Some Numerical Methods for Analysis of Transient Responses of Nonlinear Control System

Some Numerical Methods for Analysis of Transient Responses of Nonlinear Control System

By

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This paper deals with some numerical methods for the analysis of transient responses of a nonlinear control system and their computational errors, as shown in Fig. 1. Three types of mothods are proposed, which will be called the "zero-, the "first- and the "second-order data hold" types. They are all based on the same principle of replacing the output of the nonlinear element by another time function convenient to analyze. Excepting the second-order data hold type, each type consists of two alternative methods: one is useful when the indicial response of G(s) is obtained only experimentally; the other can be used when G(s) is obtained in an analytic form.

1. Introduction

The equivalent linearization method is a powerful one for the analysis of the stationary responses of nonlinear control systems. However, this method cannot be applied for the investigation of transient responses because of their non-stationarity. There is no effective method for nonlinear systems but the equivalent linearization method. Therefore, we wish to discuss here some numerical methods for the transient responses of nonlinear control systems.

2. Zero-order data hold type

Fig. 1 shows a typical single-loop nonlinear control system with a single nonlinear element, where the nonlinear element is of zero-memory type and $G(s)$

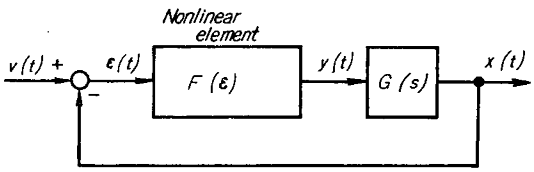

Fig. 1.

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is the linear transfer function. The relation between the input $\varepsilon(t)$ and the output $y(t)$ of the nonlinear element can be expressed as follows:

$$y(t) = F[\varepsilon(t)]. \tag{1}$$

Assume that this relation is known analytically or grapihcally.

(1) Direct method

When we wish to investigate a nonlinear system approximately, it may be natural to use a suitable piecewise-linear function as a substitute for the nonlinear characteristics of the nonlinear element as shown in Fig. 2. If this is done, this problem becomes the linear problem. But, since the solution for each linear

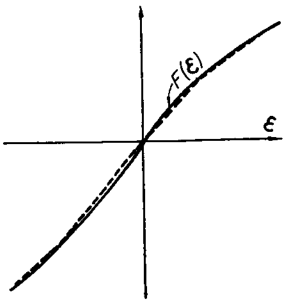

Fig. 2.

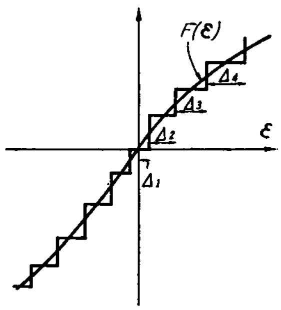

Fig. 3.

piece is different, it is necessary to connect these solutions to fit the differential equation of the system, and this procedure usually involves much labor. If we use the staircase function as shown in Fig. 3, instead of the piecewise-linear function, as an approximate function of the nonlinear characteristics of the element, it is easier to compute the transient response of the system. For the output of this element becomes the superposition of step functions of time, and the responses of linear elements for the superposition of step functions are easily found. It is desirable to make $\Delta_1, \Delta_2, \cdots$ equal from the point of view of computational labor, but it is desirable not to make them equal from the point of view of computational accuacy. In order to avoid this dilemma, it is better to approximate the output of the nonlinear elements by a staircase function of time with the same time interval T between any two adjacent jump points as shown in Fig. 4,

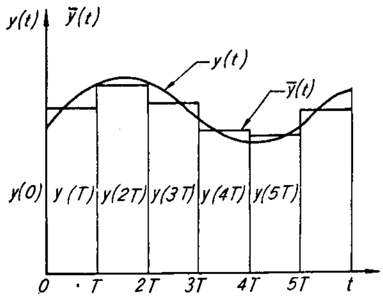

Fig. 4.

The expression of this function is given as follows:

$$\bar{y}(t) = \frac{1}{2}\Big[y(nT)+y(\overline{n+1}\,T)\Big], \quad \text{for} \quad nT<t\leq(n+1)T \quad (n=0,1,2,\cdots) \tag{2}$$

By using such the staircase approximation, the output of $G(s)$ becomes

$$x(t) = \frac{1}{2}\Big\{(y_0+y_1)f(t)+\sum_{k=1}^{n}(y_{k+1}-y_{k-1})f(t-kT)\Big\} \quad \text{for} \quad nT\leq t\leq(n+1)T \tag{3}$$

where

$$y_n = y(nT)$$

and $f(t)$ is the indicial response of $G(s)$ defined by

$$f(t) = L^{-1}[G(s)/s]. \tag{4}$$

From Fig. 1, it is seen that $\varepsilon(t)=v(t)-x(t)$, where $v(t)$ is the reference input. Then

$$\varepsilon_n = v_n - x_n \qquad [n=0,1,2\cdots] \tag{5}$$

where

$$\varepsilon_n = \varepsilon(nT)\,, \quad v_n = v(nT) \quad \text{and} \quad x_n = x(nT)\,.$$

Substituting Eq. (3) into Eq. (5), there results the following:

$$\varepsilon_0 = v_0 \qquad : n=0$$

$$\varepsilon_1+\frac{1}{2}f_1y_1 = v_1-\frac{1}{2}f_1y_0 \qquad : n=1$$

$$\varepsilon_n+\frac{1}{2}f_1y_n = v_n-\frac{1}{2}f_n(y_0+y_1)+\frac{1}{2}f_1y_{n-2}+\frac{1}{2}\sum_{k=2}^{n-1}f_k(y_{n-k-1}-y_{n-k+1}) \qquad : n\geq 2. \tag{6}$$

If the indicial response of $G(s)$ is known analytically or experimentally in the time domain, the right side of Eq. (6) can be calculated at the selected instant nT. Since the relation between $\varepsilon(t)$ and $y(t)$ is known, ε_n can be obtained successively. In this procedure, it is necessary to solve the following nonlinear equation:

$$\varepsilon_n+\frac{1}{2}f_1F(\varepsilon_n) = \text{known value} \tag{7}$$

which can be solved graphically. But, if $G(s)$ includes dead time which is longer than T, this is needless.

If $\varepsilon(t)$ is known and $f(t)$ unknown, f_n may be obtained by rearranging the terms of Eq. (6) as follows:

$$f_n = 2\Big[v_n-\varepsilon_n-\frac{1}{2}\sum_{k=1}^{n-1}f_k(y_{n-k+1}-y_{n-k-1})\Big]/(y_0+y_1) \qquad n\geq 1 \tag{8}$$

(2) Method by difference equations

On the other hand, the fact is that the use of the staircase approximation of $y(t)$ converts the system into a sampled model as illustrated in Fig. 5, where

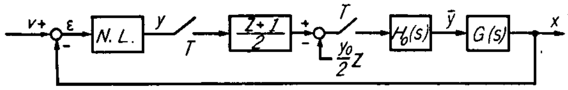

Fig. 5.

T is the sampling interval and $H_0(s)$ is the zero-order data hold expressed as follows:

$$H_0(s) = \frac{1-e^{-Ts}}{s}. \tag{9}$$

Because of the use of $H_0(s)$, the staircase approximation may be called the zero-order data hold type. The pulse transfer function $(z+1)/2$ in Fig. 5 is not physically realizable and this requires correction for the initial value of $y(t)$, $y_0 z/2$, which is inserted as input at the intermediate summing point, as shown, if y_0 is nonzero. Two samplers are operated synchronously. Thus the following relation may be derived from Fig. 5.

$$E(z) = V(z) - \frac{1}{2}(z+1)\,GH_0(z)\,Y(z) + \frac{1}{2}\,y_0 z\,GH_0(z) \tag{10}$$

where

$$E(z) = \sum_{n=0}^{\infty} \varepsilon_n z^{-n} \quad V(z) = \sum_{n=0}^{\infty} v_n z^{-n} \quad Y(z) = \sum_{n=0}^{\infty} y_n z^{-n} \tag{11}$$

and $GH_0(z)$ represents the Z-transform corresponding to the Laplace transform $G(s)\,H_0(s)$. By substituting Eq. (11) values for $E(z)$, $V(z)$ and $Y(z)$ in Eq. (10) and equating the coefficients of each order of z^{-1} in both sides of Eq. (10), there result difference equations relating the values of $\varepsilon(t)$, $v(t)$ and $y(t)$ at the selected instants of time so that ε_n can be solved numerically.

Both the direct method and the one using difference equations, which are essentially identical, may be carried out by means of desk calculators or by digital computers. It takes $n(n+3)/2$ multiplications to obtain ε_0, $\varepsilon_1 \cdots \varepsilon_n$ by the direct method, and about 3 mn by difference equations, supposing the derived difference equation is of the m th order. Accordingly, when n is a large number, it will be convenient to use the latter method in manual calculation. But, in view of programming a digital computer, the former has an advantage in that the same programming can be applied to the system whatever linear element it may include.

3. First-order data hold type

In order to improve the accuracy of the solution $\varepsilon(t)$, considerations will lead to a polygonal or a first-order data hold type approximation of $y(t)$. This approximation is shown in Fig. 6, where it is seen that the reconstructed function $\bar{y}(t)$ is obtained by connecting the sample values $y(nT)$ by straight lines. The expression for $\bar{y}(t)$ is

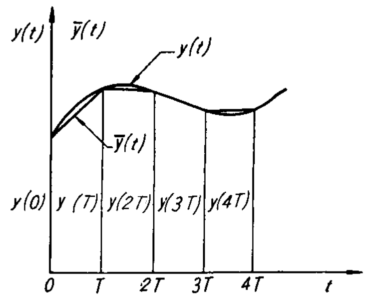

Fig. 6.

$$\bar{y}(t) = y_n + \frac{1}{T}(y_{n+1} - y_n)(t - nT) \qquad \text{for} \quad nT \leq t \leq (n+1)T. \tag{12}$$

(1) Direct method

The relation corresponding to Eq. (6) of the zero-order data hold type is

$$\varepsilon_0 = v_0, \quad \varepsilon_1 + g_1 y_1/T = v_1 - f_1 y_0 + \frac{1}{T} y_0 g_1$$

$$\varepsilon_n + g_1 y_n/T = v_n - f_n y_0 - \frac{1}{T}\Big[(y_1 - y_0) g_n + \sum_{k=1}^{n-2} (y_{k-1} - 2y_k + y_{k+1}) g_{n-k}$$
$$+ (y_{n-2} - 2y_{n-1}) g_1\Big] \qquad n \geq 2 \tag{13}$$

where

$$g(t) = L^{-1}[G(s)/s^2], \qquad g_n = g(nT) \tag{14}$$

and other notations are the same as in Eq. (6). This equation can be written as

$$\varepsilon_n + \bar{g}_1 y_n/T = v_n - f_n y_0 - \frac{1}{T} \sum_{k=1}^{n-1} (y_k - y_{k-1}) \bar{g}_{n-k+1} + \frac{1}{T} y_{n-1} \bar{g}_1, \qquad n \geq 2 \tag{15}$$

where

$$\bar{g}_n = g_n - g_{n-1}, \quad \bar{g}_1 = g_1 \tag{16}$$

However, $g(t)$ is difficult to obtain, while $f(t)$ is obtainable by experiment. In such a case it is desirable to get $g(t)$ from $f(t)$. One approach for this purpose is to assume

$$\bar{g}_n = \frac{T}{2}(f_n + f_{n-1}) \tag{17}$$

Substituting Eq. (17) for $\bar{g}_n$ in Eq. (15) and rearranging the terms, Eq. (15) becomes Eq. (6), which means that the use of trapezoidal rule reduces the first-order data hold type approximation to the zero-order data hold type one. Another approach is by Simpson's rule, as

$$\bar{g}_n = \frac{T}{6}(f_n + 4f_{n-1/2} + f_{n-1}) \tag{18}$$

which can be calculated easily. Once $\bar{g}_n$ is obtained, Eq. (15) requires the same calculation for ε_n with Eq. (6).

(2) Method using difference equations

As in the case of the staircase approximation, the use of the first-order data hold type approximation converts the continuous nonlinear control system shown

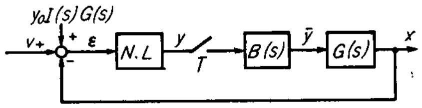

Fig. 7.

in Fig. 1 to a sampled model as illustrated in Fig. 7, where $B(s)$ is a physically unrealizable data hold whose transfer function is

$$B(s) = \frac{e^{Ts}}{Ts^2}(1-e^{-Ts})^2 \tag{19}$$

and $y_0I(s)G(s)$ which is placed as input at the summing point indicates the correction of the Z-transform for the initial condition, as was seen in Fig. 5. $I(s)$ is given by

$$I(s) = \frac{1}{Ts^2}(e^{Ts}-1)-\frac{1}{s} \tag{20}$$

According to the sampled-data theory, the corresponding relation to Eq. (10) can be written as

$$E(z) = V(z)-BG(z)\,Y(z)+y_0IG(z) \tag{21}$$

Taking the same procedure as was applied to Eq. (10), ε_n can be obtained by difference equations.

Each of the alternative methods has advantages and disavantages corresponding to those for the zero-order data hold type.

4. Second-order data hold type

Another attempt for a still more accurate approximation of $y(t)$ will be made using a second-order data hold type approximation. The reconstructed function $\bar{y}(t)$ in this approximation in the interval $nT<t\leq(n+1)T$ is the parabolic curve whose expression is given as follows:

$$\bar{y}(t) = y_0+\dot{y}_0t+\frac{1}{T^2}\left\{(y_1-y_0)-T\dot{y}_0\right\}t^2 \qquad 0<t\leq T \tag{22}$$

$$\bar{y}(t) = y_n+\frac{y_{n+1}-y_{n-1}}{2T}(t-nT)+\frac{y_{n-1}+y_{n+1}-2y_n}{2T^2}(t-nT)^2$$

$$\text{for}\quad nT\leq t\leq(n+1)T \quad (n\geq 1) \tag{23}$$

where

$$\dot{y}_0 = \left.\frac{dy}{dt}\right|_{t=+0}$$

The corresponding relation to Eq. (6) or (13) can not be obtained in a simple form. This approximation means that the system shown in Fig. 1 is converted to a sampled model shown in Fig. 8, where

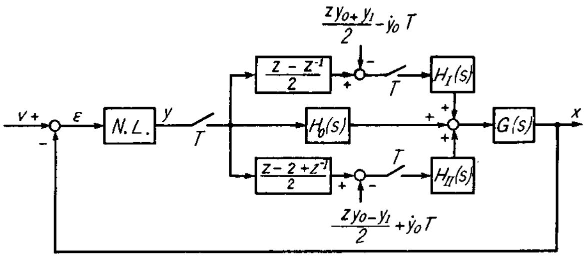

Fig. 8.

$$H_{\rm I}(s) = \frac{1-e^{-Ts}}{Ts^2} - \frac{e^{-Ts}}{s}$$
$$H_{\rm II}(s) = \frac{1}{T^2 s^3}\left\{2-e^{-Ts}-(Ts+1)^2 e^{-Ts}\right\} \tag{24}$$

and all the samplers are synchronous. The impulsive responses of data holds $H_I(s)$ and $H_{II}(s)$ are shown in Fig. 9. The pulse transfer functions $(z-z^{-1})/2$ and $(z-2+z^{-1})/2$ in Fig. 8 are the devices for construction of the coefficients of $(t-nT)$ and $(t-nT)^2$ in equation (23), respectively. As these devices are not

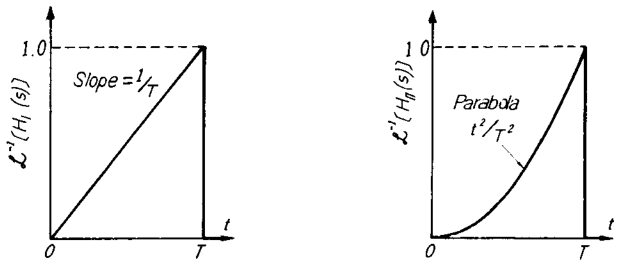

Fig. 9.

physically realizable, the correction for the initial value of $y(t)$, $y_0 z/2$, is inserted as input at the two intermediate summing points. The other correctional inputs at these summing points, $\pm y_0/2$ and $\pm \dot{y}_0 T$, are inserted in consideration of the difference of the coefficients in $\dot{y}(t)$ between Eq. (22) and Eq. (23).

Thus, referring to Fig. 8, $E(z)$ is found from the following expression:

$$E(z) = V(z) - Y(z)\left\{H_0 G(z) + \frac{z-z^{-1}}{2} H_{\rm I} G(z) + \frac{z-2+z^{-1}}{2} H_{\rm II} G(z)\right\}$$
$$+\left(\frac{zy_0+y_1}{2} - \dot{y}_0 T\right) H_{\rm I} G(z) + \left(\frac{zy_0-y_1}{2} + \dot{y}_0 T\right) H_{\rm II} G(z) \tag{25}$$

In order to obtain a numerical solution from the above equation, it is necessary to calculate $\dot{y}_0$ and to substitute it to the equation. Fortunately, $\dot{y}_0$ is easily obtainable in most cases. For example, if

$$G(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

and $m \leq n-2$, we obtain the following relation

$$\dot{y}_0 = \frac{dF(\varepsilon_0)}{d\varepsilon}\frac{dv(+0)}{dt}$$

5. Discussions of Computational Errors and Some Examples

One of the usual problems in the application of numerical methods lies in an estimation of the computational errors which are incurred. However, it is so difficult to give a complete discussion of this problem that we wish to forego any such complete discussion and to give some particular discussions instead.

Numerical methods proposed above involve computational error in the case of linear systems as well as in the case of nonlinear systems. The computational error in the case of a nonlinear element of saturation type is rather smaller than in the case of a linear element whose gain is equal to the maximum gain of a corresponding nonlinear element.

Considering the principle of the numerical method, the accuracy of the solution $\varepsilon(t)$ depends solely on the behavior of $y(t)$ and not on the characteristics of the nonlinear element, at least explicitly. Hence, for an estimate, a linear system is taken up here for convenience.

(1) Case of exponentially decreasing $y(t)$

If

$$G(s) = \frac{1}{\tau s}, \quad v(t) = \mathbf{1}(t) \quad \text{and} \quad F(\varepsilon) = K\varepsilon \tag{26}$$

$\varepsilon(t)$ in Fig. 1 is given as follows:

$$\varepsilon(t) = e^{-\alpha t}, \tag{27}$$

where $\alpha = \dfrac{K}{\tau}$.

Corresponding reference errors of zero- and first-order data hold type approximations, which may be denoted as $\varepsilon^{(0)}(nT)$ and $\varepsilon^{(1)}(nT)$, are given as follows:

$$\begin{aligned} \varepsilon^{(0)}(nT) &= e^{-\alpha_0 nT}, \\ \varepsilon^{(1)}(nT) &= e^{-\alpha_1 nT}, \end{aligned} \tag{28}$$

where

$$\alpha_0 = \alpha_1 = \frac{1}{T}\log\left(\frac{2\tau - KT}{2\tau + KT}\right). \tag{29}$$

As was seen above, both $\varepsilon^{(0)}(nT)$ and $\varepsilon^{(1)}(nT)$ have the same form with the exact reference error $\varepsilon(t)$. Since the value of integration of Eq. (2) is the same as that of Eq. (12), and $\varepsilon^{(0)}(nT)$ is equal to $\varepsilon^{(1)}(nT)$ for any $v(t)$ as far as $G(s) = \frac{1}{\tau s}$.

The corresponding second-order data hold type approximation of $\varepsilon(t)$, $\varepsilon^{(2)}(nT)$, is obtained as follows:

$$\varepsilon^{(2)}(nT) = Ae^{-\alpha_2 nT} + Be^{-\bar{\alpha}_2 nT} \tag{30}$$

where

$$\alpha_2 = \frac{1}{T}\log\frac{24+10\xi}{12-8\xi+2\sqrt{36-36\xi+21\xi^2}}$$
$$\bar{\alpha}_2 = \frac{1}{T}\log\frac{24+10\xi}{12-8\xi-2\sqrt{36-36\xi+21\xi^2}}, \qquad \xi = \frac{K}{\tau}T \tag{31}$$

and A and B are constants to be determined by

$$\varepsilon_0^{(2)} = 1, \qquad \varepsilon_1^{(2)} = \frac{1}{12+4\xi}\left\{(12-8\xi)\,\varepsilon_0 - 2\alpha^2\dot{\varepsilon}_0\right\}. \tag{32}$$

It is seen that $\varepsilon^{(2)}(nT)$ tends to zero exponentially with $n\to\infty$. α_0/α, α_1/α and α_2/α may be chosen as estimates or measures, which are functions with an argument of $\frac{KT}{\tau}$ shown in Fig. 10,

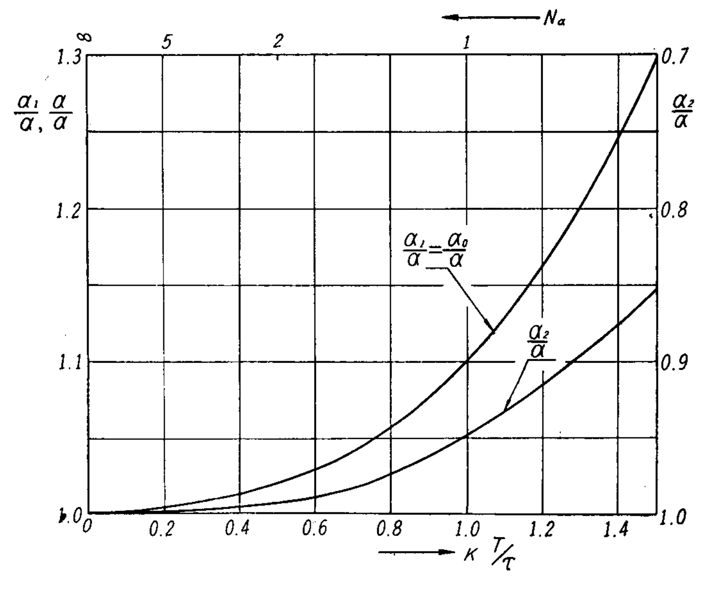

Fig. 10.

Table 1

n	$\varepsilon(nT)$	$\varepsilon^{(2)}(nT)$	$\varepsilon^{(1),(0)}(nT)$
0	1	1	1
1	0.6065	0.6071	0.6000
2	0.3679	0.3694	0.3600
3	0.2231	0.2247	0.2160
4	0.1353	0.1367	0.1296
5	0.08209	0.08320	0.07776
6	0.04970	0.05061	0.04666
7	0.03020	0.03080	0.02799
8	0.01832	0.01874	0.01680
9	0.01111	0.01140	0.01009
10	0.006738	0.006935	0.006047

$KT/\tau=0.5$

where

$$N_\alpha = \frac{\tau}{KT}. \tag{33}$$

Table 1 shows the exact and approximate numerical values of $\varepsilon(t)$ of this system for a first certain time interval. The second-order data hold approximation is the best.

(2) Case of cycling $y(t)$

If

$$G(s) = \frac{1}{\tau^2 s^2},\quad v(t) = 1(t) \quad \text{and} \quad F(\varepsilon) = K\varepsilon \tag{34}$$

there results

$$\begin{aligned} \varepsilon(t) &= \cos\beta t, \\ \varepsilon^{(0)}(nT) &= \cos\beta_0 nT, \\ \varepsilon^{(1)}(nT) &= \cos\beta_1 nT, \end{aligned} \tag{35}$$

where

$$\beta = \frac{\sqrt{K}}{\tau},\quad \beta_0 = \frac{\cos^{-1}\left(\dfrac{4\tau^2-KT^2}{4\tau^2+KT^2}\right)}{T},\quad \beta_1 = \frac{\cos^{-1}\left(\dfrac{6\tau^2-2KT^2}{6\tau^2+KT^2}\right)}{T}. \tag{36}$$

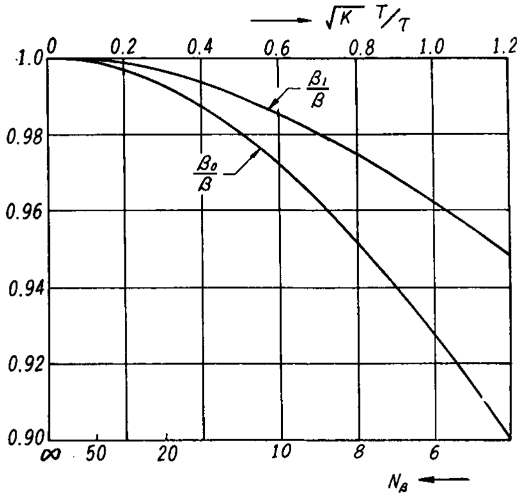

Fig. 11.

Also in this case, $\varepsilon^{(0)}(nT)$ and $\varepsilon^{(1)}(nT)$ remain the forms of exact solution. Hence, β_0/β and β_1/β may be adopted as estimates or measures. Fig. 11 shows them as functions of $\sqrt{K}\frac{T}{\tau}$, where N_β indicates how many times samplers are operated in one cycle of $y(t)$, i.e.

$$N_\beta = \frac{2\pi}{\beta T} = \frac{2\pi\tau}{\sqrt{K}T}. \tag{37}$$

$\varepsilon^{(2)}(nT)$ is obtained from the following recurrence formula:

$$\left.\begin{aligned} \varepsilon_0^{(2)} &= 1 \\ \varepsilon_1^{(2)} &= \{(48-10\eta^2)\varepsilon_0^{(2)}-24\}/(24-2\eta^2) \\ \varepsilon_2^{(2)} &= \{(48-16\eta^2)\varepsilon_1^{(2)}-(24+5\eta^2)\varepsilon_0^{(2)}\}/(24+3\eta^2) \\ \varepsilon_n^{(2)} &= \{(48-17\eta^2)\varepsilon_{n-1}^{(2)}-(24+5\eta^2)\varepsilon_{n-2}^{(2)}+\eta^2\varepsilon_{n-3}^{(2)}\}/(24+3\eta^2) \qquad n\geq 3 \end{aligned}\right\} \tag{38}$$

Table 2

n	$\varepsilon(nT)$	$\varepsilon^{(2)}(nT)$	$\varepsilon^{(1)}(nT)$	$\varepsilon^{(0)}(nT)$
0	1	1	1	1
1	0.8776	0.8776	0.8800	0.8824
2	0.5403	0.5399	0.5493	0.5571
3	0.0707	0.0692	0.0867	0.1007
4	−0.4162	−0.4197	−0.3966	−0.3792
5	−0.8011	−0.8069	−0.7848	−0.7698
6	−0.9900	−0.9976	−0.9850	−0.9797
7	−0.9365	−0.9444	−0.9489	−0.9588
8	−0.6536	−0.6598	−0.6854	−0.7125
9	−0.2108	−0.2129	−0.2577	−0.2982
10	0.2837	0.2872	0.2320	0.1861
11	0.7057	0.7183	0.6659	0.6265
12	0.9601	0.9745	0.9409	0.9196
13	0.9766	0.9927	0.9892	0.9963
14	0.7539	0.7678	0.8009	0.8385
15	0.3564	0.3547	0.4206	0.4836

$\sqrt{K}T/\tau=0.5$

where,

$$\eta = \sqrt{K}\frac{T}{\tau}.$$

The behavior of $\varepsilon^{(2)}(nT)$ for large n depends on the z-plane distribution of the roots of the next algebraic equation.

$$(24+3\eta^2)z^3-(48-17\eta^2)z^2 + (24+5\eta^2)z-\eta^2 = 0. \qquad (39)$$

It is easily seen that there is no root of Eq. (39) on the unit circle on z-plane. Hence, $\varepsilon^{(2)}(nT)$ converges to zero or diverges with $n\to\infty$. That is, the property of exact reference error is destroyed in the second-order data hold type approximation. It seems that this arises from the fact the system to be dealt with is "structurally unstable".

Table 2 shows numerical values of $\varepsilon(t)$, $\varepsilon^{(0)}(nT)$, $\varepsilon^{(1)}(nT)$ and $\varepsilon^{(2)}(nT)$ for a first certain time interval.

(3) Case of oscillatory-damped $y(t)$

If

$$G(s) = \frac{1}{\tau_1 s(1+\tau_2 s)}, \quad v(t) = \mathbf{1}(t) \quad \text{and} \quad F(\varepsilon) = K\varepsilon \qquad (40)$$

there results

$$\begin{aligned} \varepsilon(t) &= e^{-at}(\cos bt + c\sin bt), \\ \varepsilon^{(0)}(nT) &= e^{-a_0 nT}(\cos b_0 nT + c_0 \sin b_0 nT), \\ \varepsilon^{(1)}(nT) &= e^{-a_1 nT}(\cos b_1 nT + c_1 \sin b_1 nT), \end{aligned} \qquad (41)$$

where

$$a = 1/2\tau_2, \quad b = \frac{1}{2\tau_2}\sqrt{4K\frac{\tau_2}{\tau_1}-1}$$

$$a_0 = (1/2T)\log\left[\{2+\xi-\eta(1-e^{-\xi/\eta})\}/\{(2-\xi)e^{-\xi/\eta}+\eta(1-e^{-\xi/\eta})\}\right]$$

$$b_0 = \frac{1}{T}\cos^{-1}\frac{e^{-\xi/\eta}(2+\xi)+2-\xi}{2\sqrt{\{2+\xi-\eta(1-e^{-\xi/\eta})\}\{e^{-\xi/\eta}(2-\xi)+\eta(1-e^{-\xi/\eta})\}}}$$

$$a_1 = (1/2T)\log\left[\{2\xi+\xi^2-2\xi\eta+2\eta^2(1-e^{-\xi/\eta})\}/\{e^{-\xi/\eta}(2\xi-\xi^2-2\xi\eta)+2\eta^2(1-e^{-\xi/\eta})\}\right]$$

$$b_1 = \frac{1}{T}\cos^{-1}\frac{2\xi-\xi^2-2\xi\eta+2\eta^2(1-e^{-\xi/\eta})+e^{-\xi/\eta}(2\xi+\xi^2-2\xi\eta)+2\eta^2(1-e^{-\xi/\eta})}{2\sqrt{\{2\xi+\xi^2-2\xi\eta+2\eta^2(1-e^{-\xi/\eta})\}\{e^{-\xi/\eta}(2\xi-\xi^2-2\xi\eta)+2\eta^2(1-e^{-\xi/\eta})\}}}$$

$$\xi = KT/\tau_1, \quad \eta = K\tau_2/\tau_1$$

Three pairs of estimates may be considered, i.e. a_0/a, a_1/a; b_0/b, b_1/b; c_0/c, c_1/c which are functions of ξ with a parameter of η. They are illustrated in Fig. 12 and 13, where N_a corresponds to N_α and N_b to N_β, respectively.

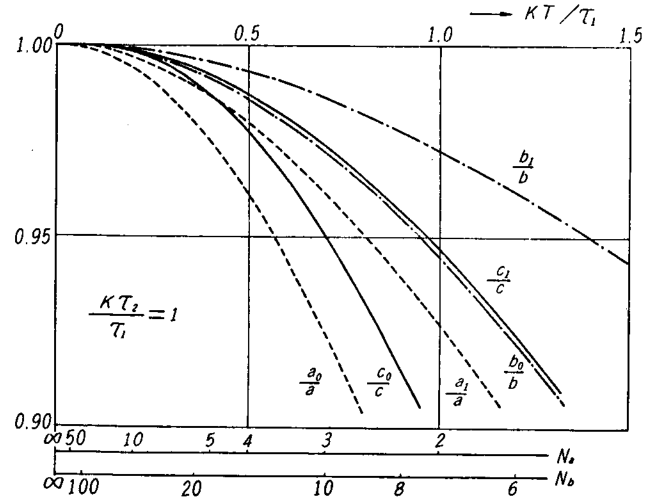

Fig. 12.

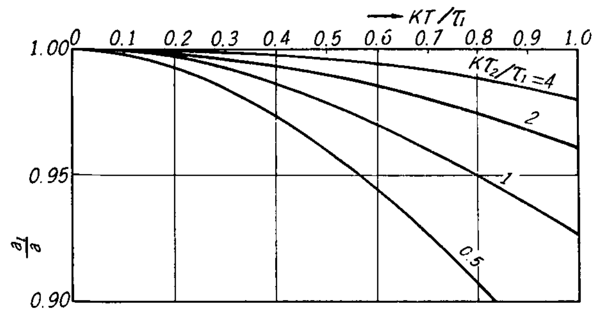

Fig. 13. (a)

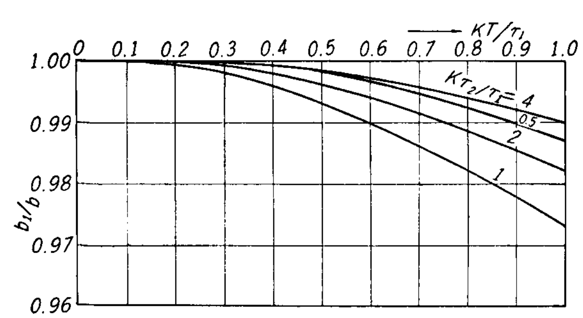

Fig. 13. (b)

(4) Case where off-set exists

If

$$G(s)=\frac{1}{1+\tau s},\quad v(t)=\mathbf{1}(t)\quad \text{and}\quad F(\varepsilon)=K\varepsilon \tag{42}$$

there results

$$\begin{aligned}
\varepsilon(t)&=\frac{1}{1+K}+\frac{K}{1+K}e^{-\frac{1+K}{\tau}t},\\
\varepsilon^{(0)}(nT)&=\frac{1}{1+K}+\frac{K}{1+K}e^{-q_0 nT},\\
\varepsilon^{(1)}(nT)&=\frac{1}{1+K}+\frac{K}{1+K}e^{-q_1 nT},\\
\varepsilon^{(2)}(nT)&=\frac{1}{1+K}+Ae^{-q_2 nT}+Be^{-\bar{q}_2 nT},
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
q_0&=(1/T)\log\left[\{2+K(1-p)\}/\{2p-K(1-p)\}\right]\\
q_1&=(1/T)\log\left[\left\{1+K\left(1-\frac{\tau}{T}+p\frac{\tau}{T}\right)\right\}\Big/\left\{p+K\left(p-\frac{\tau}{T}+p\frac{\tau}{T}\right)\right\}\right]\\
q_2&=(1/T)\log\{H/(M+N)\},\qquad p=e^{-T/\tau}\\
\bar{q}_2&=(1/T)\log\{H/(M-N)\},\\
M&=p(1+K)-2K\frac{\tau}{T}+2K(1-p)\left(\frac{\tau}{T}\right)^2\\
N&=\left[p^2(1+K)^2+2K(1+K)(1-p)\frac{\tau}{T}-K(4+3K)(1-p)^2\left(\frac{\tau}{T}\right)^2\right]^{1/2}\\
H&=2(1+K)-K(3-p)\frac{\tau}{T}+2K(1-p)\left(\frac{\tau}{T}\right)^2
\end{aligned}$$

and A and B are constants to be decided from the initial condition. The value of off-set in each approximation is equal to the exact one.

6. Conclusion

The numerical methods discussed above lend themselves only to the case of $\lim_{s\to\infty} G(s)=0$. When $\lim_{s\to\infty} G(s)\neq 0$, there arise two unknowns at the first step to determine the reference error, and so it cannot be decided.

We may compose a hold type approximation method for use with data of a higher order than the second. This can be performed if we can construct the higher order data hold whose impulsive response is $\{\mathbf{1}(t)-\mathbf{1}(t-T)\}t^n$. And we can do this easily. However, since the accuracy required is always obtained from the lower order data hold type approximation by taking the sampling period T as small as occasion demands, the higher order method is usually unnecessary unless the number of selected points required is too large for the lower order approximation.

Choice of the kind of approximation and the sampling period T must be made considering not only the accuracy of sampled values computed and the computational labor, but also the accuracy of reconstruction of the continuous curve from the discrete time sequence of sampled valves.

Reference

J. R. Ragazzini and G. F. Franklin; Sampled-data control systems, McGraw-hill Co., Inc., New York, 1958.