Two-Stage Parametric Amplifier Coupled Through Idler Wave

By

Jun-ichi IKENOUE* and Masamitsu NAKAJIMA*

It is described how a matched parametric amplifier can be obtained if two negative resistance type parametric amplifiers are cascaded so that their idler tanks are interconnected by a quarter wave length transmission line. The analysis of such an amplifier has been carried out and testified by experiment. This type of parametric amplifier has several features besides the matched characteristic at the input and the output terminal, that is, the gain is independent of the phase difference between the two parametric excitations, the sensitivity of gain vs. excitation power is very low and the bandwidth is wider than that of single ones, having a double peak characteristic.

1. Introduction

Since parametric amplification in the microwave region was suggested by H. Shul[1], tremendous investigation has been made because of its extremely low noise property together with its mechanical simplicity. Some kinds of parametric devices are now available for practical use. However, the parametric amplifier, especially of a negative resistance type, has serious disadvantages such as narrow band characteristic and gain instability. The former defect has been considerably improved by the application of a double tuning technique. And a circulator is at present indispensable for alleviating the instability of gain, but the gain variation caused by the fluctuation of pump power is only compensated by the use of a highly stable pump source. For a radical solution of such a problem, various types of parametric devices have been proposed. D. K. Adams analyzed a four-frequency parametric circuit[2], which allows both upper and lower sidebands to exist. It has the unique property that unlimited amplification gain is possible without reflecting negative input resistance. But its experimental verification has not yet been made except for a very simple case by S. Kaito. K. K. N. Chang published a paper entitled "a four-terminal parametric amplifier"[3] in which an up- and a down-converter are used at the input and the output of a direct parametric amplifier to buffer the negative resistance effect of the intermediate amplifier. The combination of an up-

* Department of Electronics

converter using a nonlinear capacitor and a down-converter using a nonlinear resistor has recently been reported by S. Kumagai et al., and later by T. Sugiura and H. Suga.

We have found in the course of basic investigation of parametric devices that a matched amplifier can be built by interconnecting two negative resistance parametric amplifiers. The analysis shows that this is realized if we use an impedance inversion element for the coupling. In this case the two negative resistances cancel each other out, and thus the input and the output conductance become positive. If the two amplifiers are coupled directly e.g. through a half wave length line, the two negative resistances act so as to compensate each other. This is no more than a parallel combination of negative conductances and brings forth no useful result.

2. Matrix Representation of Paramatric Circuits

It is possible and convenient to use matrix calculus for our purpose of analyzing a cascade connection of parametric circuits. To begin with, we shall find a matrix representation of a single parametric amplifier.

The capacitance variation of a nonlinear capacitor which is excited by a pump at frequency $\omega_p/2\pi$ will be expressed as

$$\begin{aligned} C(t) &= C_0+2|C_1|\cos(\omega_p t+\varphi) \\ &= C_0+C_1e^{j\omega_p t}+C_1^*e^{-j\omega_p t}, \end{aligned} \tag{1}$$

neglecting higher terms. As H. E. Rowe obtained[4], the admittance matrix is given by*

$$\begin{pmatrix} I_s \\ I_i^* \end{pmatrix} = \begin{pmatrix} j\omega_s C_0 & j\omega_s C_1 \\ -j\omega_i C_1^* & -j\omega_i C_0 \end{pmatrix} \begin{pmatrix} V_s \\ V_i^* \end{pmatrix}, \tag{2}$$

for the small signal voltages and currents at frequencies $\omega_s/2\pi$ and $\omega_i/2\pi$,

$$\left.\begin{aligned} v &= V_s e^{j\omega_s t}+V_s^* e^{-j\omega_s t}+V_i e^{j\omega_i t}+V_i^* e^{-j\omega_i t} \\ i &= I_s e^{j\omega_s t}+I_s^* e^{-j\omega_s t}+I_i e^{j\omega_i t}+I_i^* e^{-j\omega_i t} \end{aligned}\right\}, \tag{3}$$

where

$$\omega_s+\omega_i = \omega_p. \tag{4}$$

Next consider a quadrupole as shown in Fig. 1, where only the component at the signal frequency $\omega_s/2\pi$ is allowed to exist at the input terminal and only the component at the idler frequency $\omega_i/2\pi$ at the output terminal. Noting that the sign of V_i^* is changed the fundamental matrix of such a quadrupole is obtained by transforming Eq. (2)

* The signal components are denoted by a suffix s and the idler by i.

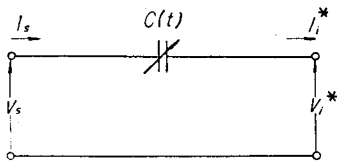

Fig. 1. Two-terminal pair representation of a parametrically excited capacitance. It is assumed that there exists only one component at the associated frequency in the input or in the out put terminal.

$$\begin{pmatrix} V_s \\ I_s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ j\omega_s C_0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/j\omega_i C_1^* \\ -j\omega_s C_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -j\omega_i C_0 & 1 \end{pmatrix} \begin{pmatrix} V_i^* \\ I_i^* \end{pmatrix}. \tag{5}$$

With this expression it is found that we can analyze the circuits as an ordinary alternating current network by incorporating the residual capacitance C_0 into the susceptance of each tank and by taking the complex conjugations for the idler components. Thus we can regard the fundamental matrix of the time-varying capacitance as

$$F_c = \begin{pmatrix} 0 & -1/j\omega_i C_1^* \\ -j\omega_s C_1 & 0 \end{pmatrix} = \frac{1}{-j\omega_i C_1^*} \begin{pmatrix} 0 & 1 \\ -g_1^2 & 0 \end{pmatrix}, \tag{6}$$

where

$$g_1^2 = \omega_s \omega_i |C_1|^2 .$$

Using Eq. (6) we now calculate the gain and bandwidth of Fig. 2, each

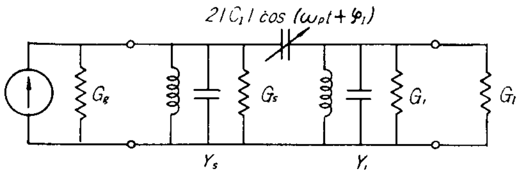

Fig. 2. Basic circuit model of the parametric amprifier—a frequency converter—

tank being tuned at ω_{s0} and ω_{i0} respectively, where $\omega_{s0}+\omega_{i0}=\omega_p$. The tank susceptance in the neighbourhood at resonance is expressed as

$$Y = G + j\omega C + \frac{1}{j\omega L} \approx G + j2\delta' B = G(1+j2\delta' Q),$$

where

$$\delta' = (\omega - \omega_0)/\omega_0 .$$

The residual capacitance C_0 of the variable capacitor is involved in C in the above equation. Introducing the quantities

$$\delta = \delta_s' = \frac{\omega_s - \omega_{s0}}{\omega_{s0}}, \quad B_i = \frac{\omega_{s0}}{\omega_{i0}}\sqrt{\frac{C_i}{L_i}}, \tag{7}$$

the admittances of the signal and the idler tank are written respectively as

$$Y_s = G_s + j2\delta B_s, \quad Y_i = G_i - j2\delta B_i.$$

The fundamental matrix of Fig. 2 is obtained as

$$F_1 = \begin{pmatrix} 1 & 0 \\ Y_{s1} & 1 \end{pmatrix} \frac{1}{-j\omega_i C_1^*} \begin{pmatrix} 0 & 1 \\ -g_1^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y_{i1} & 1 \end{pmatrix}^*$$

$$= \frac{1}{-j\omega_i C_1^*} \begin{pmatrix} G_{i1} + j2\delta B_{i1} & 1 \\ -g_1^2 + (G_{s1} + j2\delta B_{s1})(G_{i1} + j2\delta B_{i1}) & G_{s1} + j2\delta B_{s1} \end{pmatrix}. \tag{8}$$

The transducer conversion gain of this two terminal -pair network becomes*

$$G_{t1} = \frac{\omega_i}{\omega_s} \frac{4G_g G_l g_1^2}{\{(G_g + G_s)(G_l + G_i) - g_1^2 - 4\delta^2 B_s B_i\}^2 + 4\delta^2 \{B_s(G_l + G_i) + B_i(G_g + G_s)\}^2}.$$

Again introducing dimensionless quantities

$$a^2 = \frac{g_1^2}{(G_g + G_s)(G_l + G_i)} : \text{ excitation coefficient}$$

$$Q_s = \frac{B_s}{G_g + G_s}, \quad Q_i = \frac{B_i}{G_l + G_i} : \text{ loaded } Q$$

and assuming that the bandwidth be narrow, and neglecting the small losses of the tanks, the transducer gain is expressed in a simpler form as

$$G_{t1} = \frac{\omega_i}{\omega_s} \cdot \frac{4a^2}{(1-a^2)^2 + 4\delta^2 (Q_s + Q_i)^2},$$

and 3 db-down bandwidth is

$$2\delta = \frac{1 - a^2}{Q_s + Q_i}.$$

The condition for stable amplification is determined by the postulate that the absolute value of the input conductance be smaller than the internal conductance of the signal generator i.e.

$$G_g + G_s > g_1^2 / (G_l + G_i),$$

$$\therefore \quad a^2 < 1.$$

---

* The transducer gain is defined to be the ratio of the power dissipated in the load to the available power of the signal generator. The transducer gain of a four terminal network (A, B, C, D) is given by

$$G = \frac{4G_g G_l}{|AG_g + BG_g G_l + C + DG_l|^2} = \frac{4}{\left| A\sqrt{\frac{G_g}{G_l}} + B\sqrt{G_g G_l} + \frac{C}{\sqrt{G_g G_l}} + D\sqrt{\frac{G_l}{G_g}} \right|^2}$$

where G_g is the internal conductance of the signal generator and G_l of the load.

The characteristics of a direct amplifier using a circulator are similarly obtained.

3. Two-Stage Parametric Amplifier

In this section we consider generally a two stage parametric amplifier, whose coupling element is a quadrupole as shown in Fig. 3.

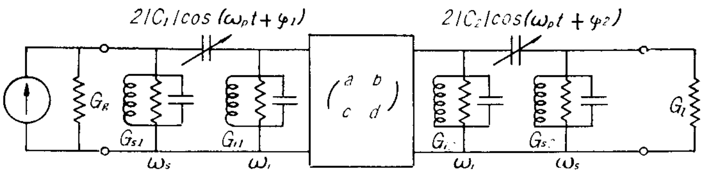

Fig. 3. Circuit model of the two-stage parametric amplifier coupled by a two-terminal pair network in the idler circuit.

The fundamental matrix of the first stage has been obtained as Eq. (8). We must take the complex conjugate for the matrix of the coupling quadrupole;

$$F_c = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* . \tag{9}$$

Interchanging the components for signal and idler, the matrix of the second stage becomes

$$F_2 = \frac{1}{j\omega_s C_2}\begin{pmatrix} G_{s2}+j2\delta B_{s2} & 1 \\ -g_2^2+(G_{i2}+j2\delta B_{i2})(G_{s2}+j2\delta B_{s2}) & G_{i2}+j2\delta B_{i2} \end{pmatrix} . \tag{10}$$

The total matrix of the amplifier is given by the multiplication from Eq. (8) to Eq. (10);

$$F_t = F_1 F_c F_2 = \frac{-j e^{j(\varphi_1-\varphi_2)}}{\omega_s \omega_i |C_1 C_2|}\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{11}$$

$$A = a^*(G_{i1}+j2\delta B_{i1})(G_{s2}+j2\delta B_{s2}) + b^*(G_{i1}+j2\delta B_{i1})\{-g_2^2+(G_{i2}+j2\delta B_{i2})(G_{s2}+j2\delta B_{s2})\}$$
$$+C^*(G_{s2}+j2\delta B_{s2}) + d^*\{-g_2^2+(G_{i2}+j2\delta B_{i2})(G_{s2}+j2\delta B_{s2})\}$$
$$B = a^*(G_{i1}+j2\delta B_{i1}) + b^*(G_{i1}+j2\delta B_{i1})(G_{i2}+j2\delta B_{i2}+c^*+d^*(G_{i2}+j2\delta B_{i2})$$
$$C = a^*\{-g_1^2+(G_{s1}+j2\delta B_{s1})(G_{i1}+j2\delta B_{i1})\}(G_{s2}+j2\delta B_{s2})$$
$$+b^*\{-g_1^2+(G_{s1}+j2\delta B_{s1})(G_{i1}+j2\delta B_{i1})\}\{-g_2^2+(G_{i2}+j2\delta B_{i2})(G_{s2}+j2\delta B_{s2})\}$$
$$+c^*(G_{s1}+j2\delta B_{s1})(G_{s2}+j2\delta B_{s2}) + d^*(G_{s1}+j2\delta B_{s1})\{-g_2^2+(G_{i2}+j2\delta B_{i2})(G_{s2}+j2\delta B_{s2})\}$$
$$D = a^*\{-g_1^2+(G_{s1}+j2\delta B_{s1})(G_{i1}+j2\delta B_{i1})\}$$
$$+b^*(G_{i2}+j2\delta B_{i2})\{-g_1^2+(G_{s1}+j2\delta B_{s1})(G_{i1}+j2\delta B_{i1})\}$$
$$+c^*(G_{s1}+j2\delta B_{s1}) + d^*(G_{s1}+j2\delta B_{s1})(G_{i2}+j2\delta B_{i2}) .$$

It is seen in Eq. (11) that the term associated with the phase difference between the two pumps is factorized from the matrix. Since the absolute value of this factor is always unity, it follows that the power gain of this type of amplifier is independent of the phase difference of the parametric excitations, which is different from the so-called traveling wave parametric amplifier. But the phase of the output wave relative to the input is linearly related to the pump phase difference.

As a particular example we now select the coupling quadrupole to be a shunt loss conductance, so that $(a, b, c, d)^* = (1, 0, G, 1)$. Neglecting smaller terms associated with losses, the transducer power gain at resonance is obtained as follows;

$$G_{t20} = \frac{4\alpha^2\beta^2}{(1-\alpha^2-\beta^2)^2}, \tag{12}$$

where

$$\alpha^2 = \omega_s\omega_i|C_1|^2/G_g G, \quad \beta^2 = \omega_s\omega_i|C_2|^2/G G_l,$$

and the bandwidth

$$2\delta = \frac{1-\alpha^2-\beta^2}{Q+Q_g(1-\beta^2)+Q_l(1-\alpha^2)}, \tag{13}$$

where

$$Q = \frac{B_{i1}+B_{i2}}{G}, \quad Q_g = \frac{B_{s1}}{G_g}, \quad Q_l = \frac{B_{s2}}{G_l}.$$

These relations are not so remarkably different from a single parametric amplifier. The gain is increased without limit if the pump $\alpha^2+\beta^2$ is increased close to unity and the bandwidth becomes unlimitedly narrow. It is seen that we can not obtain a significant amplifier only by the direct coupling of two negative resistance amplifiers.

4. Two-Stage Parametric Amplifier coupled by a quarter wave length transmission line

We can expect some interesting characteristics if we use an impedance inversion element for the coupling quadrupole as shown in Fig. 4. It is assumed to have a phase shift of $\pi/2$ for all frequencies under consideration,

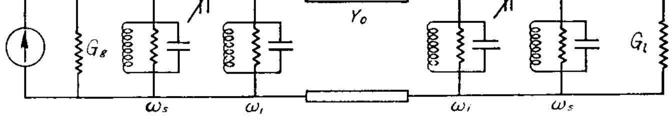

Fig. 4. Two-stage parametric amplifier with a quarter wave coupling.

$$F_c = \begin{pmatrix} 0 & jZ_0 \\ jY_0 & 0 \end{pmatrix}.$$

The substitution of this into Eq. (11) yields

$$F_t = \frac{-j e^{j(\varphi_1-\varphi_2)}}{\omega_s \omega_i |C_1 C_2|} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{14}$$

$$\begin{aligned}
A Y_0/G_{i1} G_{i2} G_l &= (1+j2\delta Q_1)\{-\beta^2+(1+j2\delta Q_2)(l_l+j2\delta Q_l)\}+\eta^2(l_l+j2\delta Q_l) \\
B Y_0/G_{i1} G_{i2} &= (1+j2\delta Q_1)(1+j2\delta Q_2)+\eta^2 \\
C Y_0/G_g G_{i1} G_{i2} G_l &= \{-\alpha^2+(l_g+j2\delta Q_g)(1+j2\delta Q_1)\}\{-\beta^2+(1+j2\delta Q_2)(l_l+j2\delta Q_l)\} \\
&\quad +\eta^2(l_g+j2\delta Q_g)(l_l+j2\delta Q_l) \\
D Y_0/G_g G_{i1} G_{i2} &= \{-\alpha^2+(l_g+j2\delta Q_g)(1+j2\delta Q_1)\}(1+j2\delta Q_2)+\eta^2(l_g+2\delta Q_g),
\end{aligned}$$

where

$$\alpha^2 = \frac{\omega_s \omega_i |C_1|^2}{G_g G_{i1}}, \quad \beta^2 = \frac{\omega_s \omega_i |C_2|^2}{G_{i2} G_l}, \quad \eta^2 = \frac{Y_0^2}{G_{i1} G_{i2}}$$

$$Q_g = \frac{B_{s1}}{G_g}, \quad Q_l = \frac{B_{s2}}{G_l}, \quad Q_1 = \frac{B_{i1}}{G_{i1}}, \quad Q_2 = \frac{B_{i2}}{G_{i2}},$$

and

$$l_g = \frac{G_{s1}}{G_g}, \quad l_l = \frac{G_{s2}}{G_l}.$$

By definition the transducer gain is given by, neglecting loss terms l_g and l_l,

$$G_t = \frac{4\alpha^2\beta^2\eta^2}{\{(1-\alpha^2)(1-\beta^2)+\eta^2\}^2+4\delta^2 F+16\delta^4 G+64\delta^6 H+256\delta^8 I}, \tag{15}$$

and the reflection gain at the input terminal is also obtained as the ratio of the reflected power to the input power,

$$G_r = \frac{\{(1+\alpha^2)(1-\beta^2)+\eta^2\}^2+4\delta^2 F'+16\delta^4 G'+64\delta^6 H'+256\delta^8 I'}{\{(1-\alpha^2)(1-\beta^2)+\eta^2\}^2+4\delta^2 F+16\delta^4 G+64\delta^6 H+256\delta^8 I}, \tag{16}$$

where

$$\begin{aligned}
F &= \{(1-\alpha^2)(Q_2+Q_l)+(1-\beta^2)(Q_g+Q_1)+\eta^2(Q_g+Q_l)\}^2 \\
&\quad -2\{(1-\alpha^2)(1-\beta^2)+\eta^2\}\{(1-\alpha^2)Q_2Q_l+(1-\beta^2)Q_gQ_1+\eta^2 Q_gQ_l+(Q_g+Q_1)(Q_2+Q_l)\} \\
G &= \{(1-\alpha^2)Q_2Q_l+(1-\beta^2)Q_gQ_1+\eta^2Q_gQ_l+(Q_g+Q_1)(Q_2+Q_l)\}^2 \\
&\quad +2Q_gQ_1Q_2Q_l\{(1-\alpha^2)(1-\beta^2)+\eta^2\} \\
&\quad -2\{Q_1Q_2(Q_g+Q_l)+Q_gQ_l(Q_1+Q_2)\}\{(1-\alpha^2)(Q_2+Q_l)+(1-\beta^2)(Q_g+Q_1)+\eta^2(Q_g+Q_l)\} \\
H &= \{Q_1Q_2(Q_g+Q_l)+Q_gQ_l(Q_1+Q_2)\}^2 \\
&\quad -2Q_gQ_1Q_2Q_l\{(1-\alpha^2)Q_2Q_l+(1-\beta^2)Q_gQ_1+\eta^2Q_gQ_l+(Q_g+Q_1)(Q_2+Q_l)\} \\
I &= (Q_gQ_1Q_2Q_l)^2.
\end{aligned}$$

The primed quantities in the expression of the reflection gain are similarly defined by changing the signs of α^2 and Q_g.

(a) Gain and Pumping

The midband gain is obtained by setting $\delta=0$ in Eq. (15);

$$G_{t0}=\frac{4\alpha^2\beta^2\eta^2}{\{(1-\alpha^2)(1-\beta^2)+\eta^2\}^2}. \tag{15)'$$

Similarly the reflection gain is

$$G_{r0}=\frac{\{(1+\alpha^2)(1-\beta^2)+\eta^2\}^2}{\{(1-\alpha^2)(1-\beta^2)+\eta^2\}^2}. \tag{16)'$$

The gain and the input conductance as a function of pump factor γ^2 in the case of equal pumping ($\alpha^2+\beta^2=\gamma^2$) are shown in Fig. 5. When the pump is not applied to the amplifier, the insertion gain is zero and all power delivered to the amplifier is reflected to the generator by the assumption that there is no loss in the signal circuit (for simplification losses in the signal tanks are neglected in the above equations, but latter this will be taken into account). As the pumps are applied, the input conductance assumes a negative value and both the insertion and the reflection gain are increased. At the pump $\gamma^2=(1-\eta)(1+\eta^2)^{\frac{1}{2}}$ the input conductance takes the maximum negative value, the reflection gain being also maximized. The absolute value of this negative maximum conductance never exceeds that of the generator conductance, so that the amplifier does not become unstable at this point. As the pumps are increased further, the negative conductance is contrarily decreased to cross the zero axis and comes to take a positive value equal to the generator conductance G_g at $\gamma^2=(1+\eta^2)^{\frac{1}{2}}$. The amplifier is namely matched at both input and output terminals. At the same time the insertion gain reaches its maximum

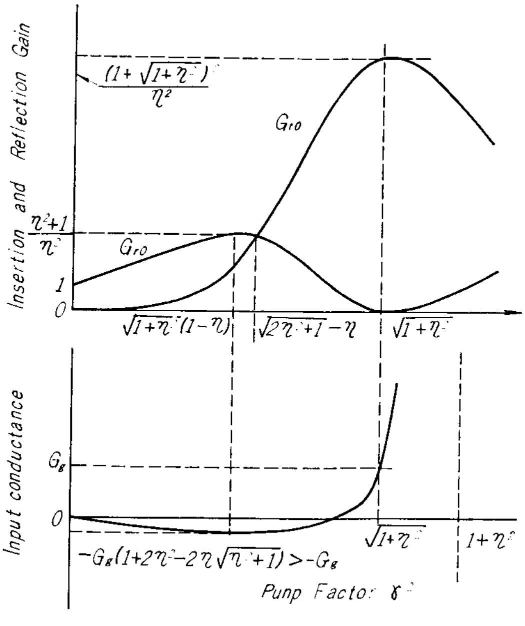

Fig. 5. Amplification gain and input conductance at the midband frequency as a function of pump faccor.

$$G_{t\,\max}=\frac{(1+\sqrt{1+\eta^2})^2}{\eta^2},$$

which is a function of the coupling factor only. Since this is a monotonous decreasing function of η^2, the looser the coupling, the higher the gain.

Next consider briefly the case of asymmetric pumping. The insertion gain of Eq. (15)′ is symmetric with respect to the pump factors α^2 and β^2, but the reflection gain of Eq. (16)′ is not. If the pump of the first stage α^2 is increased with β^2 constant, the insertion gain G_{t0} is increased provided $\beta^2<1$, but, before G_{t0} reaches infinity, the amplifier becomes unstable at frequencies other than the midband frequency. The reflection gain is increased or decreased according to the magnitude of β^2, and the matching at the input terminal is possible, if and only if the second pump β^2 lies in the region

$$1+\frac{\eta^2}{2+\eta^2}<\beta^2<1+\eta^2\,.$$

For a special case of $\beta^2=1$, the reflection gain is unity independent of the magnitude of the first pump α^2. This phenomenon is qualitatively observed in experiment. Inversely, if the pump of the second stage β^2 is varied with α^2 constant, the match at the input is always possible by selecting β^2 to be $1+\eta^2/(1+\alpha^2)$. These relations are illustrated in Fig. 6.

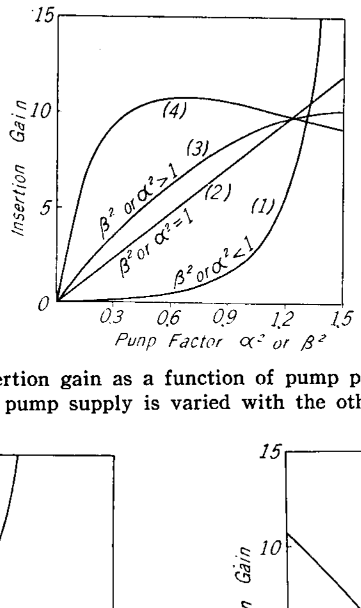

Fig. 6 (a) Insertion gain as a function of pump power, when only one pump supply is varied with the other constant.

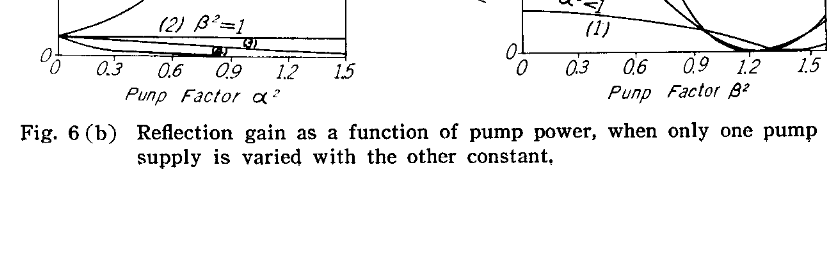

Fig. 6 (b) Reflection gain as a function of pump power, when only one pump supply is varied with the other constant.

It is necessary that $\alpha^2=\beta^2=(1+\eta^2)^{\frac{1}{2}}$ be satisfied for the amplifier to be matched at both input and output terminals.

(b) Frequency Characteristic of the Gain

It is seen from Eq. (15) that the frequency response of this type of amplifier is double peaked, since F in Eq. (15) is negative, when the pumps are large enough to meet the match condition. Eq. (16) tells that exact match is accomplished at the center frequency of the band ($\delta=0$, and $\alpha^2=\beta^2=(1+\eta^2)^{\frac{1}{2}}$). As the signal input frequency deviates from the center, the reflection appears and it reaches maximum near the band edges. The calculated frequency responses of the insertion gain and the reflection gain are both shown in Fig. 7, were $k=Q_1/Q_g=Q_2/Q_l$. The corresponding frequency response of a single

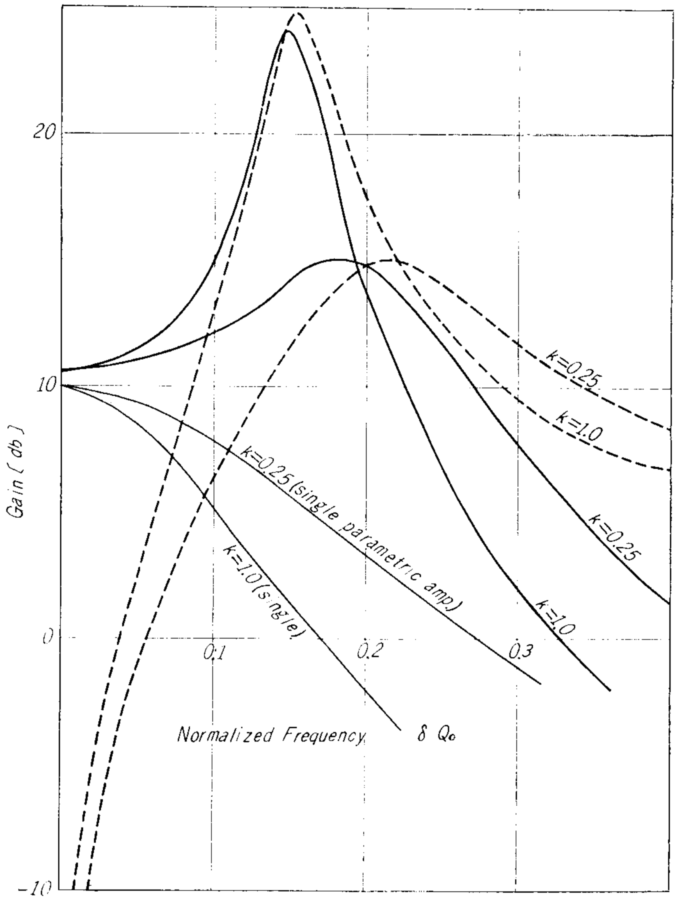

Fig. 7. Calculated frequency charactetistic of the insertion gain of the two-stage parametric amplifier. The dashed lines represent the reflection gain,

parametric amplifier using a circulator is also depicted in the same figure for comparison. The bandwidth of the two-stage parametric amplifier is wider, but the quality factor of the idler tanks should be smaller in order to obtain a smoother frequency response. This is achieved by choosing the idler frequency as high as possible since the idler quality factors in this analysis are multiplied by ω_s/ω_i as in Eq. (7). This is also desirable for lowering the noise figure as will be stated later.

(c) Stability Criterion

We must investigate the conditions for stable operation of this circuit. The characteristic equation for the stability criterion is obtained, for example, by considering the admittance or impedance at the input terminal. The input admittance of this amplifier is by Eq. (14)

$$Y_{in} = (C+DG_l)/(A+BG_l)\,,$$

so that the characteristic equation is given by $G_g+Y_{in}=0$, or

$$AG_g+BG_gG_l+C+DG_l = 0\,.$$

Normalizing this with respect to the conductances, the characteristic equation can be expressed as

$$f(j2\delta) \equiv \{-\alpha^2+(1+j2\delta Q_1)(1+l_g+j2\delta Q_g)\}\{-\beta^2+(1+j2\delta Q_2)(1+l_l+j2\delta Q_l)\} \\ +\eta^2(1+l_g+j2\delta Q_g)(1+l_l+j2\delta Q_l) = 0\,, \tag{17}$$

whose absolute square is identical with the denominator of the gain expressions (15) and (16). The characteristic equation is expressed in terms of the normalized relative frequency δ. For the complete investigation of the stability problem we may have to replace this with the original real frequency ω_s and find the complex roots $s=\sigma_s+j\omega_s$. But there will be another method. If we know by some means the stable operating limit of the amplifier, it is comparatively easy to determine which region across the boundary is stable. It is supposed that, at the stable amplification limit, the real part of the complex frequency satisfying the characteristic equation will be zero. This may not be an unreasonable assumption in the case of this parametric amplifier. Accordingly we can treat the characteristic equation in terms of the transformed frequency δ which is real, and it is no longer necessary to solve the characteristic equation to find the complex roots.

The frequency, at which the circuit will begin to oscillate when the pumps are increased, is obtained by solving

$$f(j2\delta) = 0\,.$$

For simplicity of theory we consider the circuit in the symmetric case.* Hence the imaginary part of the above equation is written as

$$j2x[\eta^2(1+l)+(1+l-\gamma^2-kx^2)\{1+k(1+l)\}]=0,$$

where

$$x=2\delta Q_0, \quad Q_0=Q_g=Q_l, \quad Q=Q_1=Q_2,$$
$$k=Q/Q_0, \quad l=l_g=l_l, \quad \gamma^2=\alpha^2=\beta^2.$$

Solving this cubic equation for x, we have

$$x=0, \quad \text{or} \quad x^2=\frac{1}{k}\left\{1+l-\gamma^2+\frac{\eta^2(1+l)}{1+k(1+l)}\right\}. \tag{18}$$

The first one is just the resonant frequency of the tanks and the second ones are a little deviated from the resonant frequency. To determine at which frequency the amplifier is to become unstable, it is necessary to investigate whether the real part of the characteristic equation can also be made zero.

First take up the solution $x=0$. Substitution of this into Eq. (17) gives

$$f(j2\delta)=(1+l-\gamma^2)^2+\eta^2(1+l)^2,$$

which is positive and non-zero for all value of γ^2. Thus the amplifier never becomes unstable at the midband frequency.

Next substitute the second solution of (18) into the characteristic equation (17), and we have

$$f(j2\delta)=-\frac{\{1+k(1+l)\}^2}{k}\left[1+l-\gamma^2+\frac{\eta^2(1+l)}{\{1+k(1+l)\}^2}\right].$$

This becomes zero, if we choose the pump factor to be

$$\gamma_l^2=1+l+\frac{\eta^2(1+l)}{\{1+k(1+l)\}^2}. \tag{19}$$

This is the limit of pumping for stable operation. In other words, if the amplifier is excited by the pump corresponding to Eq. (19), the characteristic equation is satisfied by real frequencies and consequently the amplifier will be in the state of constant amplitude oscillation at the frequencies (18). For practical use the pump should be kept under the limit.

As previously stated this amplifier is matched by the pump

$$\gamma_m^2=l+\sqrt{1+\eta^2(1-l^2)},$$

where the losses in the signal tanks are considered. In order that the match may be possible, the pump γ_m^2 for matching must lie in the stable operating region, i.e.

* This does not necessarily mean that the two elemental amplifiers should be quite same.

$$\gamma_m^2 < \gamma_t^2 .$$

Solving this inequality for k, the necessary condition for matching is obtained as

$$k < \frac{1}{1+l}\left\{\sqrt{\frac{\eta^2(1+l)}{\sqrt{1+\eta^2(1-l^2)}-1}}-1\right\}. \tag{20}$$

The quality factor of the idler tanks should be low for realization of the match. If the losses in the signal tanks are increased, this restriction is relaxed.

(d) Sensitivity of Gain

The sensitivity of gain to the variation of pump power is an important factor of a parametric amplifier. In this section consider the sensitivity of gain according to the definition by H. E. Rowe:

$$S = \frac{dG_t/G_t}{d\gamma^2/\gamma^2}.$$

In the two-stage parametric amplifier case, it is assumed that both the pump factors are taken as equal.

The sensitivity of a single parametric amplifier (either direct or conversion type) in the inverting case is given by

$$S' = \sqrt{G_t}$$

in the case of comparatively high gain. For the two-stage amplifier, the sensitivity becomes

$$S = 2+\frac{2(1-\gamma^2)}{\eta^2}\sqrt{G_t}.$$

Different from the case of a single parametric amplifier, this reduces to zero, when the pump factor γ^2 is selected equal to $\gamma_m^2=(1+\eta^2)^{\frac{1}{2}}$ which is just the condition for matching. This fact indicates that the gain has the maximum value at the matched point.

By the consideration that the pump variation is relative to that of the external circuit constants, it is concluded that the amplification gain is also insensitive to the variation of the external constants. This may be a natural consequence of matching.

(e) Noise Figure

Let us calculate the noise figure due to the losses in the amplifier. The noise figure F of an ordinary amplifier is defined as

$$F = \frac{N_0}{KTBG},$$

where N_0 is the noise dissipated in the load. G is the ratio of the power dissipated in the load to the available power of the generator, and which agrees with the available power gain when the amplifier is matched. The contribution of loss G_N to the noise figure is given by

$$F(G_N)=\frac{4G_N G_l}{|AG_N+BG_N G_l+C+DG_l|^2},$$

where (A, B, C, D) is the fundamental matrix of the circuit between G_N and the load G_l.

The calculated noise figures are as follows: for the frequency converter

$$F_1=1+\frac{G_s}{G_g}+\frac{\omega_s}{\omega_i}\cdot\frac{G_i}{\alpha^2 G_l},$$

for the direct amplifier with a circulator

$$F_2=1+\frac{G_s}{G_g}+\frac{\omega_s}{\omega_i}\cdot\frac{4\alpha^2}{(1+\alpha^2)^2},$$

and for the two-stage amplifier with a quarter wave coupling

$$F_3=1+\frac{G_{s1}}{G_g}+\frac{\omega_s}{\omega_i}\cdot\frac{(1-\alpha^2)^2+\eta^2}{\alpha^2\eta^2}+\frac{G_{s2}}{G_l}\cdot\frac{(1-\alpha^2+\eta^2)^2}{\alpha^2\beta^2\eta^2}.$$

The first two terms represent the noise figure contributions by the input signal circuit and the third by the idler and the fourth in the last equation is due to the output circuit at the signal frequency. These results show that the two-stage parametric amplifier has a larger noise figure than the single ones. But the excess noise of the fourth term is small so far as the output loss G_{s2} is not great, and can be decreased at a high gain operation with $\alpha^2\approx 1$. The third term in the last equation is minimized with the pump $\alpha^2=(1+\eta^2)^{\frac{1}{2}}$, which is exactly the match condition. And this term has a factor w_s/w_i common to the single parametric amplifier, so that the noise figure of the two-stage amplifier is also reduced by selecting an idler frequency higher than the signal.

5. Experiment

(a) Construction

As elemental amplifiers we have used non-degenerated parametric amplifiers as shown in Fig. 8. Since there exist both the signal and the idler wave in the nonlinear capacitor, it is necessary to separate them in oder to construct the two stage parametric amplifier in question. For that purpose we have used two filters of one element as shown in Fig. 9(a), each being resonant at the signal and the idler frequency respectively. Their external Q's are made low so that the resonant susceptance of the filter does not affect the resonance

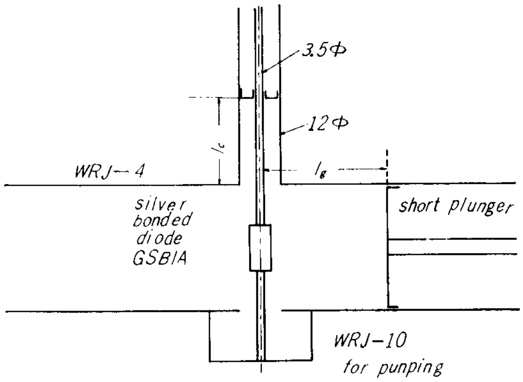

Fig. 8. Elemental non-degenerated parametric amplifier.

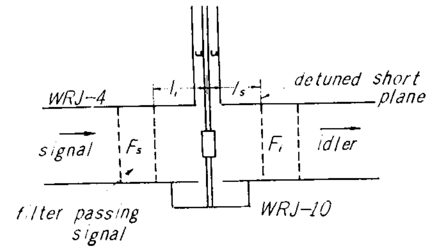

Fig. 9 (a) Separating device of the idler wave from the signal.

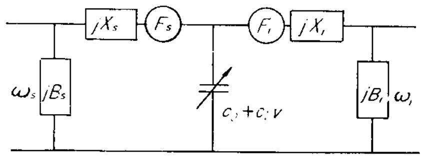

Fig. 9 (b) Equivalent circuit of the elemental amplifier. The circles represent ideal band-pass filters.

of the time-varying capacitance. The equivalent circuit of this device will be drawn as in Fig. 9 (b). The time-varying capacitance is denoted by C_0+C_1v, which is a nonlinear capacitance diode excited by the pump through coaxial line under the main wave guide. $X=j\tan\beta_c l_c$ represents the reactance of the coaxial line and $B=-j\cot B_g l_g$ the susceptance presented at the diode plane by the detuned short of the filter. These reactances, needless to say, have different values at the signal and the idler frequency.

Let us now consider this circuit at the signal frequency. The signal wave enters through the signal filter suffering no obstruction and is applied to the diode, which is resonated by the reactance X_s and susceptance B_s. If we neglect the discontinuity at the junction of wave guide and coaxial line, we can suppose that X_s and B_s are series-connected to the diode. Therefore the condition for resonance at the signal frequency can be expressed as

$$jX_s+\frac{1}{j\omega_s C_0}+\frac{1}{jB_s}=0\,.$$

At the idler frequency a similar condition must be satisfied. The idler wave generated by the time-varying capacitance comes out through the idler filter and goes into the idler port of the next stage. The two idler ports of each amplifier are coupled by a phase shifter, with which the coupling phase is adjusted to be an odd integral mutiple of $\pi/2$.

As stated in the foregoing section, the midband gain is determined only by the coupling factor η^2, which is directly related to the characteristic admittance of the coupling line. Ordinarily the characteristic admittance of a line is difficult to change. There is a method for obtaining an equivalent alteration of η^2 with a fixed transmission line. If we set an ideal transformer with turn-ratio n at the junction point of the circuit and the quarter wave transmission line, the equivalent characteristic impedance is multiplied by the turn-ratio n, that is,

$$\begin{pmatrix} n & 0 \\ 0 & 1/n \end{pmatrix} \begin{pmatrix} 0 & jz_0 \\ j/z_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & jnz_0 \\ j/nz_0 & 0 \end{pmatrix}.$$

Thus the maximum gain is varied arbitrarily by changing the coupling between the idler tank and transmission line.

The magnitude of coupling is determined by the value of X and B. But it is difficult to find a definite measure for coupling because of the complexity of the reactances. However, we can approximately estimate it by the value $1/|BX|$, which must be small in order to obtain a large amplification gain. The magnitude of coupling can not be changed so freely, since the quantities B and X are also related to the resonant frequencies of the signal and the idler.

Practically, our device had a fairly large value of B, hence small value of $1/|BX|$. The distance between the diode plane and the detuned short of the filter is close to an integer times a half wave length.

The losses in the idler tanks have also some effects on the operation of the amplifier. But we did not take much note of this in the experiment.

The experimental block diagram is shown in Fig. 10. The pump power is delivered to both stages with a single klystron V-55. Since the pump phase difference has nothing to do with the amplification gain, there arises no trouble, even if the pump power leaks through the idler coupling element. The gain is measured as the ratio of the output power when the amplifier is inserted between the signal generator and the load, to that when the load is directly connected to the generator.

For complete operation of the two-stage amplifier, each elemental amplifier is separately adjusted to operate as a direct amplifier using circulator, and, after coupling the two amplifiers, slightly readjusted.

(b) Characteristics of the two-stage parametric amplifier

(1) Gain versus pump phase difference

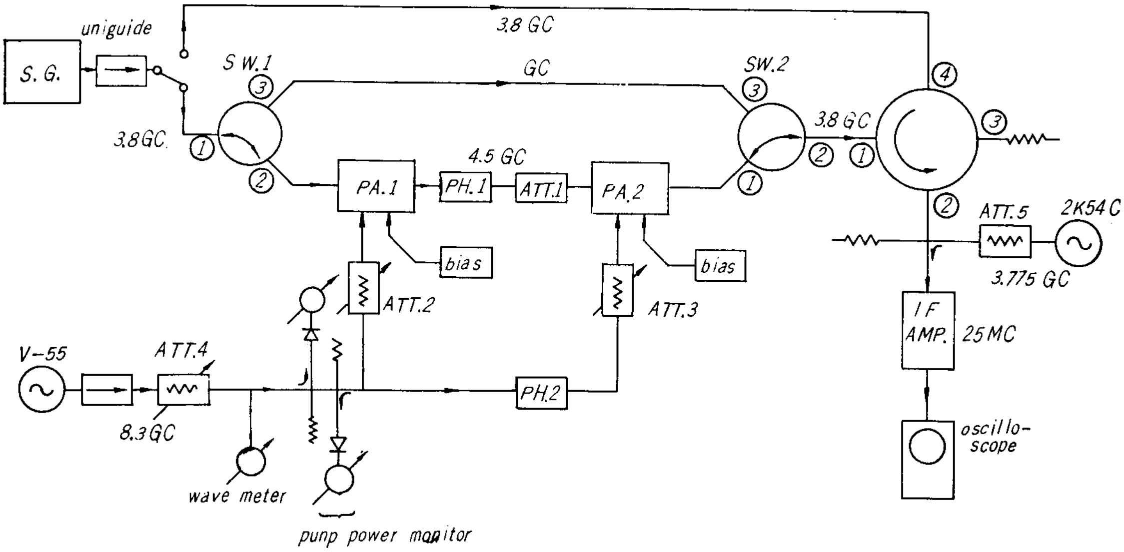

Fig. 10. Block diagram of experiment.

It was demonstrated within the limits of experimental error that the gain is independent of the phase difference of pumps. Consequently it is assured that there is no coupling wave except the idler.

(2) Phase shift in the coupling element.

Theoretically the length of the coupling line between the two idler tanks must be an integer times a quarter wave length. Though it is difficult to find the plane of the idler parallel resonance, it is comparatively easy to obtain the quarter wave coupling experimentally. Fig. 11 shows the relation of the reflection gain to the phase angle of the coupling line. The reflection gain is a periodic function of the phase shift of the coupling line, and it becomes minimum at the phase shift of 0.4 π (relative value) and the insertion gain is maximized at the same time. Here is realized the desired impedance-inversion coupling. If the pump power is properly increased, midband frequency.

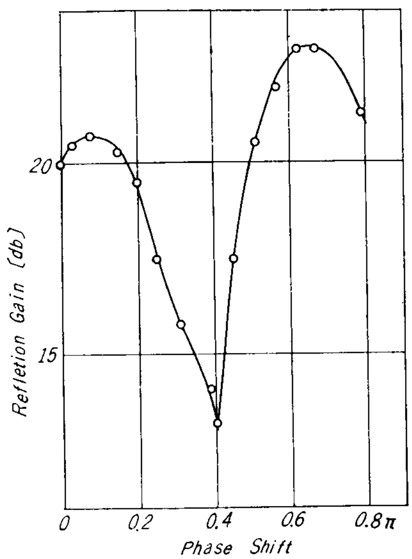

Fig. 11. Relation between the reflection gain at the input terminal and the phase shift of the coupling line.

At the coupling phase shift of 0.05π or 0.65π in Fig. 11, the reflection attains its maximum, the negative resistances of the two amplifiers being added to each other. This is just the direct coupling or a half wave coupling.

(3) Frequency characteristic of the gain.

A typical frequency response is shown in Fig. 12. The response is double-peaked when the pumps are adequately supplied. The more the pumps are increased, the higher the peaks become. Though a slight detuning of the tanks yields somewhat beter frequency response, over detuning causes rather narrow bandwidth.

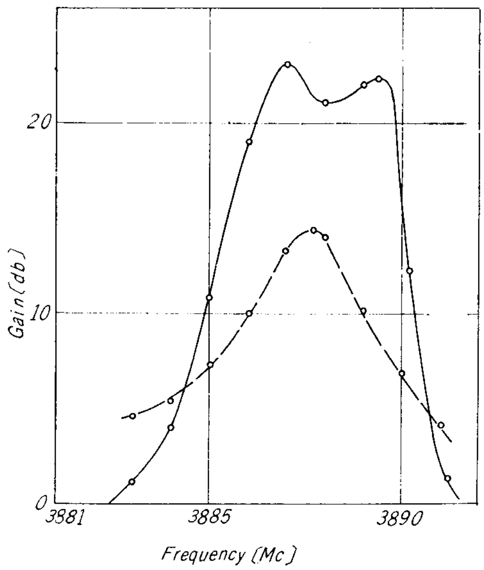

Fig. 12. Typical frequency characteristic.

For comparison, the response of a single parametric amplifier is also drawn in the same figure; one of the pair was operated as a direct amplifier using a circulator.

(4) Relation of the gain to the pumps.

To demonstrate the validity of the analysis, an example of the relationships between the gain and the pumps is shown in Fig. 13. In the figure (a), as the pump of the second is increased with the pump of the first held constant to a proper value, there exists a point where the reflection vanishes entirely at the midband frequency. If the pump is further increased beyond this point, the amplifier becomes unstable as seen in the figure. The insertion gain has the maximum value as in Fig. 13 (b), which was expected by analysis.

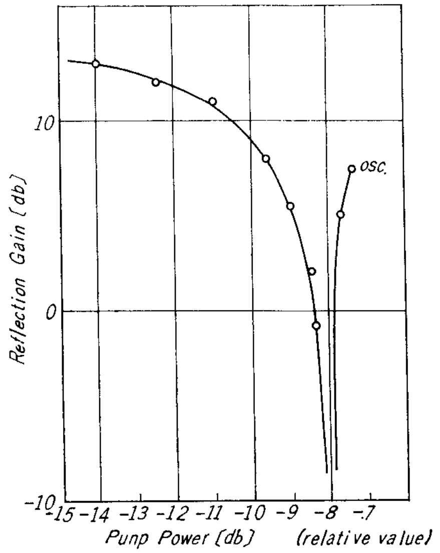

Fig. 13 (a) Reflection gain as a function of β^2 with α^2 constant.

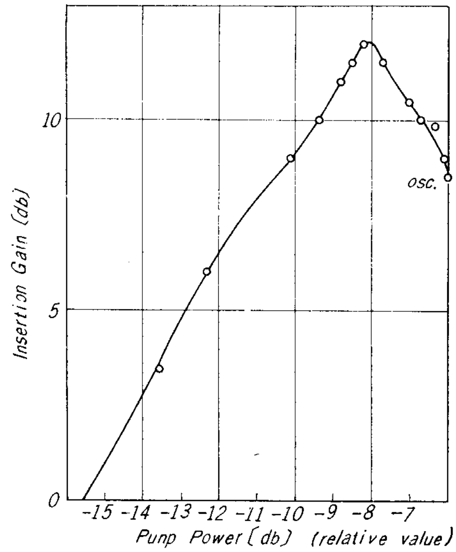

Fig. 13 (b) Insertion gain as a function of β^2 with α^2 constant.

(5) Noise Figure

The noise figures of each elemental amplifier are shown in Table 1. The reason of the bad noise figure of the first amplifier is as follows: We have made filters in order to separate the idler wave from the signal, which are designed to slide along the waveguide axis. Consequently there is considerably large loss at the contact plane with the waveguide wall. The second amplifier has a better noise figure because we designed fixed filters.

The noise figures when the two elemental amplifiers are coupled are shown in Table 2. As is commonly anticipated, the noise figure in the case where the less noisy amplifier is placed to the first stage is better than the opposite case. Though it is difficult to draw definite conclusions since there are in-

Table 1. Noise figure of the elemental amplifier.

	Gain (db)	N.F. (db)
P.A. 1	10	15.1
	15	14.4
	20	13.1
P.A. 2	10	6.0
	15	7.1
	20	7.0

Table 2. Noise figure of the two-stage parametric amplifier.

	Gain (db)	N.F. (db)
P.A. 1 → P.A. 2	15	12.6
	20	13.9
P.A. 2 → P.A. 1	10	10.7
	15	7.7
	17	9.8

evitably errors in the measurement of the noise figure, there arises no serious disadvantage with the two-stage paramatric amplifier.

6. Conclusion

It has been proved that a matched amplifier is possible by the proper connection of two parametric amplifiers. Aside from the utility of this amplifier, the phenomenon that the match is possible with negative resistance elements is interesting in itself. Formally this principle lies in that the twice negative inversion of a positive resistance is also positive. But, in order that this may be realized, the conditions for stable operation must be satisfied.

Recently a negative resistance amplifier in which two Esaki diodes are coupled by a quarter wave line was published[5]. Its characteristics have much resemblance to ours. The basic difference between them may be that the latter uses the negative impedance inversion by the time-varying capacitance rather than the negative resistance itself.

7. Acknowledgement

The authors wish to express their thanks to Mr. Kunio Komiyada of the Nippon Telegraph and Telephone Public Corporation and Mr. Toshio Takei of Kyoto University for their collaboration in this experiment.

Bibliography

1) H. Suhl; Phys. Rev. **106**, 384 (1957).
2) D. K. Adams; IRE. Trans. MTT-8, **3** 274-283 (1960).
3) K. K. N. Chang; RCA Rev. **20**, 205 (1959).
4) H. E. Rowe; Proc. IRE. **46**, 850-860 (1958).
5) J. Hamasaki; J. Inst. Elect. Communication Engrs. Japan. **44**, 1781 (1961).