC_q\int_0^1\xi^{p+q}\sin k\pi\xi\sin j\pi\xi d\xi$$

$$-\omega^2\frac{2w_cl}{g}\left\{\frac{1}{2}\delta_{kj}+\sum_{p=0}^{\infty}\sum_{q=0}^{2p}(-1)^{p+q+1}\frac{1\cdot1\cdot3\cdots(2p-3)}{2^pp!}\left(\frac{4f}{l}\right)^{2p}2^q{}_{2p}C_q\right.$$

$$\left.\times\int_0^1\xi^q\sin k\pi\xi\sin j\pi\xi d\xi\right\}$$

$$D_{kj}^{(4)} = -4E_cA_c\frac{kj\pi^2}{l^2}\int_0^l\frac{h'}{(1+h'^2)^2}\cos\frac{k\pi x}{l}\cos\frac{j\pi x}{l}dx$$

$$+4H_w\frac{kj\pi^2}{l^2}\int_0^l\frac{h'}{1+h'^2}\cos\frac{k\pi x}{l}\cos\frac{j\pi x}{l}dx$$

$$= -\frac{4\pi^2 E_c A_c}{l} kj \sum_{p=0}^{\infty} \sum_{q=0}^{2p+1} (-1)^{p+q} 2^q (p+1) \left(\frac{4f}{l^2}\right)^{p+1} {}_{2p+1}C_q \int_0^1 \xi^q \cos k\pi\xi \cos j\pi\xi \, d\xi$$

$$+ \frac{4\pi^2 H_w}{l} kj \sum_{p=0}^{\infty} \sum_{q=0}^{2p+1} (-1)^{p+q} 2^q \left(\frac{4f}{l}\right)^{2p+1} {}_{2p+1}C_q \int_0^1 \xi^q \cos k\pi\xi \cos j\pi\xi \, d\xi$$

$$D_{kj}^{(5)} = \frac{2\pi^2 E_c A_c}{l^2} kj \int_0^l \frac{1}{(1+h'^2)^2} \cos\frac{k\pi x}{l} \cos\frac{j\pi x}{l} dx + \frac{2\pi^2 H_w}{l^2} kj \int_0^l \frac{h'^2}{1+h'^2} \cos\frac{k\pi x}{l} \cos\frac{j\pi x}{l} dx$$

$$+ 2H_w \int_0^l \frac{h''}{h} \sin\frac{k\pi x}{l} \sin\frac{j\pi x}{l} dx - \omega^2 \frac{2w_c}{g} \int_0^l (1+h'^2)^{1/2} \sin\frac{k\pi x}{l} \sin\frac{j\pi x}{l} dx$$

$$= \frac{2\pi^2 E_c A_c}{l} kj \sum_{p=0}^{\infty} \sum_{q=0}^{2p} (-1)^{p+q} (p+1) 2^q \left(\frac{4f}{l}\right)^{2p} {}_{2p}C_q \int_0^1 \xi^q \cos k\pi\xi \cos j\pi \xi d\xi$$

$$+ \frac{2\pi^2 H_w}{l} kj \sum_{p=0}^{\infty} \sum_{q=0}^{2(p+1)} (-1)^{p+q} 2^q \left(\frac{4f}{l}\right)^{2p+2} {}_{2p+2}C_q \int_0^1 \xi^q \cos k\pi\xi \cos j\pi\xi \, d\xi$$

$$+ \frac{16 H_w f}{h_T l} \sum_{p=0}^{\infty} \sum_{q=0}^{p} (-1)^{p+q} \left(\frac{4f}{h_T}\right)^p {}_pC_q \int_0^1 \xi^{p+q} \sin k\pi\xi \sin j\pi\xi \, d\xi$$

$$- \frac{2w_c l}{g} \omega^2 \left\{ \frac{1}{2} \delta_{kj} + \sum_{p=1}^{\infty} \sum_{q=0}^{2p} (-1)^{p+q+1} \frac{1\cdot 1\cdot 3 \cdots (2p-3)}{2^p p!} \left(\frac{4f}{l}\right)^{2p} 2^q {}_{2p}C_q \right.$$

$$\left. \times \int_0^1 \xi^q \sin k\pi\xi \sin j\pi\xi \, d\xi \right\}$$

Appendix III

$$D_{2i+1,2j+1}^{(2)} = \omega^2 \left\{ \frac{wfl}{g2} \delta_{ji} + \frac{2w_c}{g} \int_0^l (1+h'^2)^{1/2} \sin\frac{(2i+1)\pi x}{l} \sin\frac{(2j+1)\pi x}{l} dx \right\}$$

$$- \frac{(2i+1)^2 (2j+1)^2 \pi^4}{2l^3} EI_y \delta_{ij}$$

$$- 2E_c A_c \frac{(2i+1)(2j+1)\pi^2}{l^2} \int_0^l \frac{h'^2}{(1+h'^2)^2} \cos\frac{(2i+1)\pi x}{l} \cos\frac{(2j+1)\pi x}{l} dx$$

$$- 2H_w \frac{(2i+1)(2j+1)\pi^2}{l^2} \int_0^l \frac{1}{1+h'^2} \cos\frac{(2i+1)\pi x}{l} \cos\frac{(2j+1)\pi x}{l} dx$$

$$D_{2i+1,2j+1}^{(2)} = \omega^2 \frac{2w_c}{g} \int_0^l (1+h'^2)^{1/2} \left(\frac{h}{l}\right)^2 \sin\frac{(2i+1)\pi x}{l} \sin\frac{(2j+1)\pi x}{l} dx$$

$$- 2E_c A_c \int_0^l \frac{h'^2}{(1+h'^2)^2} \left\{ \frac{(2i+1)\pi}{l} \frac{h}{l} \cos\frac{(2i+1)\pi x}{l} + \frac{h'}{l} \sin\frac{(2i+1)\pi x}{l} \right\}$$

$$\times \left\{ \frac{(2j+1)\pi h}{l^2} \cos\frac{(2j+1)\pi x}{l} + \frac{h'}{l} \sin\frac{(2j+1)\pi x}{l} \right\} dx$$

$$- 2H_w \int_0^l \frac{1}{1+h'^2} \left\{ \frac{(2i+1)\pi h}{l^2} \cos\frac{(2i+1)\pi x}{l} + \frac{h'}{l} \sin\frac{(2i+1)\pi x}{l} \right\}$$

$$\times \left\{ \frac{(2j+1)\pi h}{l^2} \cos\frac{(2j+1)\pi x}{l} + \frac{h'}{l} \sin\frac{(2j+1)\pi x}{l} \right\} dx$$

$$- 2H_w \int_0^l \frac{hh''}{l^2} \sin\frac{(2i+1)\pi x}{l} \sin\frac{(2j+1)\pi x}{l} dx$$

$$D_{2i,2j}^{(3)} = \omega^2 \frac{2w_c}{g} \int_0^l \left(\frac{h}{l}\right)^2 (1+h'^2)^{1/2} \sin\frac{2i\pi x}{l} \sin\frac{2j\pi x}{l} dx$$

$$- 2E_c A_c \int_0^l \frac{1}{(1+h'^2)^2} \left\{ \frac{h'}{l} \sin\frac{2i\pi x}{l} + \frac{2i\pi h}{l^2} \cos\frac{2i\pi x}{l} \right\}$$

$$\times\left\{\frac{h'}{l}\sin\frac{2j\pi x}{l}+\frac{2j\pi h}{l^2}\cos\frac{2j\pi x}{l}\right\}dx$$

$$-2H_w\int_0^l\frac{h'^2}{1+h'^2}\left\{\frac{h'}{l}\sin\frac{2i\pi x}{l}+\frac{2i\pi h}{l^2}\cos\frac{2i\pi x}{l}\right\}\left\{\frac{h'}{l}\sin\frac{2j\pi x}{l}+\frac{2j\pi h}{l^2}\cos\frac{2j\pi x}{l}\right\}dx$$

$$-2H_w\int_0^l\frac{hh''}{l^2}\sin\frac{2i\pi x}{l}\sin\frac{2j\pi x}{l}dx$$

$$D^{(4)}_{2i+1,2j+1}=\omega^2\frac{2w_c}{g}\int_0^l(1+h'^2)^{1/2}\left(\frac{2h}{l}\right)\sin\frac{(2i+1)\pi x}{l}\sin\frac{(2j+1)\pi x}{l}dx+H_wh''\delta_{ij}$$

$$+2E_cA_c\frac{(2i+1)(2j+1)\pi^2}{l^2}\int_0^l\frac{2h'^2}{(1+h'^2)^2}\frac{h}{l}\cos\frac{(2i+1)\pi x}{l}\cos\frac{(2j+1)\pi x}{l}dx$$

$$+2E_cA_c\int_0^l\frac{h'^2}{(1+h'^2)^2}\frac{h'}{l}\left\{\frac{(2i+1)\pi}{l}\cos\frac{(2i+1)\pi x}{l}\sin\frac{(2j+1)\pi x}{l}\right.$$

$$\left.+\frac{(2j+1)\pi}{l}\cos\frac{(2j+1)\pi x}{l}\sin\frac{(2i+1)\pi x}{l}\right\}dx$$

$$-4H_w\frac{(2i+1)(2j+1)\pi^2}{l^2}\int_0^l\frac{1}{1+h'^2}\frac{h}{l}\cos\frac{(2i+1)\pi x}{l}\cos\frac{(2j+1)\pi x}{l}dx$$

$$-2H_w\int_0^l\frac{1}{1+h'^2}\frac{h'}{l}\left\{\frac{(2i+1)\pi}{l}\cos\frac{(2i+1)\pi x}{l}\sin\frac{(2j+1)\pi x}{l}\right.$$

$$\left.+\frac{(2j+1)x}{l}\sin\frac{(2i+1)\pi x}{l}\cos\frac{(2j+1)\pi x}{l}\right\}dx$$

$$D^{(5)}_{2i+1,2j}=4E_cA_c\int_0^l\frac{h'}{(1+h'^2)^2}\left\{\frac{(2i+1)2j\pi^2}{l^2}\left(\frac{h}{l}\right)^2\cos\frac{(2i+1)\pi x}{l}\cos\frac{2j\pi x}{l}\right.$$

$$+\left(\frac{h'}{l}\right)^2\sin\frac{(2i+1)\pi x}{l}\sin\frac{2j\pi x}{l}+\frac{(2i+1)\pi hh'}{l^3}\cos\frac{(2i+1)\pi x}{l}\sin\frac{2j\pi x}{l}$$

$$\left.+\frac{2j\pi hh'}{l^3}\sin\frac{(2i+1)\pi x}{l}\cos\frac{2j\pi x}{l}\right\}dx$$

$$-4H_w\int_0^l\frac{h'}{1+h'^2}\left\{\frac{(2i+1)\pi h}{l^2}\cos\frac{(2i+1)\pi x}{l}+\frac{h'}{l}\sin\frac{(2i+1)\pi x}{l}\right\}$$

$$\times\left\{\frac{h'}{l}\sin\frac{2j\pi x}{l}+\frac{2j\pi h}{l^2}\cos\frac{2j\pi x}{l}\right\}dx$$

$$D^{(6)}_{2i+1,2j}=4E_cA_c\frac{(2i+1)\pi}{l}\int_0^l\frac{2h'}{(1+h'^2)^2}\left\{\frac{h'}{l}\cos\frac{(2i+1)\pi x}{l}\sin\frac{2j\pi x}{l}\right.$$

$$\left.+\frac{2j\pi h}{l^2}\cos\frac{(2i+1)\pi x}{l}\cos\frac{2j\pi x}{l}\right\}dx$$

$$-4H_w\frac{(2i+1)\pi}{l}\int_0^l\frac{h'}{1+h'^2}\left\{\frac{h'}{l}\sin\frac{2j\pi x}{l}+\frac{2j\pi h}{l^2}\cos\frac{2j\pi x}{l}\right\}\cos\frac{(2i+1)\pi x}{l}dx$$