General Theory on Steady Behaviours of Open Channel Flows by Means of One Dimensional Procedures of Analysis

By

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This paper concerns with the general theory of steady behaviours of open channel flows by means of the one dimensional procedures of analysis. The theory described herein is essentially formed by the transitional characteristics of steady flows in channel transitions and controls, and it can provide the unified treatment for hydraulic behaviours of gradually varied flows, which mainly treat with the determination of surface profiles of water, and of rapidly varied flows, which are counterpart of gradually varied flows and deal with local variations of flow behaviours occured by sudden changes in channel characteristics.

1. Introduction

When a definite rate of water is discharged in a channel of uniform channel characteristics, the flow will gradually approach steady and uniform in itself. Such an idealized flow is rarely occured in natural channels and even in artificial water-courses because of continuous changes in channel geometry, bottom grade, and channel roughness. In reality, the velocity and the water elevation vary from point to point, and thus the flow is classified as steady and non-uniform. The steady-state hydraulics involving uniform and non-uniform behaviours of open channel flows is evidently a specified part of unsteady behaviours in hydraulic problems. Nevertheless, the hydraulic designs of conveyance and other structures are substantially made for a definite value of design discharge determined by hydrologic and hydrographical studies, and therefore, the steady-state hydraulics becomes rather compatible with the unsteady-state hydraulics and is one of the most important branches in hydraulics of open channel flows. The development of hydraulic researches on steady behaviours of open channel flows initiated in the former half of the 19th century was mainly succeeded by a large number of scientists and engineers in the latter half of that century.

Although some of recent trends in hydraulic researches of open channel flows involve the use of the theory of boundary layer for clarification of specified problems like the entrainment of air into the flow resulting from the free surface disturbances[1],

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the establishment of hydraulic performance of broad crested weir[2)] and so on, the successful results by this approach are not obtained. The classical method of analysis in steady-state hydraulics is practically divided into the following two ways of treatment: The former is known as the hydraulics of gradually varied flows characterized by very large radii of curvature and the ignorance of higher terms depending on the surface slope. The theory of gradually varied flows is essentially subject to the determination of surface profiles of water in channels, and consequently it provides the mathematical knowledge of varied flow functions in an implicite form. The theoretical studies of gradually varied flows in uniform channels of constant grade were largely developed in the last century. In reality, however, the channel geometry, bottom grade and channel roughness are not constant throughout the whole reach of stream, so that the range of application of classical studies for the flow behaviours in uniform channels to those in actual channels with non-uniform geometry, grade and roughness is extremely limited. As a matter of fact, many hydraulic engineers have practical experience that possible solutions of surface profiles of water can not be evaluated by these classical studies. The latter is classified as the hydraulics of rapidly varied flows, in which local variations of flow behaviours occured by sudden changes in channel characteristics are treated. In the rapidly varied flows local variations in flow behaviours are so rapid that the basic dynamic principles of fluid motion are not subject to a complete mathematical form, and therefore the empirical knowledge is often involved in the analysis of rapidly varied flows. When a control section for a definite rate of discharge, however, is produced in channel controls, the simultaneous occurrence of maximum discharge and minimum energy known as the theorem of Bélanger-Böss or the generalized theory of Jaeger is evidently obtained at the control section, and all of hydraulic characteristics are uniquely determined for given discharges, with a result of the establishment of basic concept for discharge metering of flows in flumes and weirs by a single water-level measurement.

In the classical treatment of hydraulics by means of the one dimensional method, the hydraulics of gradually varied flows and that of rapidly varied flows are separately developed without establishing the interconnection between two parts of flow behaviours. If the fluid motion of open channel flows involving the curvilinear motion, the separation of flow and others as sequences of secondary influences in the basic flow pattern can be represented in a complete mathematical form by means of the one dimensional procedures of analysis, the surface profiles of steady flows must be obtained by solutions of basic non-linear equation of flows expressed by both curves of normal and critical depths. Consequently, these two curves determined by the channel and flow characteristics are the most important parameters in mathematical hydraulics of steady flows, and furthermore from the point view of engineering description, the hydraulic significance of these curves will be feasibly understood, as will be seen in the later chapter, because the basic requirement for hydraulic designs of channel and other structures are to obtain the highest efficiency in hydraulic performance.

All analytical studies on steady behaviours of open channel flows by means of the one dimensional procedures of analysis since the works of Dupuit and Bresse in the last century are those in uniform channels characterized by the constancy in cross section, bottom grade and channel roughness. Recent studies of Chow[3] and others extremely improved the classical treatment and made fruitful tabulations and charts available for varied flow functions in channels and conduits of all shapes in channel geometry, with a tremendous amounts of labours. In uniform channels, the curves of normal and critical depths are determined by the flow and channel characteristics and are independent of the distance. Furthermore, these curves can never intersect together in the whole reach of stream. In reality, however, natural channels and watercourses involve frequent changes in channel characteristics and, hydraulically speaking, the channel is classified as non-uniform. The two curves of normal flow and critical depth will intersect in non-unifom channels, with results of occurrence of singular points being in a form of 0/0. Surface profiles of water can not be then predicted by the analytical methods developed by many engineers. Usual procedures to make calculation of surface profiles are numerical and graphical methods, which have been also devised for various purposes. All of hydraulic characteristics of surface profiles of water are determined by the mathematical properties of singular points as intersections of two curves, and the results are known as the transitional characteristics of steady flows. If the transitional characteristics are not evident before calculation, the numerical and graphical procedures of computation frequently imply much errors in solutions. The tabulation and the graphic formulations for varied flow functions in channels of simplified geometry were mainly accented in past studies, so that the hydraulic significance of surface profiles of water have scarcely been treated. From this point of view, the author[4),5)] treated with the transitional characteristics of steady flows in non-uniform channels of constant grade with the aid of the geometric theory of ordinary differential equation and the most significant description obtained is that the flow changes its regime from tranquil to shooting through a saddle point and from shooting to tranquil by a nodal and focal points.

The general theory on steady behaviours of open chanel flows by means of the one dimensional procedures of analysis can be provided by the use of transitional characteristics of open channel flows. In this paper, the hydraulic behaviours of steady flows will be concerned as a conclusive statement of one dimensional method of analysis. First will be treated the hydraulic characteristics of tranquil and shooting flows which are most significant parameters in hydraulics of steady flows, following by the establishment of transitional behaviours of open channel flows. The simultaneity of maximum discharge and minimum energy known as the Bélanger-Böss theorem or the generalized theory of Jaeger is also explained and then the hydraulics of control section is clarified.

As a result of analysis described in the foregoing, the unified treatment for gradually and rapidly varied flows as an extensive theory of classical treatment for uniform channels will be established. For gradually varied flows, the method to predict the surface profiles of water in open channels will be definitely indicated, and for rapidly varied flows, the method to evaluate the hydraulic performance of control structures will be evidently established.

2. Hydraulic Characteristics of Tranquil and Shooting Flows

As the water elevation of open channel flows is one of unknowns, so a gross approximation accomplished by presuming the zone of flow to consist a single stream tube characterized at a section by a mean velocity of flow and a mean elevation prevails. The result is the one dimensional procedure of analysis, which formed the basis of classical hydraulics. When the coordinate system is so selected that the x-axis is in the downstream direction along the channel bottom and the y-axis vertically upward, as seen in Fig. 1, the one dimensional equation of motion by means of the energy conservation law is approximately represented by

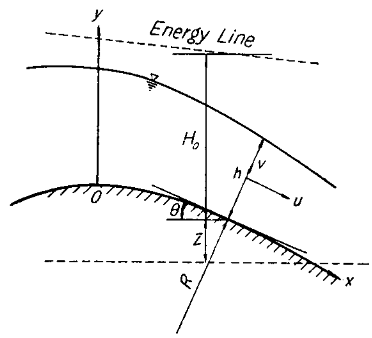

Fig. 1 Coordinate system.

$$H_0 = (1/Q)\cdot\int\{(u^2/2g)+(p/\rho g)+y\cos\theta\}udA\,, \qquad (1)$$

and

$$(dH_0/dx) = \sin\theta-(\tau/\rho gR)(u_b/u_m)\,, \qquad (2)$$

in which, H_0 : total head, Q : discharge, u : local velocity in the x-direction, u_m : mean velocity, u_b : velocity near bed, R : hydraulic radius, p : pressure, τ : shear, A : flow area, ρ : density of water, g : acceleration of gravity, and θ : inclination angle of bottom.

Introducing the energy correction coefficient of Coriolis, α, and the pressure correction coefficient of Jaeger, λ, the total head over a bottom becomes

$$H_0 = (\alpha Q^2/2gA^2)+\lambda h\cos\theta\,, \qquad (3)$$

where,

$$\alpha = (1/Q)\int(u^2/u_m^2)udA\,,\ \text{and}\ \lambda = (1/Qh\cos\theta)\cdot\int\{(p/\rho g)+y\cos\theta\}udA\,, \qquad (4)$$

The value of α is approximately unity for gradually varied flows, whereas it is not definite for rapidly varied flows. The pressure distribution of gradually varied flows is approximated by the hydrostatic pressure law, so that λ becomes unity, as seen in Eq. (5), while for rapidly varied flows in which the curvilinear motion mainly prevails, it is also not determined. The detail patterns of pressure and velocity distributions must be needed to evaluate values of α and λ for rapidly varied flows. When the flow approaches critical, the vertical acceleration of fluid motion also becomes appreciable, so that the value of λ is expressed in terms of (d^2h/dx^2) and $(dh/dx)^2$ as a secondary influence to the basic flow pattern, which will be practically ignored. The discussion of this influence is treated in the other paper of the author[6].

Differentiating H_0 expressed by Eq. (3) with respect to x, the surface profile equation is

$$\begin{aligned}(dh/dx) &= \{\sin\theta - (\tau/\rho gR)(u_b/u_m) - (\partial H_0/\partial x)\}/(\partial H_0/\partial h) \\ &= f_1(x, h)/f_2(x, h)\,, \end{aligned} \tag{5}$$

in which

$$\begin{aligned}f_1(x, h) &= \sin\theta - (\tau/\rho gR)(u_b/u_m) - (Q^2/2gA^2)(\partial\alpha/\partial x) + (\alpha Q^2/gA^3)\cdot \\ &\quad (\partial A/\partial x) - h\cos\theta(\partial\lambda/\partial x) + \lambda h \sin\theta(d\theta/dx)\,, \end{aligned} \tag{6}$$

and

$$\begin{aligned}f_2(x, h) &= \lambda\cos\theta - (\alpha Q^2/gA^3)(\partial A/\partial h) + (Q^2/2gA^2)(\partial\alpha/\partial h) + h\cos\theta\cdot \\ &\quad (\partial\lambda/\partial h)\,. \end{aligned} \tag{7}$$

(1) *Normal and critical depths of steady flows*

When the channel is completely uniform, the physical regime of steady flows will gradually become independent of x. The resistive and gravity forces are also in exact balance and the resulting surface profile is parallel to the channel bed. The condition of flow is then called normal or uniform, and the hydraulic characteristics are determined by the common empirical laws like Chézy, Manning and so on. The normal depth is evidently a function of the shape, the roughness, its slope and the given rate of discharge.

When the channel geometry is non-uniform, the flow is influenced by the non-uniformity of channels. Nevertheless, the normal condition which indicates the surface gradient is equal to the bottom gradient is obtained by $f_1(x, h)=0$, if $f_2(x, h)\neq 0$. Although Massé[7] called the curve expressed by $f_1(x, h)=0$ as the curve of quasi-normal flow and Escoffier[8] as the transition curve, the curve of $f_1(x, h)=0$ is defined as the curve of normal flow in this study.

If the denominator in Eq. (5) or $f_2(x, h)$ becomes zero, the surface gradient becomes quite large to infinity. The critical depth of steady flows in open channels is defined as a solution of $f_2(x, h)=0$ and sometimes it is called the critical condition of Bresse. Apparently, $f_2(x, h)=0$ indicates singular points, and therefore the mathematical significance of critical depth for hydraulic behaviours of surface profiles will be evident. The critical depth by means of the energy approach for steady flows is also defined by $(\partial H_0/\partial h)=0$ and $(\partial Q/\partial h)=0$. The former is known as the theorem of Böss for minimum energy (1919) whereas the latter is the theorem of Bélanger for maximum discharge presented in 1849. The other definition, which is subject to the unsteady flow, is obtained by the depth which acts as a barrier of transmittal of small ascending waves.

The mathematical verification for equivalence among the foregoing four definitions will be concerned. The general treatment for equivalence can not be made because of great difficulty to express completely the dynamic relationship of fluid motion in a mathematical form, and therefore the approximation will be indicated. Evidently seen in Eq. (5), the critical condition of Bresse is equal to that of Böss. The equivalence between the minimum energy theorem and the maximum discharge theorem is verified by the generalized theory of Jaeger, which will be later concerned. In this section, therefore, the equivalence between the critical depth for a barrier of transmittal to small disturbances and that of Böss will be explained.

The one dimensional energy equation for unsteady flows is approximately expressed by

$$(\beta/g)(\partial u_m/\partial t)+(\alpha u_m/g)(\partial u_m/\partial x)+\{\lambda\cos\theta+(u_m^2/2g)(\partial\alpha/\partial h)$$
$$+h\cos\theta(\partial\lambda/\partial h)\}(\partial h/\partial x)+(u_m/2g)(\partial\beta/\partial h)(\partial h/\partial t)+[\{(\beta-\alpha)/2g\}(u_m/A)$$
$$+h\cos\theta(1-\lambda)/u_mA](\partial A/\partial t)=\sin\theta-(\tau/\rho gR)(u_b/u_m)-\cdots \quad (8)$$

in which, β is the momentum correction coefficient of Coriolis.

Let consider a small disturbance travelling up- and downstream on the free surface of original flow. Denoting the variations of dependent variables like the velocity and so on by the prime, the depth, the velocity and others are

$$h=h_c+h',\quad u_m=u_{mc}+u',\ \cdots.$$

in which, the subscript c indicates the value of original flow, and $h_c\gg h'$, $u_{mc}\gg u'$, $\cdots$.

By the use of the equation of continuity, the disturbed wave is approximately indicated by the following linearized equation, eliminating the velocity from both equations.

$$R(\partial^2h'/\partial x^2)+S(\partial^2h'/\partial x\partial t)+T(\partial^2h'/\partial t^2)+\cdots\cdots=0, \quad (9)$$

where

$$R=(\lambda_c\cos\theta_c/A_c^3)[A_c^3\{1+(h_c/\lambda_c)(\partial\lambda/\partial h)_c\}-(Q_c^2/g\lambda_c\cos\theta_c)$$
$$\{\alpha_c(\partial A/\partial h)_c-(A_c/2)(\partial\alpha/\partial h)_c\}], \quad (10)$$

$$S=(1-\lambda_c)(h_c\cos\theta_c/Q_c)(\partial A/\partial h)_c-(\beta_c+3\alpha_c)(Q_c/2gA_c^2)(\partial A/\partial h)_c$$
$$+(Q_c/2gA_c)(\partial\beta/\partial h)_c, \quad (11)$$

$$T=-(\beta_c/gA_c)(\partial A/\partial h)_c. \quad (12)$$

If Eq. (9) is of hyperbolic type, the absolute velocity of a small disturbance, V_w, is derived by the characteristic equation of Eq. (9), and it is

$$V_w = \{S \pm (S^2 - 4RT)^{1/2}\}/2T. \tag{13}$$

The absolute velocity of a disturbance is the sum of the celerity of the wave and the undisturbed fluid velocity, so that a channel section where the flow velocity becomes equal to the celerity of ascending disturbance acts as a barrier to the transmittal, and this condition yields $R=0$, from Eq. (13). The resulting relationship between the flow characteristics and those of channels at a control barrier is

$$A_c^3\{1+(h_c/\lambda_c)(\partial\lambda/\partial h)_c\} = (Q_c^3/\lambda_c g \cos\theta_c)\{\alpha_c(\partial A/\partial h)_c - (A_c/2)(\partial\alpha/\partial h)_c\}. \tag{14}$$

This equation is equivalent to the critical condition of Bresse and Böss, as seen in Eq. (5), and the curve of critical depth for particular channels, therefore, indicates the locus of such barriers.

In the same manner, the critical condition for open channel flows by means of the momentum approach of one dimensional analysis is also obtained and it is

$$A_c\{(\partial/\partial h)(Ay_G)\}_c = (Q_c^2/\lambda_c' g \cos\theta_c)\{(\beta_c/A_c)(\partial A/\partial h)_c - (\partial\beta/\partial h)_c\} - (y_{Gc}A_c^2/\lambda_c')(\partial\lambda'/\partial h)_c, \tag{16}$$

for uniform channels, in which $\lambda' = (1/Ay_G \cos\theta)\int(p/\rho g)dA$, and y_G is the distance from the free surface to the centroid of flow area.

These two conditions for critical depth expressed by Eqs. (14) and (15) are not coincided, owing to the approximate expression in the basic relationship to the open channel flow, As Boussinesq and Jaeger did, the coincidence between two approaches will be obtained after the exact description of dynamic principles in a mathematical form is completed.

(2) *Hydraulic characteristics of tranquil and shooting flows*

Before starting the discussion of the basic characters of transitional behaviours of flows, two physical regimes of tranquil and shooting in the open channel flow will be briefly explained.

If the flow depth at a point is greater than the critical depth described in Eqs. (14) and (15), the flow is classified as tranquil, and on the contrary, the shooting flow indicates the depth of water is less than the critical depth. Apparently, two real values of tranquil and shooting are obtained for the definite value of total head and momentum flux.

As described in the foregoing, of most significance in two regimes of open channel flows is that a very small disturbance in the tranquil flow can travel upstream, whereas in the shooting flow it can not influence the flow upstream from the boundary change of critical condition. This important description for open channel flows implies the basic knowledge that the tranquil branch proceeds upstream for computation of surface profiles of water while the analysis of shooting branch proceeds downstream.

The main characteristics of tranquil flows with low kinetic energy are described as the small changes of velocity head are of great importance owing to that the flow is sustained by low gradient. When the flow becomes critical, the stage variation with the change of head and momentum flux are pronounced. Furthermore, the influence of vertical acceleration becomes appreciable, and therefore, the basic equation in the gradually varied flow also is influenced by the non-hydrostatic pressure due to the surface curvature, and the resulting formation of surface undulation is observed.

The shooting flow is generally characterized by the fact that large variations in head and momentum flux are reflected little in the value of depth, since they are due almost entirely to change in the kinetic portion of the head, and in the shooting flow many outstanding features not found in the tranquil flow are pronounced. Small changes in boundary alignment produce the shock wave and the solid boundary makes the flow unstable to the formation of roll waves, known as a result of hydraulic instability.

3. General Theory of Transitional Behaviours of Steady Flows

Nearly almost natural channels and artificial watercourses in themselves involve local changes of channel geometry and boundary characteristics, which produce a variation in flow from one uniform state to another, and such transitions will make the basic equation in a form of 0/0. In fact, the mathematical expression of 0/0 yields the transition flow from tranquil to shooting or vice versa. This type of problems was first treated by Massé[7] in the case of flow over a variable slope as the further development of Bresse theory of quasi-linear flow. In 1956, Escoffier[8] studied the transitional behaviours of flows to apply the characters of flow to the graphical method for surface profiles of water. More recently, the author[4],[5],[6] also classified the transitional charcateristics of steady flows in non-uniform channels and obtained various types of surface profiles of water in terms of the characteristics. This chapter concerns with the general characters of transitional behaviours as solutions of steady flows in the immediate vicinity of a singular point.

When the origin of coordinate system is transferred to a singular point, at which the numerator and the denominator become simultaneously zero, the surface gradient equation of (5) is also represented by the following approximate equation near the singular point.

$$(dh'/dx') = \{cx'+dh'+Q(x', h')\}/\{ax'+bh'+P(x', h')\}, \tag{16}$$

where, x' and h' are small values deviated from the new origin and P and Q are higher terms depending on x' and h'. Coefficients of a, b, c, and d are expressed as follows.

$$a = \cos\theta_c(\partial\lambda/\partial x)_c - \lambda_c \sin\theta_c(d\theta/dx)_c - (Q^2/gA_c^3)(\partial A/\partial h)_c(\partial\alpha/\partial x)_c$$
$$+ (3\alpha_c Q^2/gA_c^4)(\partial A/\partial h)_c(\partial A/\partial x)_c - (\alpha_c Q^2/gA_c^3)(\partial^2 A/\partial x\partial h)_c$$
$$-(Q^2/gA_c^3)(\partial A/\partial x)_c(\partial\alpha/\partial h)_c + (Q^2/2gA_c^2)(\partial^2\alpha/\partial x\partial h)_c$$
$$-h_c \sin\theta_c(d\theta/dx)_c(\partial\lambda/\partial h)_c + h_c\cos\theta_c(\partial^2\lambda/\partial x\partial h)_c\,, \qquad (17)$$

$$b = 2\cos\theta_c(\partial\lambda/\partial h)_c - (2Q^2/gA_c^3)(\partial A/\partial h)_c(\partial\alpha/\partial h)_c + (3\alpha_c Q^2/gA_c^4)(\partial A/\partial h)_c^2$$
$$-(\alpha_c Q^2/gA_c^3)(\partial^2 A/\partial h^2)_c + (Q^2/2gA_c^2)(\partial^2\alpha/\partial h^2)_c + h_c\cos\theta_c(\partial^2\lambda/\partial h^2)_c\,, \qquad (18)$$

$$c = \cos\theta_c(d\theta/dx)_c + (2Q^2/C_c^3 R_c A_c^2)(\partial C/\partial x)_c + (Q^2/C_c^2 R_c^2 A_c^2)(\partial R/\partial x)_c$$
$$+ (2Q^2/C_c^2 R_c A_c^3)(\partial A/\partial x)_c + (2Q^2/gA_c^3)(\partial A/\partial x)_c(\partial\alpha/\partial x)_c$$
$$-(Q^2/2gA_c^2)(\partial^2\alpha/\partial x^2)_c - (3\alpha_c Q^2/gA_c^4)(\partial A/\partial x)_c^2$$
$$+ (\alpha_c Q^2/gA_c^3)(\partial^2 A/\partial x^2)_c + 2h_c\sin\theta_c(d\theta/dx)_c(\partial\lambda/\partial x)_c - h_c\cos\theta_c(\partial^2\lambda/\partial x^2)_c$$
$$+\lambda_c h_c\cos\theta_c(d\theta/dx)_c^2 + \lambda_c h_c\sin\theta_c(d^2\theta/dx^2)_c\,, \qquad (19)$$

$$d = (2Q^2/C_c^3 R_c A_c^2)(\partial C/\partial h)_c + (Q^2/C_c^2 R_c^2 A_c^2)(\partial R/\partial h)_c + (2Q^2/C_c^2 R_c A_c^3)\cdot$$
$$(\partial A/\partial h)_c + (Q^2/gA_c^3)(\partial A/\partial h)_c(\partial\alpha/\partial x)_c - (Q^2/2gA_c^2)(\partial^2\alpha/\partial x\partial h)_c$$
$$+ (Q^2/gA_c^3)(\partial A/\partial x)_c(\partial\alpha/\partial h)_c - (3\alpha_c Q^2/gA_c^4)(\partial A/\partial x)_c(\partial A/\partial h)_c$$
$$+ (\alpha_c Q^2/gA_c^3)(\partial^2 A/\partial x\partial h)_c - \cos\theta_c(\partial\lambda/\partial x)_c - h_c\cos\theta_c(\partial^2\lambda/\partial x\partial h)_c$$
$$+h_c\sin\theta_c(d\theta/dx)_c(\partial\lambda/\partial h)_c + \lambda_c\sin\theta_c(d\theta/dx)_c\,, \qquad (20)$$

in which $(\tau/\rho gR)(u_b/u_m)$ is assumed to be (Q^2/C^2RA^2) and C is the Chézy roughness. Although all of coefficients are so complicated that their hydraulic characters are not indicated in explicite forms, derivatives of α, λ and θ with respect to the distance and the depth are commonly very small and practically ignored, as seen in the later chapter.

The surface profiles of water in the vicinity of singular points are then classified by the mathematical properties of the following characteristic equation

$$S^2 - (a+d)S + (ad-bc) = 0\,. \qquad (21)$$

When the discriminant of Eq. (21), D, is positive and $(ad-bc)$ is inversely negative, two roots of Eq. (21) are real and of opposite sign, so that the singular point is classified as a saddle point, through which two singular solutions pass. When $D>0$ and $(ad-bc)$ is also positive, the singular point becomes nodal, because two roots are real and of same sign. All solutions as surface profiles of water have the same slope at the nodal point. When D is negative, two roots are conjugate complex, and therefore the singular point is a focal point, near which solutions are logarithmic spirals.

(1) *Hydraulic significance of saddle point*

Denoting the positive and negative roots of Eq. (21) by S_2 and S_1, two slopes of singular solutions as surface profiles of water at the saddle point are

$$(dh/dx)_{c1} = -c/(S_2-a)\,, \quad \text{and} \quad (dh/dx)_{c2} = -c/(S_1-a)\,, \qquad (22)$$

and the slope of critical depth curve at the point is also

$$s_2 = -(a/b)\ . \tag{23}$$

Designating the curves obtained by the former and latter relations in Eq. (22) by the C1- and C2-curves, respectively, it is understood that the C1-curve indicates the transition curve from tranquil to shooting whereas the C2-curve the transition curve from shooting to tranquil for positive values of b, making $(dh/dx)_{ci_{i=1,2}} - s_2$ by the use of Eqs. (22) and (23). On the contrary, the reverse transitional behaviours will be indicated for negative values of b.

In the tranquil flow, on the other hand, the surface profiles of water proceed from the downstream end, and furthermore the tranquil branch of singular solutions must be an asymptote of all other solutions in the upstream reach from the saddle point. Consequently, for positive values of b, the C1-curve represents the transition curve, whereas for negative values of b, the transition curve is indicated by the C2-curve. Despite of the sign in b, therefore, of great significance in the transitional behaviours through the saddle point is that the flow changes its regime from tranquil to shooting and the transition curve of the flow is indicated by the C1-curve for positive values of b, and the C2-curve for negative values of b. Other significant indications of saddle point are that the saddle point is a starting point for computation of surface profiles of water and evidently the control section at which all hydraulic characteristics of flows are uniquely determined for given rates of discharge.

(2) *Hydraulic significance of nodal point*

Denoting the larger root of Eq. (21) by S_2, all solutions have the same slope at the nodal point and it is

$$(dh/dx) = -c/(S_2 - a)\ , \tag{24}$$

In the same manner as did in the case of saddle points, it is understood that the transition takes place from shooting to tranquil and from tranquil to shooting, depending on that the signs of of b and S_1 are same and opposite. The former transition by the nodal point indicates that the singular point is a terminal for computation, whereas the latter represents the singular point is a starting point. All curves of surface profiles, however, as solutions of the basic equation have the same slope at the nodal point, and if the nodal point is a starting point, the surface profiles of water can not uniquely be determined. Only possible transition, consequently, is that the nodal point becomes a terminal for computation of surface profiles of water and the flow changes then from shooting to tranquil. In reality, in almost all cases, the nodal point produces the hydraulic jump when the up- and downstream depths become conjugate in the momentum conservation law.

(3) *Hydraulic significance of focal point*

The surface profiles of wate near the focal point are logarithmic spirals and no definite surface slopes in direction and magnitude are obtained at a section. In the open channel flow, actually, such transitional behaviours can not be produced. Only the way possible to change the flow regime is due to the hydraulic jump and therefore the flow changes from shooting to tranquil at a section where the up- and downstream depths traced under other boundary conditions imposed by control structures and the like become conjugate.

The complete illustration of surface profiles of water classified by the transitional behaviours of flow is then obtained in terms of the channel geometry, the bottom grade, and the roughness of channels. Through the saddle point twelve transition curves expressed by *C*1– and *C*2–curves are illustrated. Among twelve curves, obtained by six *C*1-curves and three *C*2-curves are commonly observed and they are graphically indicated in the author's paper[6]. Five transition curves produced by the nodal point and four curves by the focal point are also observed.

4. Hydraulic Significance of Simultaneity Theorem of Maximum Discharge, Minimum Head and Minimum Momentum Flux Known as Generalized Theorem of Bélanger-Böss

In 1849, Bélanger[9] used the condition that the discharge became maximum for a particular value of head, when calculating the discharge over a broad crested weir, and the condition is widely known as the theorem of Bélanger for maximum discharge. In 1919, Böss[10] observed that the transition from tranquil to shooting took place when the total energy line was at its lowest level. As frequently observed in the literatures of hydraulics, the above two theorems can be proved identical to each other, if the flow is assumed of parallel streamline as a first approximation. In 1943, Jaeger[11] treated generally with the simultaneity theorem of maximum discharge and minimum energy for open channel flows, and the theorem derived is known as the general theory of Jaeger or the generalized Bélanger-Böss theorem. This theory is evidenly the basis for hydraulics of control section, which establish the theoretical knowledge on discharge metering of flows in flumes and weirs by a single water-level measurement.

The transitional behaviours of flows produced by local changes in channel characteristics are not referred in the theory of Jaeger, and in reality, the hydraulic performance of control structures can not be established by the use of the simultaneity theorem. In this chapter, the hydraulic significance of the general theory of Jaeger will be clarified through the combination of the hydraulic characteristics of transitional behaviours in channel transitions and controls to the general theory of Jaeger. In other words, this chapter will make the physical interpretation on the discharge coefficients of control structures empirically obtained through past experience.

The simultaneity theorem for maximum discharge and minimum energy is verified by Jaeger in the following. The total head is a function of the discharge, depth and others, so that generally all of these variables are connected in an implicite form of

$$F(H_0, Q, h, A, \theta, \cdots\cdots) = 0 . \tag{25}$$

Under the condition of constant discharge, the Böss theorem is

$$(\partial H_0/\partial h)_c = -\{(\partial F/\partial h)_c + (\partial F/\partial A)_c(\partial A/\partial h)_c + \cdots\cdots\}/\{(\partial F/\partial H_0)_c + (\partial F/\partial A)_c(\partial A/\partial H_0)_c + \cdots\cdots\} = 0 \tag{26}$$

In the same manner, the theorem of Bélanger for maximum discharge is

$$(\partial Q/\partial h)_c = -\{(\partial F/\partial h)_c + (\partial F/\partial A)_c(\partial A/\partial h)_c + \cdots\cdots\}/\{(\partial F/\partial Q)_c + (\partial F/\partial A)_c(\partial A/\partial Q)_c + \cdots\cdots\} = 0 . \tag{27}$$

Neither $\{(\partial F/\partial H_0)_c + (\partial F/\partial A)_c(\partial A/\partial H_0)_c + \cdots\cdots\}$ nor $\{(\partial F/\partial Q)_c + (\partial F/\partial A)_c(\partial A/\partial Q)_c + \cdots\cdots\}$ are infinite, when the flow passes through the channel, so that $\{(\partial F/\partial h)_c + (\partial F/\partial A)_c(\partial A/\partial h)_c + \cdots\cdots\} = 0$ makes $(\partial H_0/\partial h)_c$ and $(\partial Q/\partial h)_c$ zero simultaneously.

On the other hand, as seen in the foregoing, the point satisfied by the minimum energy theorem of Böss must be the saddle point. Consequently, the simultaneity theorem for maximum discharge and minimum energy is satisfied at the saddle point, through which the transition of flow takes place from tranquil to shooting. All of hydraulic characteristics of flow are uniquely determined at the saddle point, so that up- and downstream surface profiles of water are also evaluated. With the aid of foregoing description on the hydraulic significance of Jaeger's theory, the hydraulic performance of various control structures like weirs and flumes can be established. In reality, however, the basic equation of (5) is evidently non-linear and still a mathematical approximation for the exact physical behaviours, so that the mathematical formulation of discharge coefficients is of great difficulty.

When the channel is of uniform geometry, the simultaneity theorem of maximum discharge and minimum momentum flux is also established by means of the one dimensional procedure of momentum approach. Herewith, one important but deeply difficult problem of the connection between both approaches of energy and momentum flux in open channel flows will be arised. As seen in Eqs. (14) and (15) for the critical depth, the simultaneity theorem of minimum energy and minimum momentum flux can not be explained. This is due to the difference in the dynamic treatment of basic principles between the energy approach as a scalar equation and the momentum approach as a vector equation and to the great difficulty and complexity to express the complete feature of the physics of flow in an exact mathematical form as indicated by Boussinesq and Jaeger, and the perfect connection between both theorems will be obtained after the real establishment in the clarification of basic physics of flow is completed.

5. Computation of Surface Profiles of Water as Hydraulics of Gradually Varied Flows

The oldest hydraulic researches on fluid flows in open channels since the middle part of 19th century were the hydraulics of gradually varied flows in uniform channels. The gradually varied flow is defined as the flow in which the change is anything but sudden, following by the description of Posey[12], and the purpose of study is mainly to predict the flow pattern for particular discharge and channel geometry. Although the resulting surface profiles of water are commonly called as back water curves for engineering uses, the essential basis is the classification of all the possible types of water surface profiles in gradually varied flows.

The methods of analysis for gradually varied flows are the following two procedures. One is the step by step method of integration of original surface profile equation, numerically or graphically. It is a common method of calculation applicable to all shapes in channel section and especially in non-uniform channels. The other is analytical one obtained by the exact integration of original equation. As the surface profile equation can not explicitely expressed in terms of the depth for all types of channels, so it is considered as a particular case of the former procedure.

The earlier studies have been mostly developed in cases of specific sections like two-dimensional, rectangular and so on. Bakhmeteff[13] first extended the classical theory to the refined form in all shapes of channels and thereafter many hydraulic engineers have been enforced to obtain a renewed and complete form of characteristics of varied flow functions. As natural channels and artificial watercourses involve frequent changes of channel geometry and grade as well as channel roughness, so the surface profiles of water in non-uniform channels can not exactly be computed by the method briefly described in the foregoing. In non-uniform channels, the surface profiles of water are usually determined by calculating separately and successively the variation in surface elevation in each of a large number of reaches, which are divided from the whole part so short as to reduce errors of calculation resulted from the approximation to a sufficient magnitude for engineering purpose, and therefore the choice of reaches must be carefully made.

A broad varieties of step by step computation procedures for surface profiles of water have been proposed by many hydraulic engineers. The numerical method of calculation is usually an analysis widely known as the standard step method, and it provides the surface profiles of gradually varied flows by the finite difference method without negligence of terms in the basic equation. Famous graphical methods are the Escoffier-Raytchine-Chatelain method[14] and the method of Silber[15], and the latter is extremely advantageous to understanding of the general features of surface profiles in non-uniform channels. The accuracy, however, is limited owing to the graphical representation, and therefore the desirable solution for hydraulic design can not be expected.

Herewith, the hydraulic significance of transitional characteristics of steady flows described in the foregoing will be arised, to estimate the possible surface profiles of water in a permissible accuracy. A saddle point changes the flow from tranquil to shooting and the surface profiles of water can be uniquely evaluated. Furthermore, all other profiles, speaking in hydraulic description, regulated by other control structures, and mathematically, located in each quadrant separated by two possible singular solutions, can not pass to the other zone without changing regime by sudden variation in elevation as the hydraulic jump. Nodal and focal points produce the hydraulic jump locally. The estimation of surface profiles, near the singular point must be carefully made, bearing in mind the transitional characteristics of flows.

(1) Trace carefully the normal flow curve and the critical depth curve throughout the whole reach of flow under investigation.

As explained in the foregoing, two curves of normal flow and critical depth are of basic importance to describe all the flow patterns in open channels, so that the complete estimation of two curves must be needed to evaluate the mathematical properties and, thus, the hydraulic knowledge of surface profiles of water. The term "gradually varied", however, is characterized by very large radii of surface curvature and the hydrostatic law in pressure distribution as a first approximation satisfied for almost all purposes, and therefore λ becomes unity and α also is a constant approximated by 1.0. The expressions of two curves are then

$$f_1(x, h) \risingdotseq i-(Q^2/C^2RA^2)+(Q^2/gA^3)(\partial A/\partial x) , \tag{28}$$

and

$$f_2(x, h) \risingdotseq 1-(Q^2/gA^3)(\partial A/\partial h) . \tag{29}$$

(2) Determine locations of singular points as intersections of two curves of $f_1(x, h)=0$ and $f_2(x, h)=0$.

The transitional characteristics of steady flows are determined by the mathematical properties of singular points, so that the careful determination of locations must be needed. Usually, it is insufficient to determine the location by given hydraulic and topographical data in the project, so that supplementary observations are required near the singular point, which will be expected in the preliminary design.

(3) Classify the singular point by the foregoing description.

The singular points produced by the mathematical properties of basic flow equation indicate their specific characteristics obtained by their singularities, as frequently described in the foregoing. Coefficients of a, b, c and d, which are expressed in Eqs. (17)~(20), are approximated as follows.

$$a = (3Q^2/gA_c^4)(\partial A/\partial h)_c(\partial A/\partial x)_c - (Q^2/gA_c^3)(\partial^2 A/\partial x\partial h)_c, \quad (30)$$

$$b = (3Q^2/gA_c^4)(\partial A/\partial h)_c^2 - (Q^2/gA_c^3)(\partial^2 A/\partial h^2)_c, \quad (31)$$

$$c = (2Q^2/C_c^3R_cA_c^2)(\partial C/\partial x)_c + (Q^2/C_c^2R_c^2A_c^2)(\partial Q/\partial x)_c + (2Q^2/C_c^2R_cA_c^3)(\partial A/\partial x)_c - (3Q^2/gA_c^4)(\partial A/\partial x)_c^2 + (Q^2/gA_c^3)(\partial^2 A/\partial x^2)_c, \quad (32)$$

$$d = (2Q^2/C_c^3R_cA_c^2)(\partial C/\partial h)_c + (Q^2/C_c^2R_c^2A_c^2)(\partial R/\partial h)_c + (2Q^2/C_c^2R_cA_c^3)\cdot(\partial A/\partial h)_c - (3Q^2/gA_c^4)(\partial A/\partial x)_c(\partial A/\partial h)_c + (Q^2/gA_c^3)(\partial^2 A/\partial x\partial h)_c, \quad (33)$$

Three kinds of singular points, saddle, nodal and focal, are then classified by the signs of discriminant of characteristic equation and $(ad-bc)$.

(4) Compute the surface profiles of water by solving the basic nonlinear equation numerically.

When no singular point is produced in the whole reach under investigation, the calculation procedure is rather simple. In the tranquil branch, the computation proceeds upstream from the downstream control, whereas in the shooting branch, it proceeds downstream. Many formulas, tabulations and graphical solutions for varied flow function are available for the estimation of surface profiles of water in each of the whole reach.

The saddle point is evidently a control section from which the computation procedure will be started, and the nodal and focal points are terminals. When the saddle point is produced in the flow, surface profiles of water are started up- and downstream from this point under the condition obtained by the analysis of hydraulic characteristics at the singular point.

Some of examples for possible surface profiles of water as solutions of the basic equation will be seen in the paper of the author[6], and it is easily understood that the complete analysis of transitional characteristics of flow produced by the singular point is needed.

6. Evaluation of Hydraulic Performance of Control Structures as Hydraulics of Rapidly Varied Flows

The rapidly varied flow is defined as the flow in which local variations are suddenly occured by the change of channel geometry, the grade and the roughness. The basic dynamic principle of rapidly varied flows is so difficult that the complete mathematical description on the basic flow pattern is not obtained, and therefore the empirical knowledge is often involved in the analysis of rapidly varied flows. In the control section is occured in the flow, all the hydraulic characteristics at the section are uniquely determined by the simultaneity theorem of Bélanger-Böss. A weir as discharge metering devices by a single water-level measurement is an example of such structures through which the rapidly varied flow is carried. Conversely speaking, the control structures must involve the control section for all discharges.

The hydraulic characteristics of flow over a control structure are then completely established and therefore the theoretical evaluation of discharge coefficients have largely obtained by the empirical studies can be possible. In this chapter, some examples of hydraulic performance of control strnctures will be briefly described, though the details will be seen in the paper of the author[6].

(1) *Hydraulic performance of round crested weir*

The hydraulic performance of a round crested weir, which is formed by round and solid boundaries in the direction of flow, will be explained. The flow over a weir is approximated by the irrotational and invicid motion, and the resulting surface profile equation is

$$dh/dx = f_1(x, h)/f_2(x, h)\ , \tag{34}$$

in which

$$f_1(x, h) = \sin\theta\{1+h(d\theta/dx)\}+(q^2/g)(R+h)^{-3}\{\log(1+h/R)\}^{-3}\cdot (dR/dx)\{\log(1+h/R)-(h/R)\}\ , \tag{35}$$

$$f_2(x, h) = \cos\theta-(q^2/g)(R+h)^{-3}\{\log(1+h/R)\}^{-3}\{1+\log(1+h/R)\}\ , \tag{34}$$

q: discharge per unit width, and R: local radius of curvature of boundary.

The first equation indicates the location of singular point and the second one gives the discharge characteristics of a weir, because the weir must serve as the control structure for all discharges. If the simplest case in which the curvature is constant and the weir is of circular shape is concerned, the hydraulic performance of the weir is feasibly obtained. The singular point is classified as saddle and located at the weir crest, and therefore all requirements for control structures are obtained.

As the discharge coefficient in terms of the total head H_0 from the reference line above the weir crest, C, is expressed by

$$q = (2/3)(2g)^{1/2}CH_0^{3/2}\ , \tag{37}$$

so the solution for C is obtainable as a function of H_0 by

$$C = (3/2)(1-h_c/H_0)^{1/2}(R/H_0+h_c/H_0)\log(1+h_c/R)\ , \tag{38}$$

and

$$2(1-h_0/H_0)\{1+\log(1+h_0/R)\} = (R/H_0+h_c/H_0)\log(1+h_c/R)\ , \tag{39}$$

in which h_c is the critical depth at the singular point. The experimental studies at the Hydraulics Laboratory, Kyoto University, verified that the preceding theoretical analysis was valid for almost all discharges, and the graphical representation is seen in the paper of the author[6].

In the same manner, the hydraulic performance of control structures of overflow spillways will be treated. In this case, the local curvature is a variable of distance from the crest, and therefore the evaluation of discharge characteristics is provided by a tremendous amout of labours to solve the basic non-linear equation. The approximate treatment for this case[6], however, indicates a good agreement with the actual observations made by various models and prototypes. In past, the popular representation of C was of power type of the head, and the results were not established in a generalized formula. If the present analysis is applied to the flow over a control structure of overflow spillway, the discharge characteristics are approximately indicated by a single parametric representation of (R/H_0), and it is seen that the indication is a better one than any other formula in past.

(2) *Hydraulic performance of sharp crested weir*

A sharp crested weir is also one of most popular control structures for discharge metering by a single water-level measurement. The basic flow pattern, however, is so difficult as to be not subject to a complete mathematical form, because the flow is perfectly curvilinear and free so that the distributions of velocity and pressure are not easily determined. The establishment of empirical formulas has been widely made, and the results are famous as formulas of Bazin, Rehbock and others.

If the same procedure as did for a round crested weir, the discharge characteristics of free flow from a sharp crested weir will be approximately evaluated, and the resulting equations become

$$C = (3/2)\{1-(z_c/H_0)-(h_c/H_0)-(h_c/H_0)\}^{1/2}(R_c/H_0+h_c/H_0)\log(1+h_c/R_c)\ , \qquad (40)$$

for discharge coefficient,

$$(2R_c/H_0+h_c/H_0)(z_c/H_0) = (1-h_c/H_0)(2R_c/H_0+h_c/H_0)-(R_c/H_0)^2\ , \qquad (41)$$

for pressure condition, and

$$2\{1+\log(1+h_c/R_c)\}(z_c/H_0) = 2(1-h_c/H_0)-(R_c/H_0+3h_c/H_0-2)\cdot \log(1+h_c/R_c)\ , \qquad (42)$$

for discharge equation derived by the Bélanger-Böss theorem, in which z_c is the vertical distance from the crest level to the lower nappe at the singular point.

Being different from the foregoing case, R_c and z_c are variables, so that the value of C is not calculated, if one variable is not assumed. Rehbock obtained through his experimental study that (z_c/H_0) was constant for all values of discharge. Nevertheless, many experimental data indicate Rehbock's conclusion is not valid. Although it will be expected another principles for free flow from a weir exists, the present knowledge can not establish the relationship. In approximations, however, the discharge characteristics are obtained, if (z_c/H_0) is assumed, and the study of the author[6] indicates the assumption is approximately valid for a particular weir. The final establishment for the hydraulic performance of a sharp crested weir, however, is still most difficult.

The performance of control sill considered to be a zero height weir is also analyzed by evaluating the limiting case of transition from tranquil to shooting, and the theoretical results[6] treated by the present procedure is closely agreed with the empirical formulas of Rouse[16], and Kandaswamy-Rouse[17].

The engineering application of hydraulics of gradually varied flows to discharge regulating devices is also seen in the use of flumes like those of Parshall, de Marchi and Inglis. The same principles and treatments to evaluate the discharge characteristics are available. In this case, however, the discharge coefficient can not be determined in a simple fashion, because the transition curve as the surface profile of water must be computed upstream from the singular point produced in the channel control to the gauging well. Many hydraulic engineers often propose their own empirical formulas of discharge coefficients for a particular flume through their experimental studies and field observations. Theoretical expressions of discharge coefficients are not possible, because of non-linearity of basic flow equation, and only the possible mean to express the discharge characteristics is to provide the tabulations and the graphical representation for a specified model. The universal formula of discharge coefficients in a flume is never establshed.

7. Conclusion

The steady-state behaviours of open channel flows are commonly treated by means of the one dimensional procedures of analysis, presuming the zone of flow to consist a single stream tube, and the resulting equation of motion is expressed in terms of the slope of free surface. This paper treated with the general theory of steady flows in open channels by the application of geometric theory of ordinary differential equation to the one dimensional equation of motion, and generally the behaviours concerned are called as the transitional characteristics of flow which produce the flow from tranquil to shooting and vice versa.

The most essential features in steady-state hydraulics are formed by the curves of normal and critical depths, and especially singular points as intersections of two curves provide the complete hydraulic features for all the possible surface profiles of water, and therefore the emphasis of the analysis is put on the mathematical and hydraulic properties of singular points.

Another singificant establishment in hydraulics of steady flows is that the simultaneity theorem of maximum discharge, minimum energy and minimum momentum flux known as the generalized theorem of Bélanger-Böss is valid at the saddle point where the smooth transition from tranquil to shooting takes place.

When the present knowledge on steady-state hydraulics of open channel flows is applied to engineering problems encountered in hydraulic design of gravity projects, all the hydraulic characteristics of flow are uniquely deterimined and thus the prediction of flow pattern is also obtained. Consequently, the unified treatment for both flows of gradually and rapidly varied is made by means of the general theory on the steady behaviours of open channel flows described herein.

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