Hydraulic Significance of Transitional Behaviours of Flows in Channel Transitions and Controls : Application of Geometric Theory of Differential Equation to Surface Configuration of Transition Flow in Divergent or Convergent Channels

AUTHOR(S):

IWASA, Yoshiaki

Hydraulic Significance of Transitional Behaviours of Flows in Channel Transitions and Controls

—Application of Geometric Theory of Differential Equation to Surface Configuration of Transition Flow in Divergent or Convergent Channels—

By

Yoshiaki IWASA

Department of Civil Engineering

(Received August 30, 1958)

Abstract

As one of the research projects of clear formulation in hydraulic characteristics of transitional behaviours from subcritical to supercritical or vice versa, this paper deals with the hydraulic behaviours of channel characteristics in geometry and boundary to the functional variety of channel transitions and controls in divergent or convergent channels.

The gradually varied flow in divergent or convergent channels changes its flow regime at a singular point of surface profile equation imposed by particular discharge and channel characteristics, and consequently the theoretical approach in analysis of hydraulic behaviours of transition flow will be established with the application of the geometric theory of differential equation, which is known in problems of non-linear mechanics.

The classification of singular points as transitional points and the significance of transitional characteristics of flow passing through a transitional point, and especially from a standpoint of practical hydraulic engineering, those for flows of Chézy and Manning are presented.

The present procedure of analysis is applied not only to the flow behaviours discussed in this study but also to the flow with side in- or outflows and similar problems in the hydraulics of open channel flows.

1. Introduction

Local changes of channel geometry and boundary resistance like channel shape, roughness and bed slope produce a variation in flow from one uniform state to another, and a flow passing through such transitions may be defined as a transition flow.

Evidently in the hydraulics of open channel flows, the physical regimes of open channel flow are imposed by the existence of free surface and defined as subcritical and supercritical, so that the dynamical characteristics of transition flow are classified[1] into

(1) transitions from a subcritical to another subcritical state of flow,

(2) transitions from subcritical to supercritical flow past a channel control in a sense of classical hydraulics,

(3) transitions from a supercritical to another supercritical state of flow,

(4) transitions from supercritical to subcritical flow.

The configuration of water surface in open channel flows passing through a channel of constant channel geometry and boundary characteristics is described by equations of Bresse and Tolkmitt or similar equations of flow commonly expressed in terms of the Vedernikov power law as the boundary resistance, and continuous endeavours have been made of clear formulation and tabulation of hydraulic behaviours in the physics of surface profiles by a number of scientists and engineers[2]. Nearly almost natural channels and artificial water courses in themselves, however, involve local changes of geometrical and boundary characteristics, and an actual configuration of water surface for a particular discharge may be considered a superposition of local surface profiles resulted from changes in channel geometry and resistance. Careful treatment of hydraulic behaviours of transition flow in the light of past experience and hydraulic knowledge in basic principles of flow, therefore, will bring a real success in pertinent practice of channel design. Usual procedures to trace the surface profile of flow in channel design related to the practical hydraulic engineering are the scale model test and the analytical calculation by numerical or graphical methods. The step by step method of integration as a simple tool for numerical analysis is most practical for hydraulic engineers, neverthless much errors for calculation are involved in the solution in the immediate vicinity of a singular point.

Supercritical flows are characterized by the fact that the small changes in the channel geometry cannot have any influence on the flow upstream from the section at which such changes take place, and transitions from supercritical to subcritical flow, therefore, are classified as the rapidly varied flow in the first approximation of open channel flows. A transitional point from subcritical to supercritical flow is usually called a control section which is the class of transition for which the elevation of the water surface can be uniquely predicted at a particular geometrical characteristics for a given discharge as a barrier to the transmittal of small disturbances, and the hydraulic property of a channel control, which leads to a consistent relationship between the head and the discharge, is familiar to hydraulic engineers.

In this paper, as a first step of the research project to clear formulation of the hydraulic characteristics of channel transitions and controls and their dynamical significance, the theoretical analysis of transitional behaviours of geometrical and boundary characteristics to transition flow in divergent or convergent channels is discussed. This problem has already treated by M. Homma[3], without establishing it by rigorous analysis.

As seen in the later section, mathematically speaking, the singular point of basic equation of gradually varied flows in divergent or convergent channels is located at $h=\infty$, $h=h_c$ (h_c is the critical depth for a particular discharge) and a point at which both of numerator and denominator become simultaneously zero. The transitional behaviours of flow at the critical point are known as the rapidly varied flow characterized by the hydraulic jump. The third point at which both terms of basic equation are zero is classified as a singular point in the geometric theory of differential equation applied to the problems in non-linear mechanics. In the hydraulics of open channel flows, the curve of normal flow and the critical depth curve for particular characteristics in channel geometry and discharge intersect together, so that the change of flow regime indicated by the behaviour of transition profile will be supposed to take place. Consequently, the purpose of this study is to reveal the topological properties of singular points and the hydraulic significance of surface profile as a solution of basic equation of gradually varied flow in the immediate vicinity. of the singular point.

It is known that in 1939, P. Massé[4] first studied the application of geometric theory of ordinary differential equation to the flow in channels with variable slope as the further development of Bresse equation and a brief summary is cited, in the literature of C. Jaeger[4]. More recently, in 1956, F. F. Escoffier[6] studied the same transitional behaviours of flow for his intention of application to the graphical method of tracing of surface profiles of flow.

The first section of this paper deals with the location of transitional point and its hydraulic significance to transition flow in divergent or convergent channels, and the classification of transitional behaviours from subcritical to supercritical or vice versa and the influence of geometrical and boundary characteristics to transitional behaviours are followed. From a practical point of view, the analysis of this subject in the Chézy and Manning flows followed by the tracing procedure of surface profiles and its contribution to the hydraulics of open channel flows is briefly discussed and the details will be presented in the other publication.

The present procedure of analysis is applied not only to the flow in divergent or convergent channels but also to many hydraulic problems like the flow with side in- or outflows, transition profiles of water surface in estuaries and the hydraulic process of boundary layer growth in open channel flows[7,8]. After the real success of clear formulation of transitional behaviours in channels is attained, the pertinent hydraulic design in channels and hydraulic structures will be possible.

2. Transitional Point of Flow in Divergent or Convergent Channels and its Hydraulic Significance

Taking the x-axis in the downstream direction along the channel bed, and denoting Q: discharge, h: water depth, A: flow area, s: wetted perimeter, τ: shear along the channel bed, g: acceleration of gravity, ρ: density of water, θ: inclination angle of bed, and β: momentum correction factor of Corioli, the law of momentum conservation in the one dimensional approach of hydraulics yields

$$\frac{1}{A}\frac{d}{dx}\left(\frac{\beta Q^2}{A}\right)+g\cos\theta\frac{dh}{dx}=g\sin\theta-\frac{\tau s}{\rho A}. \tag{1}$$

Although the pressure distribution in the fluid flow is assumed hydrostatic, the non-hydrostatic influence of flow is appreciable in the flow near critical regime or along the curved boundaries. As a first approximation of hydraulics, Eq. (1) is still valid for curved streamlined flows.

For the sake of simplicity, the shape of cross-section of channel is assumed rectangular and the boundary resistance is expressed in terms of the Chézy law, and consequently, Eq. (1) is transformed into

$$\frac{dh}{dx}=\frac{g\sin\theta-\frac{gQ^2}{C^2b^2h^3}\left(1+\frac{2h}{b}\right)+\frac{\beta Q^2}{b^3h^2}\frac{db}{dx}}{g\cos\theta-\frac{\beta Q^2}{b^2h^3}}=\frac{f_1(x,h)}{f_2(x,h)}, \tag{2}$$

where, b: width of channel and C: Chézy's coefficient of roughness.

A transitional point at which the flow regime will be changed is a point of solution derived by $f_1(x,h)=0$ and $f_2(x,h)=0$ in Eq. (2). Denoting the value at the transitional point by the subscript c, $f_2(x,h)=0$ leads to the following relation of

$$b_c^2h_c^3=\frac{\beta Q^2}{g\cos\theta}. \tag{3}$$

Evidently, Eq. (3) indicates the hydraulic relation of critical regime in flow. Inserting Eq. (3) into $f_1(x,h)=0$, the ratio of critical depth to width of channel at the transitional point is

$$\frac{h_c}{b_c}=\frac{i-\left(\frac{g}{\beta C_c^2}\right)}{2\left(\frac{g}{\beta C_c^2}\right)-\left(\frac{db}{dx}\right)_c}, \tag{4}$$

in which, i is the slope of channel bed equals to $\tan\theta$.

Introducing the local critical slope i_c as in the uniform channels, it is

$$i_c=\frac{g}{\beta C_c^2}\left\{1+2\left(\frac{h_c}{b_c}\right)\right\}. \tag{5}$$

From Eqs. (4) and (5), h_c/b_c in terms of the geometrical characteristics of channels is obtained in the following.

$$\frac{h_c}{b_c} = -(i-i_c)/\left(\frac{db}{dx}\right)_c. \tag{6}$$

This is an important relationship for the existence of transitional point in transition flows and characterized by the following conclusion.

(1) A transitional point of flow is produced under the geometrical condition that the channel is divergent and of mild slope.

(2) On the contrary, in steep channels, the geometrical condition for channels to produce a transitional point is that channels are convergent.

3. Classification of Transitional Points in Transitional Flows

The foregoing analysis revealed the location of a singular point as a transitional point in divergent or convergent channels, produced by the local change in channel geometry and boundary resistance. This section concerns with the classification of transitional points of gradually varied flows by means of the geometric theory of ordinary differential equation, though the general theory of steady flows has been completely studied and will be published in the very near future.

(1) *Water surface equation of transition flow in the immediate vicinity of transitional point*

Before the classification of transitional points of gradually varied flow is discussed, the variation equation of Eq. (2) in the neighbourhood of a transitional point will be derived for the further development of the study. Let assume the channel geometry and boundary resistance change continuously and put the distance, the water depth and the other as follows;

$$x = x_c + x', \quad h = h_c + h', \cdots\cdots\cdots\cdots, \tag{7}$$

in which, $x_c \gg x'$, $h_c \gg h'$, $\cdots\cdots\cdots\cdots$.

Inserting the relations of Eq. (7) into Eq. (2) and transforming the origin of coordinate system to the transitional point, the variation equation of Eq. (2) is in a form of

$$\frac{dh'}{dx'} = \frac{(i-i_c)\left[6i(i-i_c)+\left(\frac{\partial b}{\partial x}\right)_c(i_c-3i)-\left\{\left(\frac{\partial b}{\partial x}\right)_c-2(i-i_c)\right\}h_c\left(\frac{\partial^2 b}{\partial x^2}\right)_c\Big/\left(\frac{\partial b}{\partial x}\right)_c\right]}{-2\left\{\left(\frac{\partial b}{\partial x}\right)_c-2(i-i_c)\right\}(i-i_c)\,x'}$$

$$\frac{x'+\left[\frac{2h_c i_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\left\{\left(\frac{\partial b}{\partial x}\right)_c-2(i-i_c)\right\}+\left(\frac{\partial b}{\partial x}\right)_c(2i+i_c)-4i(i-i_c)\right]h'+Q(x',h')}{+3\left\{\left(\frac{\partial b}{\partial x}\right)_c-2(i-i_c)\right\}h'+P(x',h')}, \tag{8}$$

where, $P(x', h')$ and $Q(x', h')$ are higher terms depending on the squares and products of x' and h'. Omitting the prime in Eq. (8), for the convenience of notation, and introducing the following dimensionless expressions of

$$\alpha = i/i_c, \quad \beta = (\partial b/\partial x)_c/i_c, \quad \text{and} \quad m = 3h_c^2(\partial^2 b/\partial x^2)_c/b_c i_c^2,$$

Eq. (8) becomes

$$\frac{dh}{dx} = \frac{cx+dh+Q(x, h)}{ax+bh+P(x, h)}, \tag{9}$$

in which,

$$a = -2i_c(\alpha-1)\left\{\beta-2(\alpha-1)\right\},$$

$$b = 3\left\{\beta-2(\alpha-1)\right\},$$

$$c = i_c^2\left[(\alpha-1)\left\{6\alpha(\alpha-1)-\beta(3\alpha-1)\right\}+(m/3)\left\{\beta-2(\alpha-1)\right\}\right],$$

$$d = i_c\left[(2h_c/C_c)(\partial C/\partial h)_c\left\{\beta-2(\alpha-1)\right\}+\beta(2\alpha+1)-4\alpha(\alpha-1)\right].$$

This is the basic homogeneous equation for the analysis of transitional behaviours, in which the change of flow regime will be predicted at the origin of the present coordinate system.

(2) *Classification of transitional points*

As seen in the basic equation (9), the coefficients a, b, c and d vary with the local change of channel geometry and boundary resistance for a particular discharge, so that the resulting surface profile of transition flow as a solution of Eq. (9) will be influenced by the geometrical and boundary characteristics of channels. The mathematical properties of Eq. (9) and the hydraulic behaviour of surface profiles are obtained by the geometric theory of ordinary differential equation. If $y=dx/dt$ is substituted for h, the procedure will become the same as the vibration problem in non-linear mechanics.

In the characteristic equation of Eq. (9),

$$S^2-(a+d)S+(ad-bc) = 0, \tag{10}$$

let denote two roots of equation by S_1 and S_2.

(a) When S_1 and S_2 are real and of opposite sign, a singular point is called a saddle point, through which two singular solutions defined as the transition profiles in this study pass.

(b) S_1 and S_2 are real and of same sign, so that a singular point is classified as a nodal point, at which all surface profiles of fluid flows have a certain definite direction determined by the geometrical and boundary characteristics of channel transitions.

(c) If S_1 and S_2 are conjugate complex, a singular point becomes a focal point, and all surface profiles in the immediate vicinity of the focal point are logarithmic spirals and approach the point.

The classification of singular points as the transitional point of flows from subcritical to supercritical or vice versa in channel transitions and controls, therefore, is described in the following:

$$\begin{array}{lll} \text{for saddle point,} & D \geqq 0, & ad-bc<0, \\ \text{for nodal point,} & D \geqq 0, & ad-bc>0, \\ \text{for focal point,} & D<0, & ad-bc>0, \end{array}$$

in which,

$$\begin{aligned} D = 4i_c^2&\left[5\alpha^2-2\left\{5+\frac{2h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\right\}\alpha+\frac{11-4m}{4}-\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\left\{\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c-1\right\}\right] \\ \times&\left[\beta^2-\frac{2(\alpha-1)\left[10\alpha^2-\left\{15+\frac{8h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\right\}\alpha+2(1-m)-\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\left\{2\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c-3\right\}\right]}{\left[5\alpha^2-2\left\{5+\frac{2h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\right\}\alpha+\frac{11-4m}{4}-\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\left\{\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c-1\right\}\right]}\beta\right. \\ &\left.+\frac{4(\alpha-1)^2\left[5\alpha^2-\left\{5+\frac{4h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\right\}\alpha-(1+m)-\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\left\{\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c-2\right\}\right]}{\left[5\alpha^2-2\left\{5+\frac{2h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\right\}\alpha+\frac{11-4m}{4}-\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c\left\{\frac{h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c-1\right\}\right]}\right], \end{aligned} \tag{11}$$

and

$$\begin{aligned} ad-bc = &\left\{5(\alpha-1)^2-\frac{4h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c(\alpha-1)-m\right\}\left\{\beta-2(\alpha-1)\right\} \\ &\times\left[\beta-\frac{(\alpha-1)\left\{10\alpha(\alpha-1)-\frac{8h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c(\alpha-1)-2m\right\}}{\left\{5(\alpha-1)^2-\frac{4h_c}{C_c}\left(\frac{\partial C}{\partial h}\right)_c-m\right\}}\right]. \end{aligned} \tag{12}$$

Although the above equation seems to be of much complexity, it will be seen in the later section that Eqs. (11) and (12) become simple, if the empirical resistance law of power type like formulas of Chézy and Manning are substituted in Eqs. (11) and (12). Furthermore, it is readily understood that the bed slope, the channel geometry and the boundary resistance as the geometrical and boundary characteristics of channel transitions and controls for a given discharge are represented by the dimensionless parameters of α, β and m, and C.

4. Hydraulic Significance of Transitional Point and Change of Flow Regime by Transitional Point

(1) *Curves of normal and critical depths of transition flow at transitional point*

When the numerator of basic equations, $f_1(x, h)$, becomes zero, (dh/dx) becomes also zero, and therefore, the local surface profile of transition flow is parallel to the channel bed. The curve of $f_1(x, h)=0$ is defined as the curve of normal depth or normal flow for a given discharge in the present study, though Massé called it the curve of quasi-normal flow and Escoffier the transition curve. The curve of $f_2(x, h)=0$ for a particular discharge is the well known critical depth curve in the hydraulics of open channel flows. Denoting the slopes of the foregoing two curves of flow at the transitional point by s_1 and s_2, they are derived by the linear variation equation of (9) as follows.

$$s_1 = -(c/d)\,, \tag{13}$$

and

$$s_2 = -(a/b)\,. \tag{14}$$

Consequently, the values of s_1 and s_2 are characterized by the channel geometry and the boundary resistance. The subtraction of s_2 from s_1 yields

$$s_1 - s_2 = (ad-bc)/bd\,. \tag{15}$$

If b and d are of same sign, the sign of (s_1-s_2) corresponds with that of $(ad-bc)$, and if b and d are opposite, (s_1-s_2) and $(ad-bc)$ are of opposite sign. Furthermore, the sign of $(ad-bc)$ is negative for the saddle point and positive for the nodal and focal points as seen in the foregoing section.

On the other hand, when (s_1-s_2) is positive, the geometric properties of these two curves are classified as

(a) $s_1,\ s_2>0,\quad s_1>s_2,$
(b) $s_1>0,\quad s_2<0,$
(c) $s_1,\ s_2<0,\quad |s_2|>|s_1|,$

and all of the above geometric properties indicate that the curve of normal depth passes through the lower half plane divided by the critical depth curve and approaches the transitional point at which both curves intersect together.

When (s_1-s_2) is negative, the classification of geometric situation is in forms of

(d) $s_1,\ s_2>0,\quad s_2>s_1,$
(e) $s_1<0,\quad s_2>0,$
(f) $s_1,\ s_2<0,\quad |s_1|>|s_2|.$

In contrast with the foregoing case, the curve of normal depth passes through the upper half plane divided by the critical depth curve.

Consequently, the relationship between the classification of transitional points and the geometry of two curves of normal flow and critical depth, which are the most important curves in the hydraulics of open channel flows, is described in the following, by means of the channel characteristics.

When b and d are of same sign, (s_1-s_2) is negative for saddle points and the curve of normal depth passes through the upper half plane divided by the critical depth curve and approaches the saddle point, and the opposite hydraulic behaviours in the transition flow are imposed by focal and nodal points.

On the contrary, when b and d are of opposite sign, three kinds of singular points indicate the opposite behaviours of the foregoing conclusion.

(2) *Transition slope as solution of transition flow at transitional point*

At the transitional point in this study, both of numerator and denominator become simultaneously zero and the resulting transition slope of water surface derived by the original equation is indeterminate. When the transitional point is classified as the control of channel, it is commonly a starting point of calculation for the tracing of surface profile in hydraulic design problems, and consequently, the value of (dh/dx) at the point must be known for the calculation procedure.

The usual procedure to evaluate the value of (dh/dx) at the singular point is to use the method of form 0/0 in the differential calculus known as the rule of L. Hospital. Or simply, $(dh/dx)_{x,\, h\to 0}=(h/x)_{x,\, h\to 0}=\text{const.}$, provided these values at the transitional point are certain definite, so that Eq. (9) becomes

$$b\left(\frac{dh}{dx}\right)_c^2+(a-d)\left(\frac{dh}{dx}\right)_c-c=0\,. \tag{16}$$

The transition slope is, then,

$$\left(\frac{dh}{dx}\right)_c=\frac{-(a-d)\pm\sqrt{(a-d)^2+4bc}}{2b}\,. \tag{17}$$

The possible slope of transition profile is evidently one of solutions of Eq. (17). The rather empirical study of Homma[3)] on the behaviours of transition flow with the Chézy law of resistance in wide channels indicates that the possible slope of transition profile is given by a negative root of Eq. (17) for divergent channels and a positive one for convergent channels. As the product of two transition slopes described in the foregoing is $-(c/b)$ by means of the expression of channel characteristics, so two roots are of same sign if c and b are opposite, and therefore, the foregoing conclusion of Homma on the possible transition slope at the transitional point is essentially insufficient to define the slope of transition flow.

The first consideration to make the behaviours of transition slope and transitional surface profile clear is oriented to the transition flow passing through the saddle point. Denoting the positive root by S_2 and the negative one by S_1 in the characteristic equation, in which S_1 and S_2 are of opposite sign, the canonical form of the variation equation (9) is derived with the use of the linear transformation being in forms of

$$\left.\begin{aligned} x &= (S_2-a)\,\xi+(S_1-a)\,\eta\,, \\ h &= -c\xi-c\eta\,, \end{aligned}\right\} \tag{18}$$

and it is

$$\frac{d\xi}{S_1\xi+\cdots\cdots\cdots}=\frac{d\eta}{S_2\eta+\cdots\cdots\cdots}\,. \tag{19}$$

Consequently, the hydraulic behaviour of transition flow in the immediate vicinity of transitional point is approximately indicated by

$$(d\eta/d\xi) = -|S_2/S_1|\,(\eta/\xi)\,. \tag{20}$$

The solution of Eq. (20) describes a family of hyperbolas, among which only two singular curves of transition profiles pass through the saddle point, and the transition slopes at the origin are

$$(d\eta/d\xi)_c = 0\,, \quad \text{or} \quad \infty\,.$$

Tranforming to the original coordinate system, the transition slopes are, then,

$$(dh/dx)_{c1} = -c/(S_2-a)\,, \tag{21}$$

or

$$(dh/dx)_{c2} = -c/(S_1-a)\,. \tag{22}$$

The expression of Eqs. (21) and (22) in terms of the channel characteristics becomes the same relation of Eq. (17), and it is

$$\left(\frac{dh}{dx}\right)_{c1} = \frac{-\{a-d+\sqrt{(a-d)^2+4bc}\}}{2b}\,, \tag{23}$$

and

$$\left(\frac{dh}{dx}\right)_{c2} = \frac{-\{a-d-\sqrt{(a-d)^2+4bc}\}}{2b}\,. \tag{24}$$

Next consideration to the behaviour of surface slope of transition flow is directed to the sign of $(dh/dx)_{c1}$.

If $(a-d)$ is positive, the sign of $(dh/dx)_{c1}$ depends only on the sign of b, so that $(dh/dx)_{c1}$ becomes positive for convergent channels and negative for divergent channels, and the condition that $(a-d)$ is positive is expressed as

$$\begin{aligned} &a, d>0\,, \quad a>d\,, \\ &a>0\,, \quad\;\; d<0\,, \\ &a, d<0\,, \quad |d|>|a|\,, \end{aligned}$$

On the contrary, if $(a-d)$ is negative, of which condition is

$$\begin{array}{ll} a, d > 0, & d > a, \\ a < 0, & d > 0, \\ a, d < 0, & |a| > |d|, \end{array}$$

the divergent channel makes that the value of transition slope becomes negative or positive for $c \gtreqless 0$, and the convergent channel yields $(dh/dx)_{c1} \gtreqless 0$ for $c \gtreqless 0$.

The sign of $(dh/dx)_{c2}$ is determined by the diversity of $(dh/dx)_{c1}$ in signs by means of making the product of both slopes. As

$$(dh/dx)_{c1} \cdot (dh/dx)_{c2} = -(c/b), \tag{25}$$

so for divergent channels two values of transition slopes are opposite or same depending on $c \gtreqless 0$, and on the other hand, $c \gtreqless 0$ leads that two values are same or opposite for convergent channels.

The geometric property of two possible singular solutions, which intersect together at the saddle point, is characterized by the following relation of

$$(dh/dx)_{c1} - (dh/dx)_{c2} = -\sqrt{(a-d)^2 + 4bc}/b . \tag{26}$$

In Eq. (26), the numerator is positive for any characteristic of channel geometry and boundary, and consequently the classification of geometric properties between two singular curves is described in the following.

(a) for divergent channels,

(i) $(dh/dx)_{c1} > 0$, $(dh/dx)_{c2} > 0$, $(dh/dx)_{c2} > (dh/dx)_{c1}$,

(ii) $(dh/dx)_{c1} < 0$, $(dh/dx)_{c2} > 0$,

(iii) $(dh/dx)_{c1} < 0$, $(dh/dx)_{c2} < 0$, $|(dh/dx)_{c1}| > |(dh/dx)_{c2}|$,

(b) for convergent channels,

(i) $(dh/dx)_{c1} > 0$, $(dh/dx)_{c2} > 0$, $(dh/dx)_{c1} > (dh/dx)_{c2}$,

(ii) $(dh/dx)_{c1} > 0$, $(dh/dx)_{c2} < 0$,

(iii) $(dh/dx)_{c1} < 0$, $(dh/dx)_{c2} < 0$, $|(dh/dx)_{c2}| > |(dh/dx)_{c1}|$.

In case of (a), $c1$-curve defined as the singular curve of which the transition slope is $(dh/dx)_{c1}$ passes through the upper half plane divided by $c2$-curve, while the case of (b) indicates $c1$-curve passes through the lower half plane.

Of most significance in practical problems of channel design is to select the possible transition profile between two values of (dh/dx), of which characteristics have been described, for a particular discharge and given channel characteristics in geometry and boundary. The basic property of open channel flows, in which the supercritical flow must be traced from the upstream end and the subcritical flow from the downstream end, makes the determination of possible slope. The detailed behaviours in the transition profile will be treated for a particular channel resistance as seen in the later section.

Furthermore, the other surface slope $(dh/dx)_{c2 \text{ or } c1}$ is also significant in channel design. Two singular curves as transition profiles intersecting together at the saddle point divide the whole plane into four domains, and a surface profile curve traced under certain definite boundary conditions in each domain cannot be transmitted to the other domain without changing its flow regime, and therefore, the engineering contribution of singular curves to the evaluation of backwater zone resulted from the construction of control structures will be readily understood.

The same procedure described in the foregoing is directed to the behaviours of nodal point. The characteristic equation (10) has two real roots of same sign. Let denote the greater root by S_2, and thus, (S_2/S_1) is always greater than unity. By means of the linear transformation of Eq. (18), the canonical form becomes

$$\eta = c\xi(S_2/S_1)\,.$$

The transition slope at the nodal point is uniquely determined and it is

$$(dh/dx)_c = -c/(S_2-a)\,. \tag{27}$$

Eq. (27) for nodal points is the same form as for saddle point, Eq. (21), and therefore, the behaviour of Eq. (27) is also similar to that of Eq. (21), if S_2 is defined as the greater root of Eq. (10).

(3) *Hydraulic significance of transitional behaviours*

The most significant subject of study on the transitional behaviours of flow and associated problems in channel design is to formulate the type of change in flow regime for the particular design discharge through the transitional point in channel transitions and controls. When a transition curve as a configuration of water surface approaches the transitional point after passing through the upper half plane divided by the critical depth curve, and thereafter passes through the lower half plane, the flow changes its flow regime from subcritical to supercritical. On the contrary, the transition curve passes from the lower half plane to the upper half plane, so that the flow regime changes from supercritical to subcritical. Consequently, the transitional behaviours of transition flow from subcritical to supercritical or vice versa are solved by the establishment of geometric property between the transition curve and the critical depth curve for the design discharge.

The case, in which the transitional point is classified as a saddle point and the transition slope is assumed to be $(dh/dx)_{c1}$, will be first treated. The difference of both curves of transition and critical depth at the transitional point is obtained, with the use of Eqs. (14) and (21).

$$(dh/dx)_{c1} - s_2 = S_1/b\,, \tag{28}$$

in which, S_1 is negative by the definition of saddle point, and therefore, the sign of Eq. (28) is dependent on the behaviour of b. Expressing in terms of the channel geometry, the geometric property is classified as follows:

For divergent channels, which are characterized by the positive value of b,

(i) $(dh/dx)_{c1} > 0,\quad s_2 > 0,\quad s_2 > (dh/dx)_{c1}\,,$

(ii) $(dh/dx)_{c1} < 0,\quad s_2 > 0\,,$

(iii) $(dh/dx)_{c1} < 0,\quad s_2 < 0,\quad |(dh/dx)_{c1}| > |s_2|\,,$

and the transition flow changes its flow regime from subcritical to supercritical at the transitional point.

For convergent channels, on the other hand,

(iv) $(dh/dx)_{c1} > 0,\quad s_2 > 0,\quad (dh/dx)_{c1} > s_2\,,$

(v) $(dh/dx)_{c1} > 0,\quad s_2 < 0\,,$

(vi) $(dh/dx)_{c1} < 0,\quad s_2 < 0,\quad |s_2| > |(dh/dx)_{c1}|\,,$

and consequently, the flow regime of transition flow is changed from supercritical to subcritical.

If $(dh/dx)_{c2}$ becomes the transition slope of flow, the difference between $(dh/dx)_{c2}$ and s_2 is

$$(dh/dx)_{c2} - s_2 = S_2/b\,. \tag{29}$$

S_2 is always positive, so that the channel characteristics in channel transitions and controls induce the transition flow to make the opposite behaviours in its flow regime.

Consequently, the transitional behaviours of flow passing through the transitional point classified as the saddle point are thoroughly investigated by the foregoing analysis of geometric property among curves of normal flow, critical depth and two singular curves of a family of hyperbolas as solutions of surface profiles of transition flow. The hydraulic behaviours of solutions and the resulting transition profiles for gradually varied flow of a particular resistance law will be discussed in the later section.

When the transitional point is known as a nodal point, Eq. (28) is still valid if S_1 is assumed the smaller root of the characteristic equation (10). In this case, however, the sign of S_1 changes with the change of channel characteristics. As $(S_1+S_2)=(a+d)$, and S_1 and S_2 are of same sign, so the sign of S_1 depends on the sign of $(a+d)$. Finally, the transitional behaviours of nodal point is described in the following.

(a) When $(a+d)>0$, S_1 becomes positive, and the flow changes from supercritical to subcritical through the nodal point in divergent channels and from subcritical to to supercritical in convergent channels.

(b) When $(a+d)<0$, S_1 becomes negative, and therefore, the transitional behaviours of nodal point are the same as the $c1$-curve at the saddle point does.

As the last part of this section, the transitional behaviours of focal point will be explained. The surface profile as a mathematical solution of Eq. (9) in the immediate vicinity of the focal point is a logarithmic spiral and the resulting water depth at a certain definite distance near the transitional point is indeterminate. Such a transitional behaviour in the hydraulics of open channel flows will be only possible by the hydraulic jump. Consequently, when the upstream depth from the transitional point and the downstream depth become sequent in the momentum conservatisn law, the hydraulic jump will occur and the focal point is not substantially the transitional point, through which the flow changes its flow regime.

(4) *Slope of surface·profile near transitional point*

In the foregoing analysis of transitional behaviours, the surface profile under consideration is the singular curve which passes through the transitional point determined uniquely by given values of discharge and channel characteristics in geometry and boundary. This class of transition, of which functional diversity has been discussed, is known as the channel control as seen in the introduction, and consequently it is also understood that the possible class of channel control is saddle and nodal points. When the elevation of water surface for a particular discharge is regulated by the other control structures like the sluice gate or the weir, the resulting surface profile cannot sometimes pass the transitional point as the control section, which is uniquely predicted by given discharge and channel characteristics. In this case, the local change of channel geometry simply produces a variation in flow elevation, so that the channel becomes a transition.

The analysis of the hydraulic behaviours of surface profiles in the vicinity of transitional point is essentially important to evaluate the variation in flow due to the channel transitions and controls. The increase or decrease of surface elevation of flow with the increase of distance at a point is estimated by the geometric properties of $f_1(x, h)=0$ and $f_2(x, ,h)=0$ in the basic relationship of (2).

First attention is directed to the behaviours of critical depth curve, $f_2(x,h)=0$. For the sake of simplicity in discussion, the behaviours of $f_2(x,h)=0$ associated with the increase of water elevation at the location of $x=x_c$ will be treated. From Eq. (2),

$$f_2(x_c, h) = g\cos\theta-(\beta Q^2/b_c^2h^3) \gtrless 0\,,$$

in which, the upper and lower inequalities indicate the positive and negative domains of $f_2(x_c, h)$. The insertion of Eq. (3) into the above equation yields

$$g\cos\theta(h-h_c)\left\{(h+h_c/2)^2+3h_c^2/4\right\}\Big/h^3 \gtrless 0\,. \tag{30}$$

Consequently, at $x=x_c$, the critical depth makes $f_2(x_c, h)$ zero and in the upper half plane $f_2(x, h)$ is positive and in the lower plane negative. The same behaviour of critical depth curve to $f_2(x, h)$ will be extended to the neighbouring domain of transitional point.

In the same manner, the behaviour of normal depth curve to $f_1(x, h)$ will be discussed. At $x=x_c$, $f_1(x, h)$ is

$$f_1(x_c, h) = g\sin\theta - (gQ^2/C^2b_c^2h^3)\Big\{1+(2h/b_c)\Big\} + (\beta Q^2/b_c^3h^2)(db/dx)_c \gtrless 0\,.$$

With the aid of Eqs. (3), (5) and (6) and dimensionless parameters of α and β, it is indicated as follows.

$$f_1(x_c, h) = \frac{g\sin\theta}{h^3}\left[h^3 + \frac{2\left(\frac{C_c}{C}\right)^2(\alpha-1)-(\alpha-1)\Big\{\beta-2(\alpha-1)\Big\}}{\alpha\Big\{\beta-2(\alpha-1)\Big\}}h_c^2h - \frac{\beta}{\alpha\Big\{\beta-2(\alpha-1)\Big\}}\left(\frac{C_c}{C}\right)^2h_c^3\right] \gtrless 0\,. \tag{31}$$

In contrast with the foregoing case of critical depth, the positive and negative domains will not be explicitely determined if the resistance formula of boundary like laws of Chézy or Manning are not assumed. Furthermore, the attention must be directed to the sign of bed slope or $\sin\theta$.

In the immediate vicinity of transitional point, the sign of $f_1(x, h)$ and $f_2(x, h)$ is simply determined by the mathematical behaviours of variation equation. The straight line, $ax+bh=0$, indicates $f_2(x, h)=0$ as the approximation near the transitional point, and therefore, $ax+bh=0$ divides the whole plane into the positive and negative domains. If b is positive and thus the channel is divergent, the upper half plane indicates the positive domain of $f_2(x, h)$, and vice versa.

The sign of two domains divided by $f_1(x, h)=0$ is also determined by the sign of d. Finally, the combination of behaviours of $f_1(x, h)$ and $f_2(x, h)$ in sign determines the sign of slope of transition curve positive or negative. The details of surface profiles for Chézy flows will be explained in the following section.

5. Transitional Characteristics of Chézy Flows

The investigation of hydraulic characteristics of the Chézy flow was initiated by Bresse in 1860 and thereafter many refined treatments were completed by a large number of scientists and engineers. The study, however, is limited to the problem of hydraulic behaviours and the classification of surface profiles resulted from the local change of water elevation in a constant channel geometry. Homma first treated with the transitional behaviours of Chézy flow in wide channels by means of the rather empirical procedure without rigorous mathematical knowledge. The detailed discussion of the classification of transitional points and the hydraulic significance of transition flow for the resistance law of Chézy in wide channels, as an extension of the Bresse theory of gradually varied flows, is explained in the paper of the present author[9].

This section deals with the common transitional characteristics of Chézy flows in channel transitions and controls.

(1) *Characteristics of channel geometry and boundaries*

Before discussing the classification of transitional points and their hydraulic significance, the behaviours of coefficients of a, b, c and d in the variation equation is concerned.

Putting C=constant in the expressions of coefficients of variation equation, a, b, c and d become in the following.

$$\left.\begin{aligned} a &= -2i_c(\alpha-1)\Big\{\beta-2(\alpha-1)\Big\},\\ b &= 3\Big\{\beta-2(\alpha-1)\Big\},\\ c &= i_c^2\Big[(\alpha-1)\Big\{6\alpha(\alpha-1)-\beta(3\alpha-1)\Big\}+(m/3)\Big\{\beta-2(\alpha-1)\Big\}\Big],\\ d &= i_c\Big\{\beta(2\alpha+1)-4\alpha(\alpha-1)\Big\},\end{aligned}\right\} \tag{32}$$

in which α, β and m represent the channel and boundary characteristics.

The positive or negative domains of coefficient a are influenced by two curves of $\alpha=1$ and $\beta=2(\alpha-1)$ in the $\alpha-\beta$ plane. From Eq. (6) which describes the existence theorem of transitional point in channel transitions, the divergent channel must be of mild or adverse slope and the convergent channel steep in slope. Consequently, selecting α-axis as the abscissa and β-axis as the ordinate in the $\alpha-\beta$ plane, the possible domains for the existence of transitional points are the second and fourth quardrants in the plane of new coordinate system, of which ordinate and abscissa are located at $\alpha=1$ and $\beta=0$. Finally, it is understood that in all possible domains the sign of a is positive.

The characteristics of b in sign are readily determined and b is positive in divergent channels and negative in convergent channels.

The expression of d in Eq. (32) is transformed into

$$d = i_c(2\alpha+1)\Big\{\beta-4\alpha(\alpha-1)/(2\alpha+1)\Big\}, \tag{33}$$

and evidently, $\alpha=-0.5$ is a barrier of change in sign of d. The upper and lower half planes divided by the curve of $\beta=4\alpha(\alpha-1)/(2\alpha+1)$ become positive and negative domains, corresponding with $\alpha \gtrless -0.5$. The combination of the foregoing results and the existence theorem determines the behaviours of d in the following and Fig. 1 indicates the behaviours of a, b and d in terms of the bed slope and channel diversity.

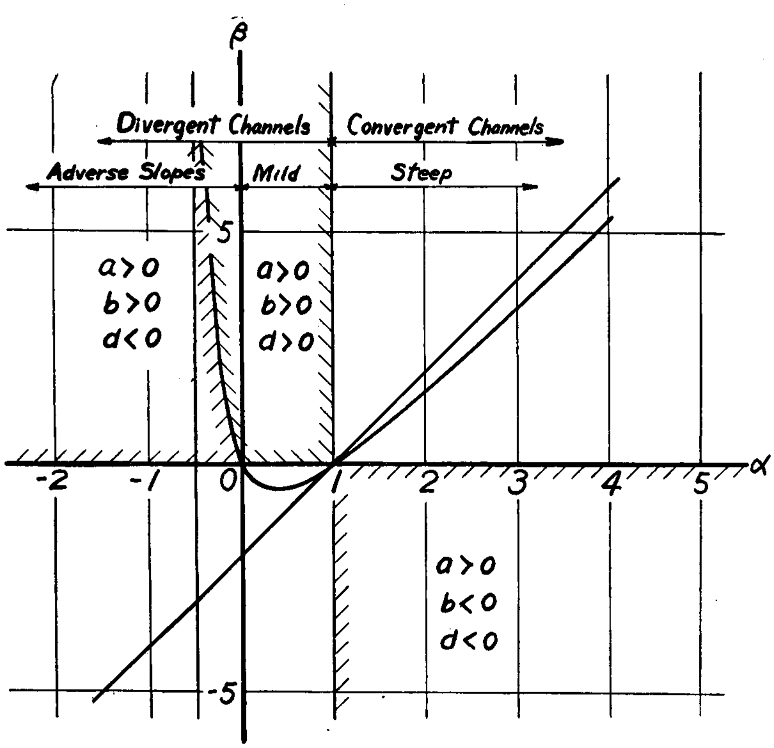


Fig. 1. Behaviours of Boundary Characteristics to a, b and d for Chézy flows.

(a) In steep channels, d is negative.
(b) In mild and gently adverse channels, d becomes positive.
(c) In adverse channels, d is commonly negative.

On the other hand, c is a function of α, β and m, so that the behaviour of c is expressible with the parameter of m in the $\alpha-\beta$ plane.

The expression of c is

$$c = -3i_c^2\left\{(\alpha-2/3)^2-(1+m)/9\right\}\left[\beta-\frac{2(\alpha-1)\left\{(\alpha-1/2)^2-(9+4m)/36\right\}}{(\alpha-2/3)^2-(1+m)/9}\right], \tag{34}$$

and the behaviour of c is classified as follows.

(a) $m \leqq -2.25$;

$$\left.\begin{aligned} c &= -3i_cM\left\{\beta-(2N/M)(\alpha-1)\right\}, \\ M &= (\alpha-2/3)^2-(1-m)/9>0, \\ N &= (\alpha-1/2)^2-(9-4m)/36\geqq 0. \end{aligned}\right\} \tag{35}$$

The negative domain of c in the $\alpha-\beta$ plane is the upper half plane divided by the curve of $\beta=(2N/M)(\alpha-1)$. Furthermore, $(2N/M)\gtreqless 0$ and $\beta=(2N/M)(\alpha-1)$ passes the point of $(1, 0)$. Consequently, c is negative for divergent channels and positive for convergent channels.

(b) $-1>m>-2.25$;

$$\left.\begin{aligned} c &= -3i_c^2 M\left\{\beta-2(\alpha-1)(\alpha-\alpha_3)(\alpha-\alpha_4)/M\right\}, \\ \alpha_3 &= \left\{1+\sqrt{1+(4m/9)}\right\}/2, \quad \alpha_4 = \left\{1-\sqrt{1+(4m/9)}\right\}/2. \end{aligned}\right\} \tag{36}$$

For the range of $-1>m>-2.25$, $1>\alpha_3>\alpha_4$, as seen in Fig. 2, which indicates the relation between α and m, so that it is evidently seen that c is positive for steep channels and mild channels in which β is of small value and negative for adverse and mild channels, as seen in Fig. 3.

(c) $0>m\geqq-1$;

$$\left.\begin{aligned} c &= -3i_c^2(\alpha-\alpha_1)(\alpha-\alpha_2)\left\{\beta-\frac{2(\alpha-1)(\alpha-\alpha_3)(\alpha-\alpha_4)}{(\alpha-\alpha_1)(\alpha-\alpha_2)}\right\}, \\ \alpha_1 &= (2+\sqrt{1+m})/3, \quad \alpha_2 = (2-\sqrt{1+m})/3. \end{aligned}\right\} \tag{37}$$

Evidently in Fig. 2, $1>\alpha_3>\alpha_1\geqq\alpha_2>\alpha_4>0$, and therefore, for $\alpha>\alpha_1$ and $\alpha<\alpha_2$, c is positive in the lower half plane divided by the curve of β in Eq. (37) and negative in the upper half plane. On the other hand, for $\alpha_1>\alpha>\alpha_2$ the upper plane is the positive domain of c. As seen in Fig. 4, the positive domain for mild channels increases gradually with the increase of m.

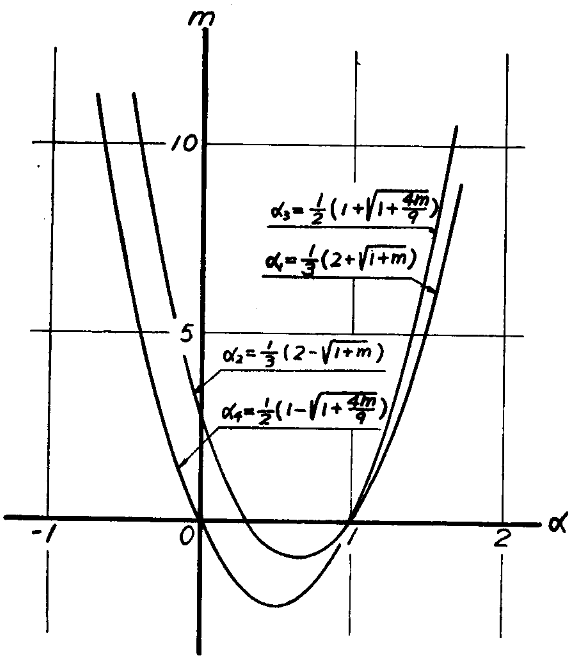


Fig. 2. Relation between α and m.

(d) $m\geqq 0$;

For positive values of m, the behaviour of c is also expressed by Eq. (37). The order of magnitude of α, however, is changed and it is expressible in a form of

$$\alpha_1>\alpha_3\geqq 1>\alpha_2>\alpha_4.$$

Consequently, both positive and negative domains of c exist for steep and adverse channels and especially, in the case of $m > 3.0$, in mild channels, c is positive.

The behaviours of slopes of critical depth and normal flow curves at the transitional point, s_1 and s_2, are also determined by the channel characteristics by means of a, b, c and d. As the value of s_2 is independent of m, so it is uniquely determined by the channel characteristics in α and β, and for steep channels s_2 is positive and negative for mild and adverse channels. On the other hand, s_1 is influenced by m and the common characteristics are described in the following.

(a) For steep channels, s_1 is positive in cases of negative values of m. The negative domain of s_1, however, exists in the neighbourhood of critical slope in case of positive values of m and it increases with the increase of m.

(b) For mild channels, s_1 is positive in the case of $m < -2.25$. The negative domain developes with the increase of m and finally it is always negative in the case of $m \geqq 3.0$.

(c) For adverse channels, the magnitude of adverseness divides into two characteristics in sign of s_1. For gently adverse channels, s_1 indicates the same behaviour as for mild channel and conversely the opposite behaviour is made for steeply adverse

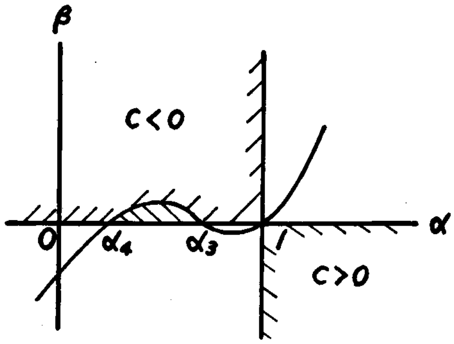


Fig. 3. Behaviour of Boundary Characteristics to Sign of c in Chézy Flows ($-1 > m > -2.25$).

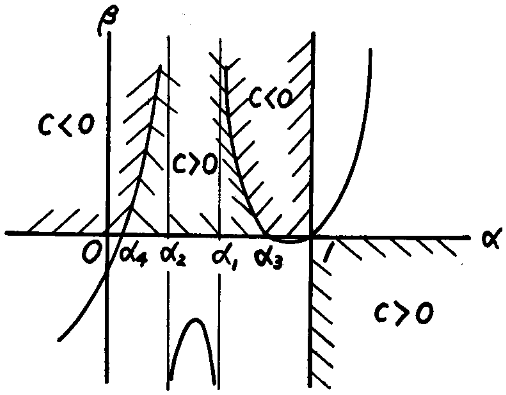


Fig. 4. Behaviour of Boundary Characteristics to Sign of c in Chézy Flows ($0 > m \geqq -1$).

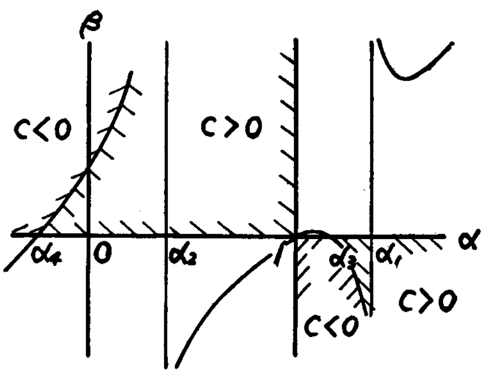
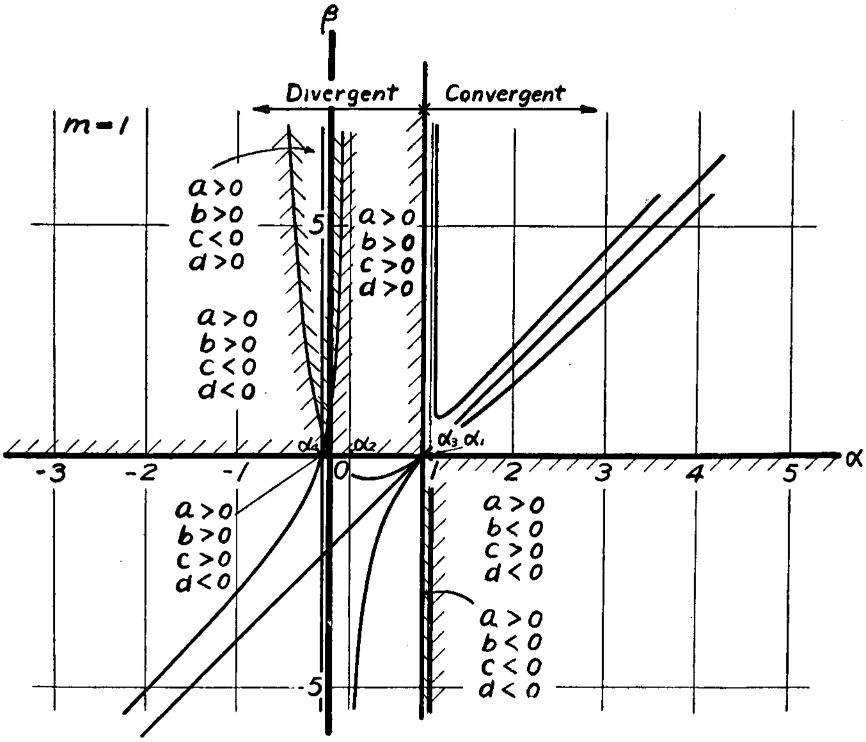
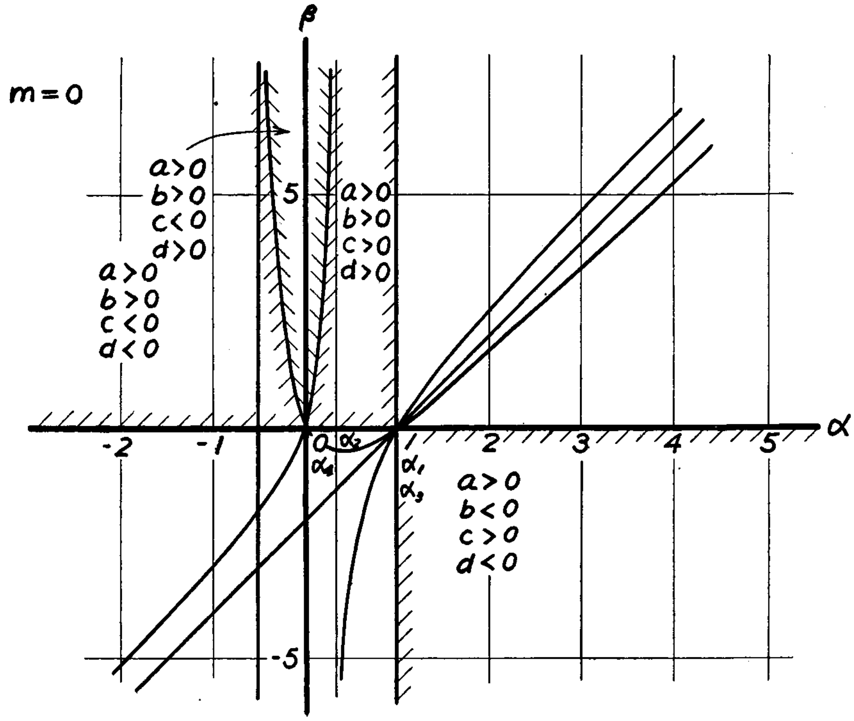


Fig. 5. Behaviour of Boundary Characteristics to Sign of c in Chézy Flows ($m \geqq 0$).


β
Divergent
Convergent
m=1
a>0
b>0
c<0
d>0
a>0
b>0
c>0
d>0
a>0
b>0
c<0
d<0
α4 α2 α3 α1
-3 -2 -1 0 1 2 3 4 5 α
5
-5
a>0
b>0
c>0
d<0
a>0
b<0
c>0
d<0
a>0
b<0
c<0
d<0


β
m=0
a>0
b>0
c<0
d>0
a>0
b>0
c>0
d>0
a>0
b>0
c<0
d<0
5
-2 -1 0 α2 1 2 3 4 5 α
α4 α1 α3
a>0
b<0
c>0
d<0
-5

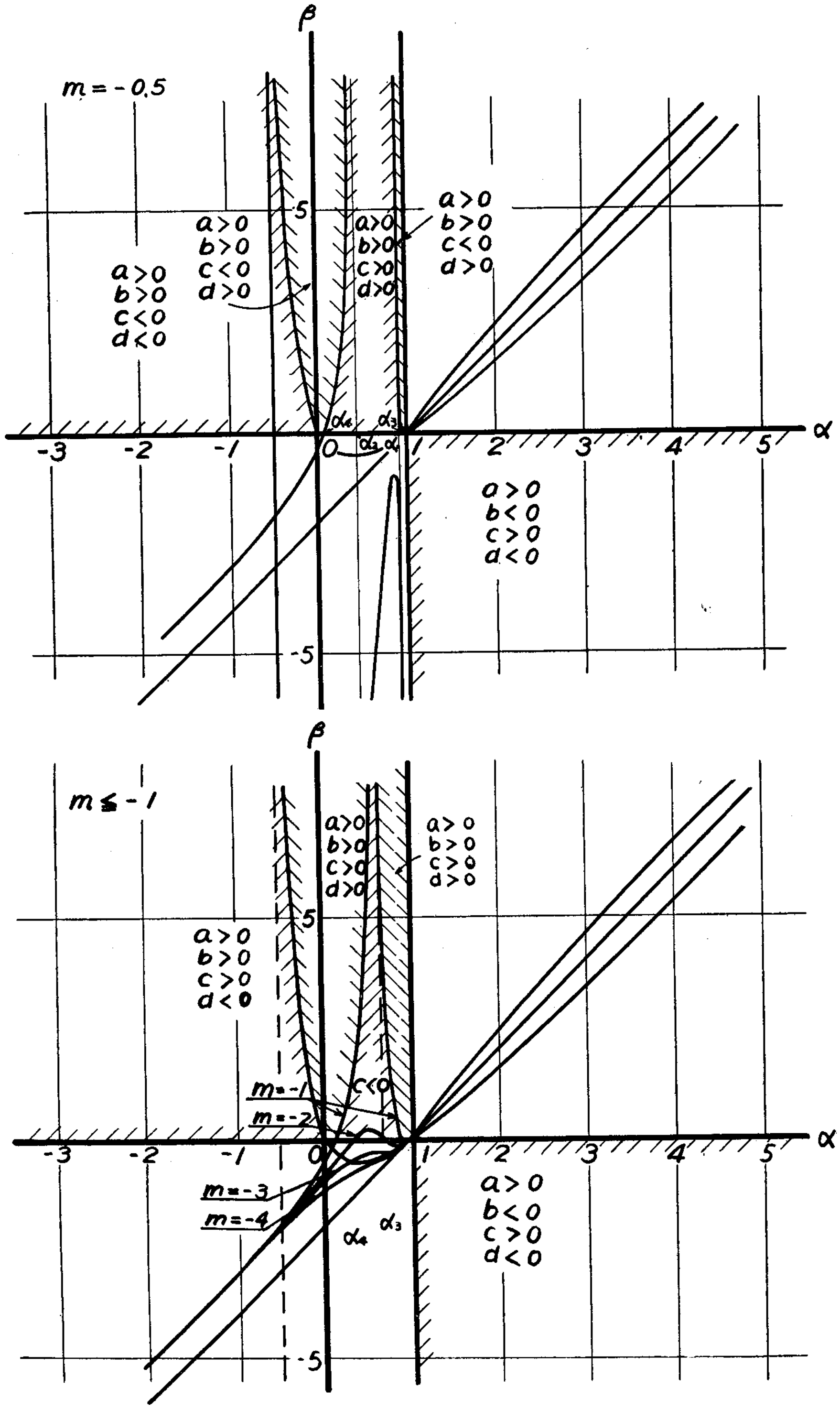


Fig. 6. Behaviours of Boundary Characteristics to a, b, c and d for Chézy flows; (i) $m=1$, (ii) $m=0$, (iii) $m=-0.5$, (iv) $m \leqq -1$.

channels. Fig. 6 indicates some examples of behaviours of boundary characteristics to a, b, c and d for Chézy flows.

The geometric property between the normal depth curve and the critical depth curve at the transitional point is determined by $(ad-bc)$ and bd. The sign of $(ab-bc)$ classifies the singular point and therefore, the sign of bd formulates the geometric relation between both curves. Fig. 7 indicates the positive and negative domains of the product bd. Evidently in inspection of Fig. 7, $bd>0$ for steep, mild and gently adverse channels, so that the normal depth curve passes through the upper plane

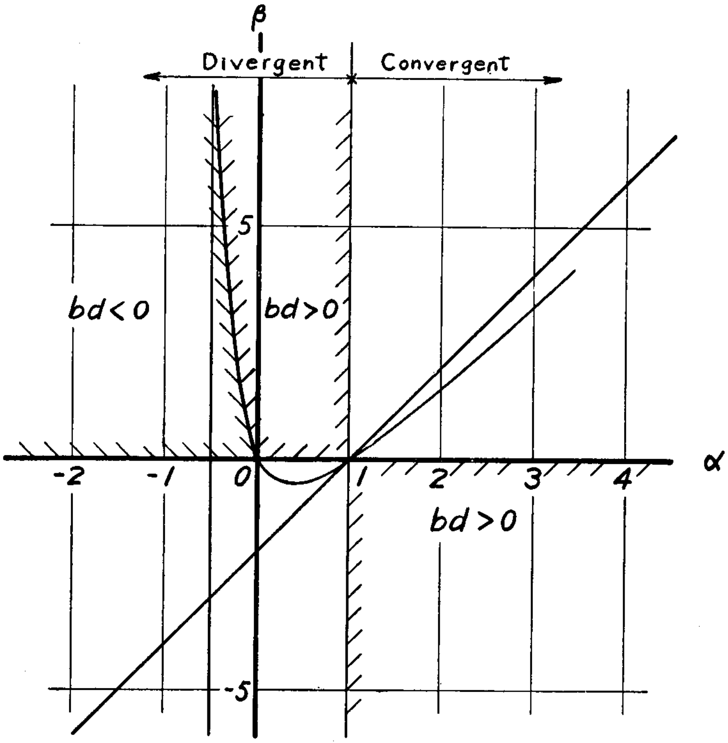


Fig. 7. Behaviours of Boundary Characteristics to Sign of bd.

divided by the critical depth curve and approaches the saddle point, while it passes through the lower plane and approaches the nodal and focal points. On the contrary, for steeply adverse channels, bd becomes negative, and the opposite relations between both curves are obtained.

(2) *Classification of transitional points*

As the characteristic equation in terms of the channel characteristics in geometry and boundary of Chézy flows is derived in a form of

$$S^2-i_c\left\{3\beta-4(\alpha-1)\right\}S+i_c^2\left\{\beta-2(\alpha-1)\right\}\left[5(\alpha-1)^2(\beta-2\alpha)-m\left\{\beta-2(\alpha-1)\right\}\right]=0, \tag{38}$$

so the classification of transitional points as focal, or saddle and nodal is obtained by the sign of discriminant of Eq. (38). The discriminant is

$$D=-20i_c^2\left(\alpha^2-2\alpha+\frac{11-4m}{20}\right)\left\{\beta^2-\frac{8(\alpha-1)(10\alpha^2-15\alpha+2-2m)}{(20\alpha^2-40\alpha+11-4m)}\beta + \frac{16(\alpha-1)^2(5\alpha^2-5\alpha-1-m)}{(20\alpha^2-40\alpha+11-4m)}\right\}\gtrless 0. \tag{39}$$

Or, transforming Eq. (39) becomes

$$D=-20i_c^2\left\{(\alpha-1)^2-\frac{9+4m}{20}\right\}\left[\left[\beta-\frac{(\alpha-1)(10\alpha^2-15\alpha+2-2m)}{5\left\{(\alpha-1)^2-\frac{9+4m}{20}\right\}}\right]^2 - \frac{24(\alpha-1)^2\left\{\alpha-\frac{3+\sqrt{1+(8m/15)}}{4}\right\}\left\{\alpha-\frac{3-\sqrt{1+(8m/15)}}{4}\right\}}{\left\{(\alpha-1)^2-\frac{9+4m}{20}\right\}}\right]\gtrless 0. \tag{40}$$

It is, therefore, seen that the sign of the discriminant depends on the behaviours of α_1, α_2, α_3 and α_4 curves expressed as

$$\alpha_1=1+\sqrt{(9+4m)/20}, \qquad \alpha_2=1-\sqrt{(9+4m)/20},$$
$$\alpha_3=\left\{3+\sqrt{1+(8m/15)}\right\}/4, \quad \alpha_4=\left\{3-\sqrt{1+(8m/15)}\right\}/4.$$

The classification of positive and negative domains of the discriminant is finally in the following.

(a) $m<-2.25$;

Two roots of α_1 and α_2 are conjugate complex and thus the first bracket of Eq. (40) becomes positive and the second bracket has real roots of

$$\left.\begin{matrix}\beta_1\\ \beta_2\end{matrix}\right\}=\frac{\alpha-1}{5\left\{(\alpha-1)^2-(4m+9)/20\right\}}\left[(10\alpha^2-15\alpha+2-2m)\pm\sqrt{30\left\{(\alpha-3/4)^2-(15+8m)/240\right\}}\right].$$

Consequently, the inner domain between $\beta_1(\alpha, m)$ curve and $\beta_2(\alpha, m)$ curve in the $\alpha-\beta$ plane makes the discriminant positive and therefore the transitional point is classified as saddle or nodal while the outer domain yields the transitional point focal.

(b) $-1.875>m\geq-2.25$;

For this range of m, the discriminant becomes

$$D = -20i_c(\alpha-\alpha_1)(\alpha-\alpha_2)(\beta-\beta_1)(\beta-\beta_2)\,,$$

and consequently, for $\alpha>\alpha_1$ and $\alpha<\alpha_2$, the same conclusion as in the case of (a) is obtained. On the contrary, for $\alpha_1>\alpha>\alpha_2$, in the outer domain between both curves of β_1 and β_2, D becomes positive and the domain in which D is negative is the inner part.

(c) $m \geqq -1.875$;

For the range of $m \geqq -1.875$, the characteristics of β depend on the values of α, and furthermore the relation between α and m in the boundary characteristics is seen in Fig. 8, so that the classification of D in sign is as follows.

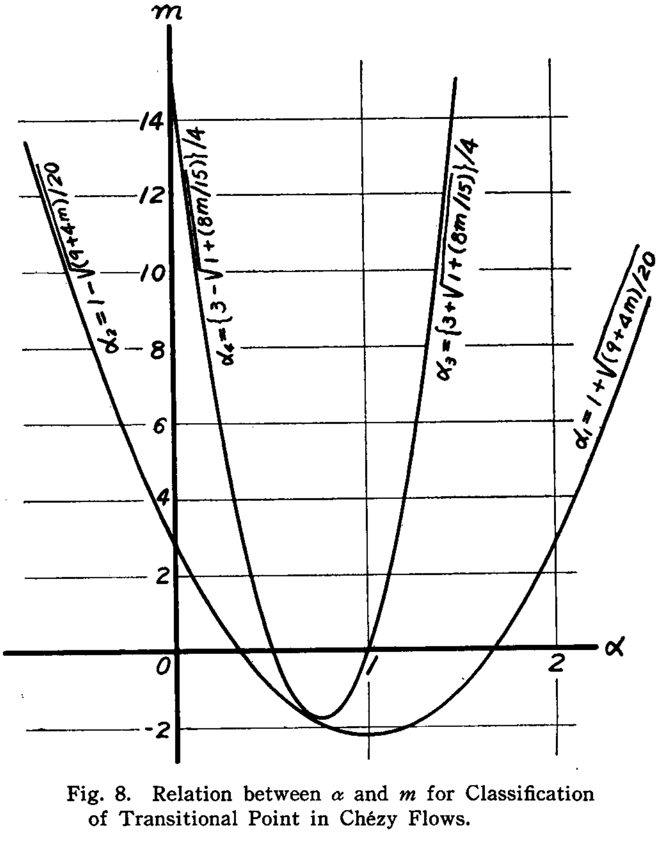


Fig. 8. Relation between α and m for Classification of Transitional Point in Chézy Flows.

As $\alpha_1>\alpha_3>\alpha_4>\alpha_2$, so for $\alpha \geqq \alpha_1$ and $\alpha \leqq \alpha_2$, the second bracket has

$$\begin{matrix}\beta_3\\ \beta_4\end{matrix} = \frac{\alpha-1}{5(\alpha-\alpha_1)(\alpha-\alpha_2)}\left\{(10\alpha^2-15\alpha+2-2m)\pm\sqrt{30(\alpha-\alpha_3)(\alpha-\alpha_4)}\right\}$$

and the inner domain makes D positive for $\alpha>\alpha_1$ and $\alpha<\alpha_2$, and also D becomes positive in the outer domain for $\alpha_4>\alpha>\alpha_2$ and $\alpha_1>\alpha>\alpha_3$.

On the other hand, for $\alpha_3 \geqq \alpha \geqq \alpha_4$, the first bracket is negative and the second one positive, and therefore D is always positive without the change of diversity in channel geometry.

The discrimination between saddle and nodal points is obtained by the sign of two real roots of the characteristic equation, and it is indicated as follows.

$$i_c^2\left\{\beta-2(\alpha-1)\right\}\left[5(\alpha-1)^2(\beta-2\alpha)-m\left\{\beta-2(\alpha-1)\right\}\right] \gtrless 0, \tag{41}$$

in which the lower and upper inequalities indicate the condition for saddle and nodal points respectively. Eq. (41) is also transformed into

$$5i_c^2\left\{(\alpha-1)^2-m/5\right\}\left\{\beta-2(\alpha-1)\right\}\left[\beta-2(\alpha-1)(\alpha^2-\alpha-m/5)\Big/\left\{(\alpha-1)^2-m/5\right\}\right] \gtrless 0, \tag{42}$$

and the classification between saddle and nodal points is described in the following.

(i) For $m \geqq 0$, the transitional point is classified as the saddle point in the inner domain enclosed by two curves of $\beta=2(\alpha-1)$ and $\beta=2(\alpha-1)(\alpha^2-\alpha-m/5)/\{\alpha-(1+\sqrt{m/5})\}\{\alpha-(1-\sqrt{m/5})\}$ for the range of $\alpha>(1+\sqrt{m/5})$ and $\alpha<(1-\sqrt{m/5})$ and the outer domain for $(1+\sqrt{m/5})>\alpha>(1-\sqrt{m/5})$. The classification for nodal points is obtained under the opposite condition of foregoing conclusion.

(ii) For the negative values of m, the first bracket of Eq. (42) is positive for any value of α, and thus the inner and outer domains represent the condition for saddle and nodal, respectively.

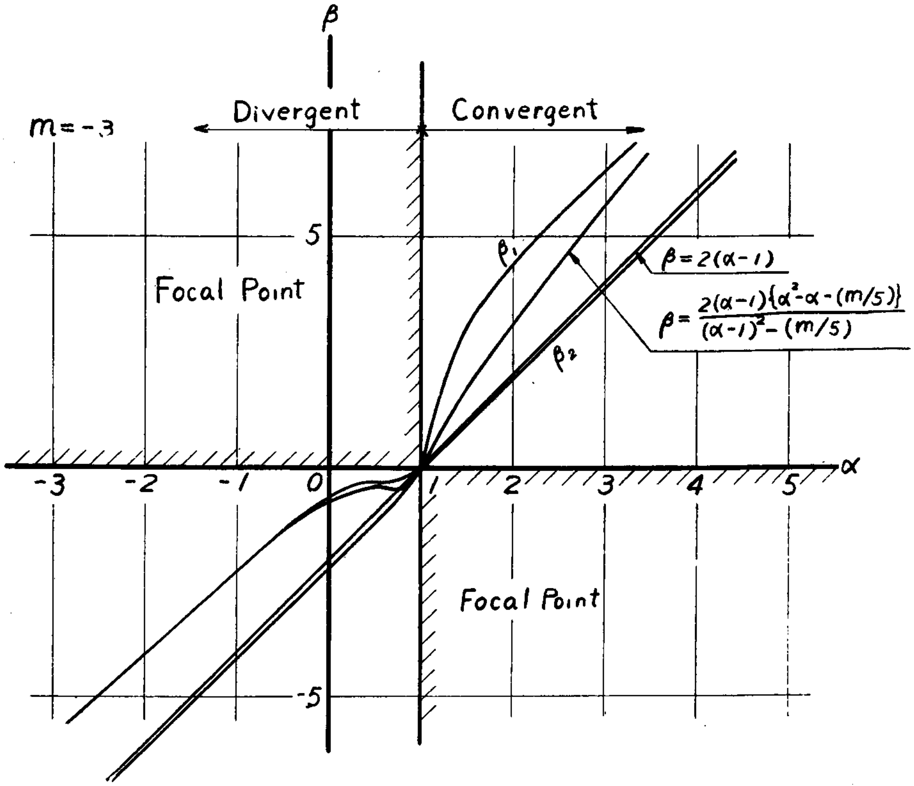
Following the foregoing conclusion for the classification of transitional points in channel transitions and controls, the retationship between the classification of transitional points and the boundary characteristics in channels are obtained, and Fig. 9 describes some examples of the hydraulic behaviours of boundary characteristics to transitional characteristics for Chézy flows.

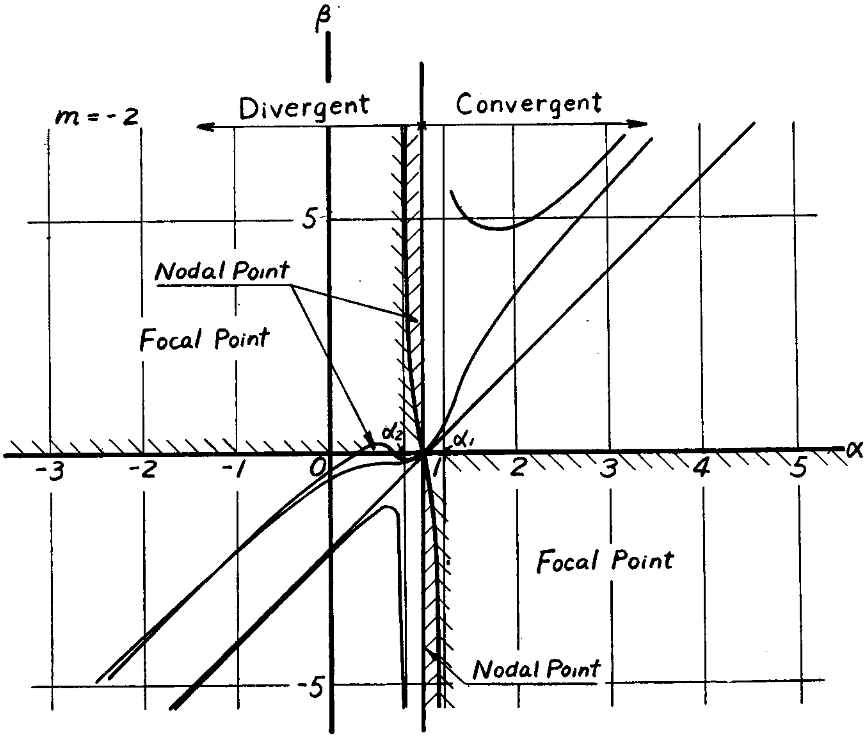
(3) *Common Properties of Transitional Characteristics*

When the transitional point is classified as saddle or nodal, the slope of transition profile at the point is uniquely evaluated by Eqs. (23) and (24). When the slope of transition curve at the transitional point is assumed to be $(dh/dx)_{c1}$, the sign of this slope depends on b, $(a-d)$ and $\sqrt{(a-d)^2+4bc}$. Making the subtraction of d from a with the aid of Eq. (32), it becomes

$$a-d = -i_c(4\alpha-1)\left\{\beta-(\alpha-1)(2\alpha-1)/(4\alpha-1)\right\}. \tag{42}$$

The behaviours of $(a-d)$ in sign, hence, is divided into two different domains by the straight line, $\alpha=0.25$, and for $\alpha>0.25$, the upper domain of $\beta=(\alpha-1)(2\alpha-1)/(4\alpha-1)$ makes $(a-d)$ negative and for $\alpha<0.25$, $(a-d)$ becomes positive in the same


β
m=-3
Divergent
Convergent
5
Focal Point
β1
β=2(α-1)
β= 2(α-1){α²-α-(m/5)} / (α-1)²-(m/5)
β2
-3
-2
-1
0
1
2
3
4
5
α
Focal Point
-5


β
m=-2
Divergent
Convergent
5
Nodal Point
Focal Point
α2
α1
-3
-2
-1
0
1
2
3
4
5
α
Focal Point
Nodal Point
-5

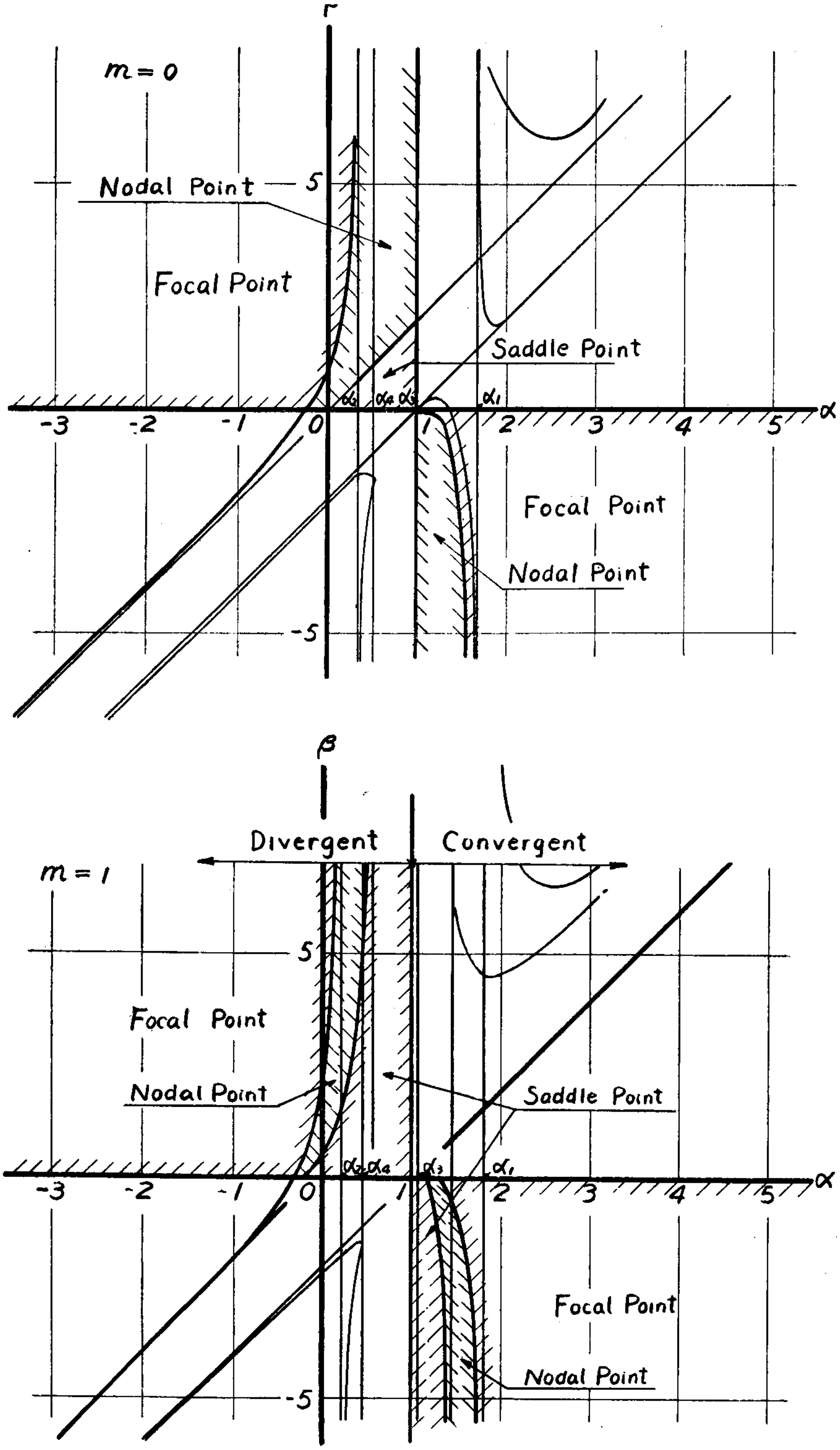


Fig. 9. Hydraulic Behaviours of Boundary Characteristics to Transitional Characteristics for Chézy Flows; (i) $m=-3$, (ii) $m=-2$, (iii) $m=0$, (iv) $m=1$.

domain. On the other hand, $\sqrt{(a-d)^2+4bc}$ is positive, and b is positive for divergent channels and negative for convergent channels. Finally the sign of $(dh/dx)_{c1}$ is classified as follows, depending on the channel characteristics.

For steep channels, $\sqrt{(a-d)^2+4bc}$ and $(a-d)$ are positive and b is negative, and thus $(dh/dx)_{c1}$ becomes positive. For mild slopes, $\sqrt{(a-d)^2+4bc}$ and b are positive, while $(a-d)$ is negative, and thus $(dh/dx)_{c1}$ becomes negative or positive, depending on the magnitudes of $(a-d)$ and $\sqrt{(a-d)^2+4bc}$. For mild channels, which are of slopes less than 0.25, and adverse channels, $\sqrt{(a-d)^2+4bc}$, b and $(a-d)$ are positive, and thus $(dh/dx)_{c1}$ is negative. Fig. 10 indicates the behaviour of $(dh/dx)_{c1}$ in sign.

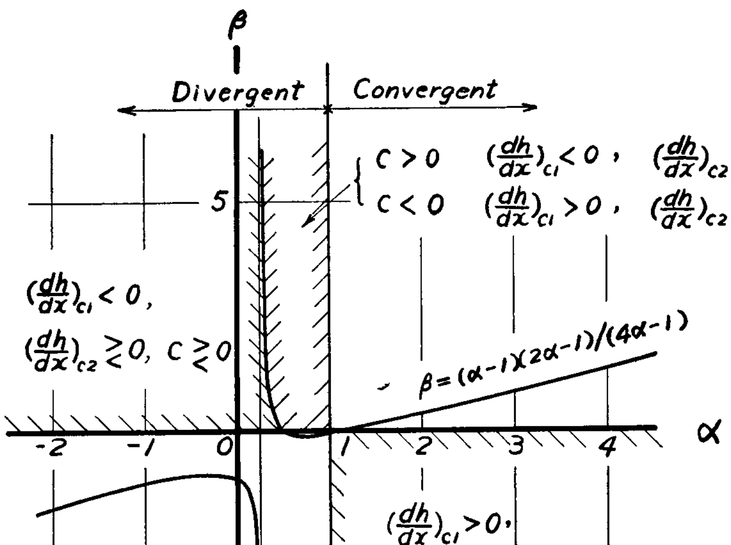


Fig. 10. Relation between Boundary Characteristics and Sign of Slope of Transition Curve.

The sign of the other slope of transition curve, $(dh/dx)_{c2}$, is readily determined. The product of both slopes is $-(c/b)$. If c is positive, $(dh/dx)_{c1}$ and $(dh/dx)_{c2}$ are of opposite sign for divergent channels and of same sign for convergent channels. On the contrary, if c is negative, $(dh/dx)_{c1}$ and $(dh/cx)_{c2}$ are of same sign for divergent channels and of opposite sign for convergent channels. Fig. 10 also indicates the behaviours of $(dh/dx)_{c2}$.

The hydraulic significance of transitional behaviours of flow is imposed by the change of flow regime in transition flows. The change of flow regime at the saddle point has been already discussed in the foregoing section and the brief summary is in the following. When $(dh/dx)_{c1}$ is assumed to be the slope of transition curve, the flow in divergent channels changes from subcritical to supercritical and in convergent channels the transition in flow regime from supercritical to subcritical is produced. On the other hand, $(dh/dx)_{c2}$ as the transition slope reduces to the opposite behaviours of transition in flow regime, and the possible transition slope between $(dh/dx)_{c1}$ and $(dh/dx)_{c2}$ is determined by the basic principle of open channel flows, in which the subcritical flow must be traced from the downstream end and the supercritical flow from the upstream end.

The transitional behaviours of flow by the nodal point is followed by the sign of $(a+d)$ already indicated. As $(a+d)$ for Chézy flows becomes in a form of

$$a+d = 3i_c\left\{\beta-4(\alpha-1)/3\right\},$$

so $(a+d)$ and thus S_1 become negative for steep slopes and positive for mild and adverse slopes. Consequently, keeping in mind that b is positive for divergent channels and negative for steep channels, the flow regime is changed from supercritical to subcritical for any characteristic in channel geometry and boundary.

(4) *Slope of surface profile near transitional point*

The critical depth curve divides the domain into the upper one, in which (dh/dx) is positive, and the lower one of negative values in slope of surface profile. The behaviour of normal depth curve in the vicinity of the transitional point is described by Eq. (31) and it is transformed into, for Chézy flows,

$$f_1(x_c, h) = g\sin\theta\left\{(t-1)/t^3\right\}\left[t^2+t+\beta\Big/\alpha\left\{\beta-2(\alpha-1)\right\}\right],$$

in which $t=h/h_c$. The positive and negative domains of $f_1(x_c, h)$ are characterized by the critical depth $t_3=1$, and two roots of the second bracket, t_1 and t_2, which are real or conjugate complex. Mathematical characteristics of the second bracket are classified in the following.

(a) For steep and mild slopes: The outer domain enclosed by two curves of $\beta=2(\alpha-1)$ and $\beta=2\alpha(\alpha-1)/(\alpha-4)$ for $\alpha>4$ and the inner domain for $4>\alpha>0$ yield two real roots and the larger root, t_1, is less than unity.

(b) For adverse slopes: The second bracket has two real roots in the outer domain enclosed by $\beta=2(\alpha-1)$ and $\beta=2\alpha(\alpha-1)/(\alpha-4)$ and conjugate complexes in the inner domain. Furthermore, t_1 is less than unity in the inner domain for $0>\alpha>-0.5$ and in the outer domain for $\alpha<-0.5$.

The behaviour of $f_1(x_c, h)$ is, thus, in the following as seen in Fig. 11.

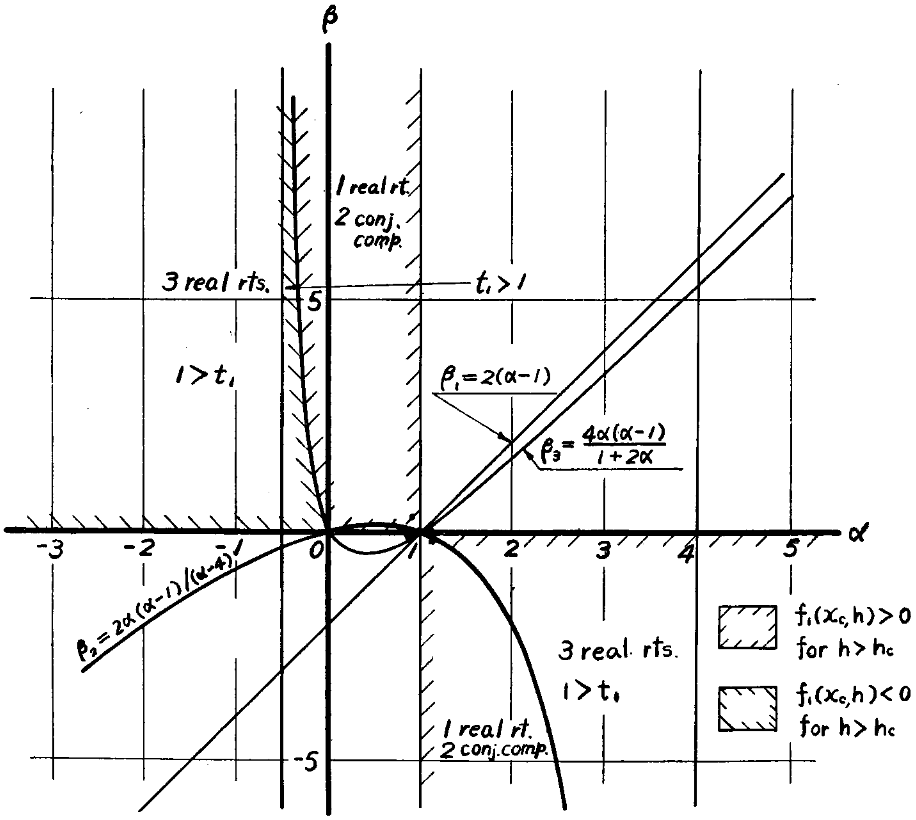


Fig. 11. Behaviour of $f_1(x_c, h)$ to Boundary Characteristics.

The behaviour of $f_1(x_c, h)$ is finally characterized by the following conclusion. The upper domain divided by the critical depth curve makes $f_1(x_c, h)$ positive for steep, mild and gently adverse slopes and negative for steeply adverse slopes.

Consequently, the schematic figures of surface profiles traced under various boundary conditions are readily drawn in the vicinity of transitional points, and the transitional behaviours of flow are also determined.

(5) *Surface profiles of flow and transition curve in the vicinity of transitional point of Chézy flows*

As the conclusive characteristics of transitional behaviours of flows at and in the vicinity of transitional points in channel transitions and controls, the following table and figures are presented by the combined features of geometrical and boundary characteristics in channels.

Table 1 indicates the transitional characteristics of Chézy flows produced by the saddle point and Figs. 12 (i)–12 (v) also describe surface profiles involving the transition curve in particular cases of channel characteristics.

Table 1. Transitional Characteristics of Chézy Flows at Saddle Point.

Transitional Character-istics	a	b	c	d	s_1	s_2	$\left(\frac{dh}{dx}\right)_{c1}$	$\left(\frac{dh}{dx}\right)_{c2}$	Slope of T.C.	(dh/dx)		Transitional Feature	Remarks
										Inner	Outer		
Steep Slopes	+	−	+	−	+	+	+	+	$(dh/dx)_{c2}$	−	+	Fig. 12 (i)	
	+	−	−	−	−	+	+	−	"	−	+	(ii)	
Mild and Gently Adverse Slopes	+	+	+	+	−	−	−	+	$(dh/dx)_{c1}$	−	+	(iii)	
	+	+	−	+	+	−	+	+					$ad-bc \nless 0$
	+	+	−	+	+	−	−	−					"
Adverse Slopes	+	+	+	−	+	−	−	+	$(dh/dx)_{c1}$	+	−	(iv)	
	+	+	−	−	−	−	−	−	"	+	−	(v)	

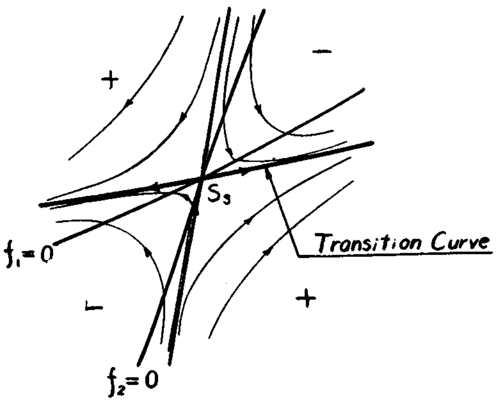


Fig. 12 (i). Surface Profiles of Transition Flows near Saddle Point in Steep Slopes.

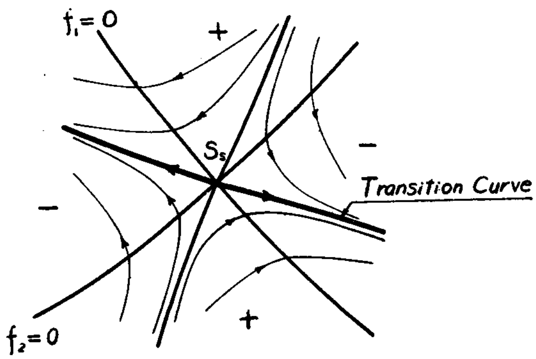


Fig. 12 (ii). Surface Profiles of Transition Flows near Saddle Point in Steep Slopes.

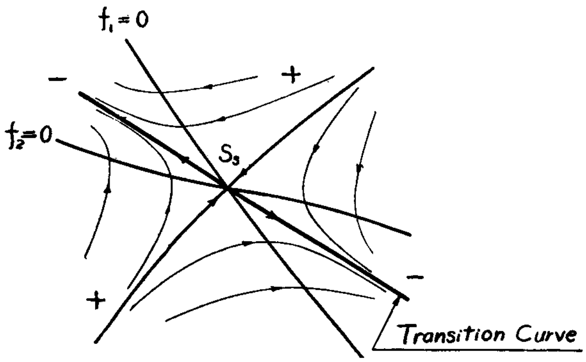


Fig. 12 (iii). Surface Profiles of Transition Flows near Saddle Point in Mild and Gently Adverse Slopes.

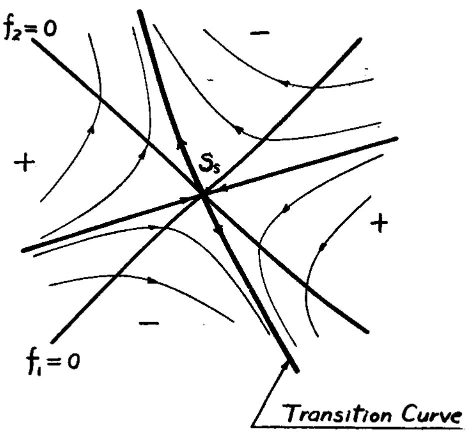


Fig. 12 (iv). Surface Profiles of Transition Flows near Saddle Point in Adverse Slopes.

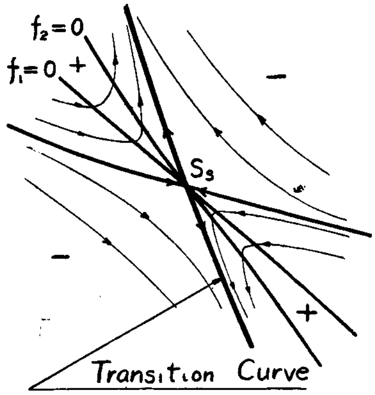


Fig. 12 (v). Surface Profiles of Transition Flows near Saddle Point in Adverse Slopes.

Table 2 indicates the same characteristics at the nodal point and Figs. 12 (vi)–12 (x) describe surface profiles. The sign of smaller root of the characteristic equation, S_1, is also indicated in the remarks.

Table 2. Transitional Characteristics of Chézy Flows at Nodal Point.

Transitional Characteristics	a	b	c	d	s_1	s_2	$\left(\frac{dh}{dx}\right)_{c1}$	(dh/dx) Inner	(dh/dx) Outer	Transitional Feature	Remarks
Steep Slopes	+	−	+	−	+	+	+	−	+	Fig. 12 (vi)	$S_1<0$
	+	−	−	−	−	+	+				$ad-bc \not> 0$
Mild and Gently Adverse Slopes	+	+	+	+	−	−	−	−	+	(vii)	$S_1>0$
	+	+	−	+	+	−	+	−	+	(viii)	$S_1>0$
	+	+	−	+	+	−	−	−	+	(ix)	$S_1>0$
Adverse Slopes	+	+	+	−	+	−	−				$ad-bc \not> 0$
	+	+	−	−	−	−	−	+	−	(x)	$S_1>0$

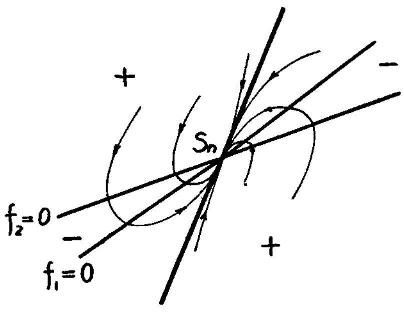


Fig. 12 (vi). Surface Profiles of Transition Flows near Nodal Point in Steep Slopes.

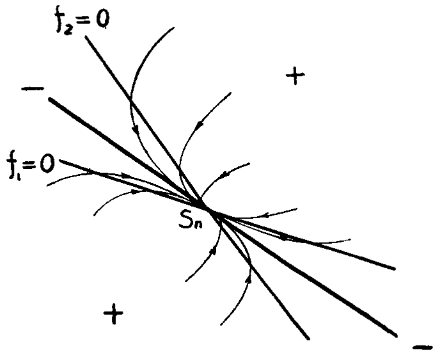


Fig. 12 (vii). Surface Profiles of Transition Flows near Nodal Point in Mild and Gently Adverse Slopes.

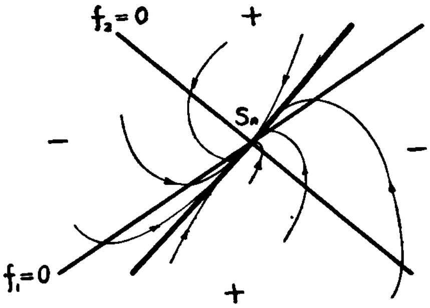


Fig. 12 (viii). Surface Profiles of Transition Flows near Nodal Point in Mild and Gently Adverse Slopes.

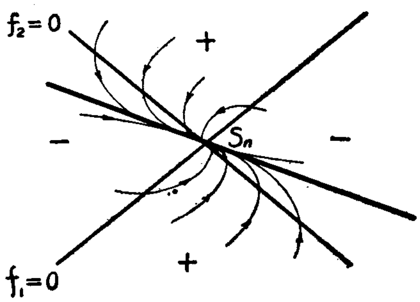


Fig. 12 (ix). Surface Profiles of Transition Flow near Nodal Point in Mild and Gently Adverse Slopes.

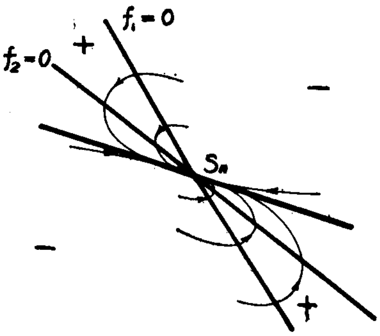


Fig. 12 (x). Surface Profiles of Transition Flows near Nodal Point in Steeply Adverse Slopes.

Table 3 indicates the hydraulic characteristics of transition flow at the focal point and Figs. 12 (xi)–12 (xiii) describe surface profiles near the point. As seen in the foregoing section, surface profiles traced from up- and downstream ends never pass the focal point and therefore the hydraulic jump is produced under the condition that both depths of upstream and downstream become sequent.

Table 3. Transitional Characteristics of Chézy Flows at Focal Point.

Transitional Characteristics	a	b	c	d	s_1	s_2	Transitional Feature	Remarks	
								$ad-bc$	Discriminant
Steep Slopes	+ +	− −	+ −	− −	+ −	+ +	Fig. 12 (xi)	$ad-bc<0$	$D>0$
Mild and Gently Adverse Slopes	+ +	+ +	+ −	+ +	− +	− −	(xii)		$D>0$
Adverse Slopes	+ +	+ +	+ −	− −	+ −	− −	(xiii)	$ad-bc<0$	$D>0$

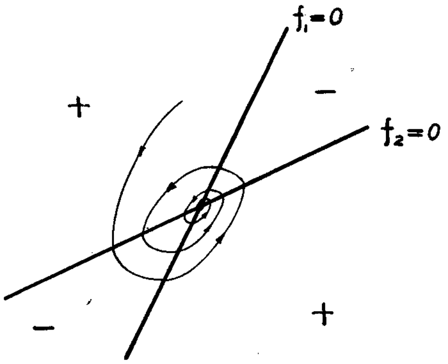


Fig. 12 (xi). Surface Profiles of Transition Flows near Focal Point in Steep Slopes.

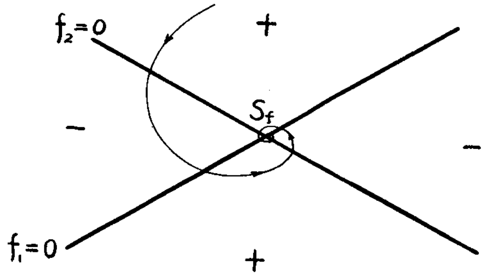


Fig. 12 (xii). Surface Profiles of Transition Flows near Focal Point in Mild and Gently Adverse Slopos.

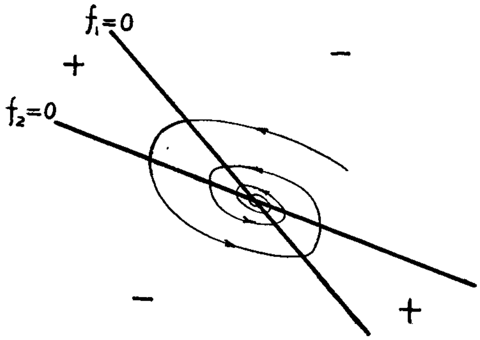


Fig. 12 (xiii). Surface Profiles of Transition Flows near Focal Point in Adverse Slopes.

In figures, the arrow indicates the direction of tracing of surface profiles in both regimes of sub- and supercritical flows.

Local changes in channel geometry and boundary resistance as channel transitions and controls are involved in natural channels and even in artificial water courses, and a combined feature of surface profile determined by the foregoing transitional characteristics for particular discharge and channel characteristics will become an actual transition curve of flow. The detailed behaviour of transition curve will be seen in some examples of the later section.

6. Transitional Characteristics of Manning Flows

The resistance law of Manning is more available for the open channels flows and, especially, the flow in alluvial channels than that of Chézy. In this section, therefore, the transitional characteristics of Manning flows are treated. Hydraulic characteristics of Manning flows, however, are commonly similar to those of Chézy flows, and consequently the summary of analysis will be described.

The Manning roughness n is closely related to the Chézy roughness C through the relation of $C=R^{1/6}/n$, and

$$(h_c/C_c)(\partial C/\partial h)_c = (h_c/C_c)(\partial C/\partial R)_c(\partial R/\partial h)_c = \beta\Big/6\Big\{\beta-2(\alpha-1)\Big\}. \tag{43}$$

In the flow of Manning, therefore, a, b, c and d bocome

$$\left.\begin{aligned} a &= -2i_c(\alpha-1)\Big\{\beta-2(\alpha-1)\Big\}, \\ b &= 3\Big\{\beta-2(\alpha-1)\Big\}, \\ c &= i_c^2\Big[(\alpha-1)\Big\{6\alpha(\alpha-1)-\beta(3\alpha-1)\Big\}+(m/3)\Big\{\beta-2(\alpha-1)\Big\}\Big], \\ d &= i_c\Big\{\beta(2\alpha+4/3)-4\alpha(\alpha-1)\Big\}. \end{aligned}\right\} \tag{44}$$

The characteristics of c are different from those in Chézy flows, as seen in Eq. (44). $\alpha=-0.667$ is a barrier of change in sign from positive to negative or vice versa for Manning flows, compared with $\alpha=-0.5$ for Chézy flows.

The discriminant of characteristic equation is also

$$D = -\frac{i_c^2}{9}(45\alpha^2-96\alpha+26-9m)\Big\{\beta^2-\frac{6(\alpha-1)(30\alpha^2-47\alpha+7-6m)}{(45\alpha^2-96\alpha+26-9m)}\beta + \frac{36(\alpha-1)^2(5\alpha^2-5\alpha-1-m)}{(45\alpha^2-96\alpha+26-9m)}\Big\}. \tag{45}$$

Consequently, the positive domain of discriminant is classified in the following.

(a) From $m<-2.80$;

The inner domain enclosed by two curves of β_1 and β_2 expressed as

$$\begin{matrix}\beta_1\\ \beta_2\end{matrix} = \frac{\alpha-1}{15\Big\{\Big(\alpha-\frac{16}{15}\Big)^2-\frac{126+45m}{225}\Big\}}\Big[(30\alpha^2-47\alpha+7-6m)\pm\sqrt{369\Big\{\Big(\alpha-\frac{29}{41}\Big)^2-\frac{53136+26896m}{620289}\Big\}}\Big],$$

makes D positive.

(b) For the range of $-1.9756>m\geqq-2.80$, the inner domain for $\alpha>\alpha_1$ and $\alpha<\alpha_2$ and the outer domain for $\alpha_1>\alpha>\alpha_2$ make D positive. The expressions of α_1 and α_2 are in forms of

$$\begin{matrix}\alpha_1\\ \alpha_2\end{matrix} = (1/15)(16\pm3\sqrt{14+5m}).$$

(c) When m is equal to and larger than -1.9756, putting

$$\begin{matrix}\alpha_3\\ \alpha_4\end{matrix} = (1/41)\Big\{29\pm\sqrt{144+(656m/9)}\Big\}$$

and

$$\begin{matrix}\beta_1\\ \beta_2\end{matrix} = \frac{\alpha-1}{15(\alpha-\alpha_1)(\alpha-\alpha_2)}\left\{(30\alpha^2-47\alpha+7-6m)\pm\sqrt{369(\alpha-\alpha_3)(\alpha-\alpha_4)}\right\},$$

the positive domains are the inner part of β_1 and β_2 curves for $\alpha_1 \leqq \alpha$ and $\alpha \leqq \alpha_2$ and the outer part of two curves for $\alpha_4 > \alpha > \alpha_2$ and $\alpha_1 > \alpha > \alpha_3$, and especially for $\alpha_3 \geqq \alpha \geqq \alpha_4$, D is always positive.

The classification between saddle and nodal is obtained by

$$5\left\{\left(\alpha-\frac{16}{15}\right)^2-\frac{1+45m}{225}\right\}\left\{\beta-2(\alpha-1)\right\}\left[\beta-\frac{2(\alpha-1)\left\{5\alpha(\alpha-1)-m\right\}}{5\left\{\left(\alpha-\frac{16}{15}\right)^2-\frac{1+45m}{225}\right\}}\right]\gtrless 0, \tag{47}$$

and the domain for saddle points is as follows.

(i) When $m < -0.022$, the transitional point becomes saddle in the inner domain of $\beta = 2(\alpha-1)$ and $\beta = 2(\alpha-1)\{5\alpha(\alpha-1)-m\}/5\{(\alpha-16/15)^2-(1+45m)/225\}$.

(ii) When $m \geqq -0.022$, the inner domain for $\alpha > \alpha_1$ and $\alpha < \alpha_2$ and the outer domain for $\alpha_1 > \alpha > \alpha_2$, in which

$$\begin{matrix}\alpha_1\\ \alpha_2\end{matrix} = (16\pm\sqrt{1+45m})/15,$$

make transitional points saddle.

On the other hand, the other domains separated from the whole domain by the domain for saddle points indicate the part for nodal points, provided the discriminant given by Eq. (47) is positive.

7. Surface Profiles in Channel Transitions and Controls

The surface profile of flow for a particular design discharge must be traced with a sufficient accuracy in hydraulic design problem of channels and water courses, which carry safely the design discharge. When the flow under investigation is gradually varied, the problem is classified as the Cauchy problem and the surface profile is also calculated under given boundary conditions. Fruitful tabulations for back water and draw down curves proposed by many scientists and engineers are available for the estimation of surface profiles, if the flow has no transitional points.

Natural channels characterized by continuous change in channel geometry and boundary produce transitional points in flows as a combined mixture of three kinds of singular points, and thus the transitional behaviours described in the foregoing analysis become important factors to estimate an actual surface profile in channels and water courses. The influence of transitional behaviours to control structures in channels to the actual surface profiles and the tracing procedure of surface profiles are explained with the aid of some examples.

(a) When a saddle point first appears and a nodal point continues in the transition flow, a family of surface profiles involving the transition curve is indicated in Fig. 13. Such profiles of water surface are observed in gradually divergent channels.

If no control structures like weir, dam and gate exist in channels, the resulting surface profile becomes the transition curve indicated in the figure. The flow regime is classified as subcritical in the upstream reach from the saddle point and in the downstream reach from the nodal point. Between the saddle and nodal points, on the contrary, the flow is supercritical.

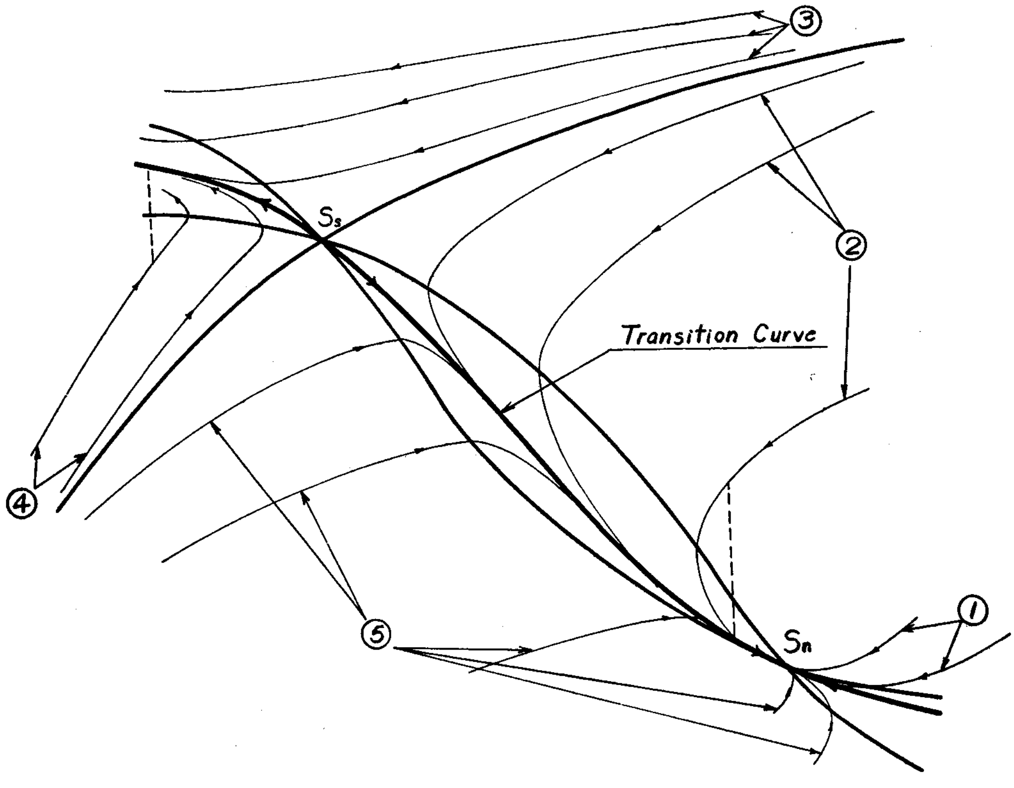


Fig. 13. Water Surface Profiles of Transition Flows in Divergent Channels of Mild Slopes.

The procedure to calculate the surface profile starts from the saddle point to both directions of up- and downstream. Numerical values required for the calculation procedure are evaluated by the foregoing analysis. Of special interest is the hydraulic behaviour of nodal point. The possibility of hydraulic transition from supercritical to subcritical without the hydraulic jump is observed in gradually divergent channels. Two types of different transitional behaviours are divided in the following. When the surface profile traced from the downstream reach intersects the critical depth curve for the design discharge before it approaches the nodal point, as seen in the curve of ② of Fig. 13, the hydraulic jump is observed at the location where both depths become sequent in the momentum conservation law of rapidly varied flows. On the other hand, the curve ① approaches the nodal point without intersecting the critical depth curve and the flow regime changes smoothly from supercritical to subcritical. Consequently, the nodal point in the latter case is classified as control of channels.

Next discussion is directed to surface profiles of flows traced from the overflow structure like dams constructed in the downstream reach. When the surface elevation of flow resulted from daming up by control structures is less than the depth indicated by the curve started from the saddle point, surface profiles becomes ① and ② in Fig. 13 and their hydraulic behaviours are also described in the foregoing. When the height of control structures is high, the resulting surface profile is represented by the curve ③. In this case, the saddle point as the transitional point of flow can not exert the hydraulic behaviour of surface profile, and the back water region stretches to the upstream reach beyond the saddle point. Consequently, usual procedures of calculation is applied to this case.

When the control structure like a gate, by which the underflow is produced, is constructed in the upstream reach, resulting surface profiles are represented by curves of ④ and ⑤. The flow expressed by the curve ④ changes from supercritical to subcritical by the hydraulic jump and thereafter is expressed by the transition curve. After passing through the saddle point, the flow changes again to supercritical. On the other hand, the flow represented by the curve of ⑤ is supercritical, until the flow passes the nodal point or unless the flow is drowned by the regulation of stage due

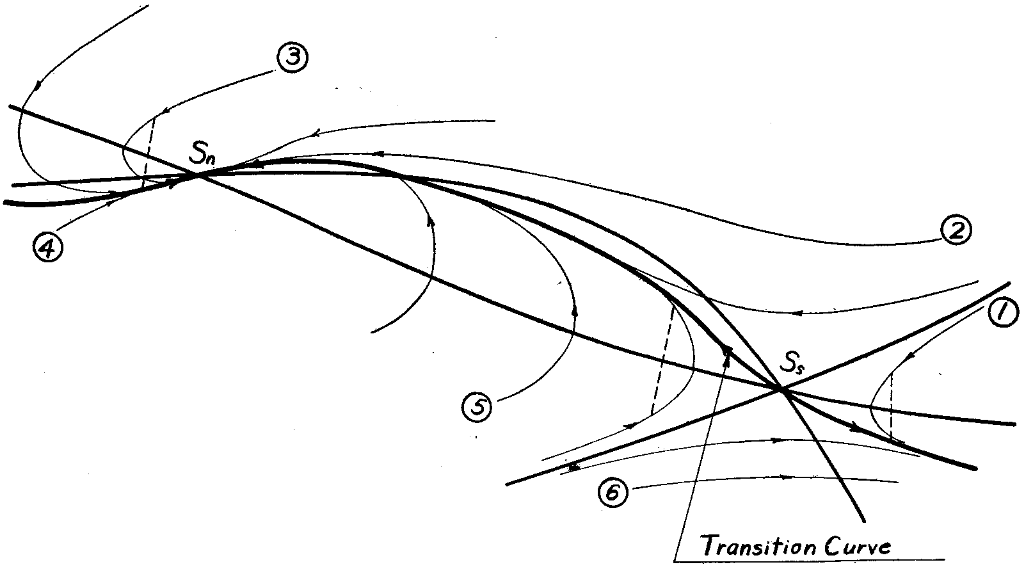


Fig. 14. Water Surface Profiles of Transition Flows in Convergent Channels.

to other control structures in the down stream reach.

(b) When a nodal point first appears and a saddle point follows, which is observed in convergent channels, a family of surface profiles is represented in Fig. 14. In the same manner as described in (a), the hydraulic behaviours of surface profiles are discussed by means of the foregoing analysis.

The details of transitional behaviours of flow in channels are treated in the other paper of the author[10]. As seen in the above examples, however, the following attention must be called.

(1) Before the calculation is applied to estimate the surface profile in divergent or convergent channels for design discharge, two curves of normal depth and critical depth must be traced.

(2) When the singular point is not located in the flow, the usual procedure of calculation is applied. On the other hand, the singular point appeared must be classified and the transitional behaviours are treated by means of the foregoing analysis.

8. Conclusion

This paper deals with the hydraulic characteristics of transitional behaviours of flow in channels involving local changes in channel geometry and boundary by means of the geometric theory of differential equation. As the conclusion of the present research, the following summaries are expressible.

(1) The transitional point produces the change of flow regime from subcritical to supercritical or vice versa and it is characterized by Eq. (6).

(2) The characteristics of transitional point are classified as saddle, nodal and focal, resulted from the geometric properties of the variation equation of surface profile near the transitional point.

(3) Saddle point is the most important to determine the transitional characteristics of flow and also a starting point to trace the surface profile.

(4) Nodal point is characterized by the point at which the flow changes from supercritical to subcritical.

(5) Focal point produces the hydraulic jump and therefore surface profiles can not pass the point.

(6) Saddle and, under some conditions, nodal points become controls of channels for which the elevation of surface profile is uniquely predicted.

(7) In relation with the application of this analysis to problems in hydraulic engineering, the transitional characteristics for Chézy and Manning flows are discussed.

Furthermore, the application of the analysis to the discharge measurement by control structures will make a real promotion to formulate a functional diversity of control structures, and this problem is treated in the other literature[10]. The analysis expressed in the study may be considered as an extension of the theory of gradually varied flow in uniform channels developed by a large number of investigators for some centuries. The systematic experiments are now in progress at the Hydraulics Laboratory, and a fruitful contribution to the hydraulics of open channel flows will be obtained after successful completion of research program.

Acknowledgments

The author wishes his grateful appreciation to Dr. Tojiro Ishihara, Director of the Hydraulics Laboratory, for performance of the present theoretical program. Acknowledgments are hereby made of the continuous assistance of Messrs. S. Kinukawa, K. Ihda, T. Kadoya and A. Murakami.

References

1) Ippen, A. T., Channel Transitions and Controls, *Engineering Hydraulics*, edited by H. Rouse, John Wileys, 1950.
2) Chow, V. T., Integrating the Equation of Gradually Varied Flow, *Proc. ASCE*, Separate Paper No. 838, Nov., 1955.
3) Homma, M., *Hydraulics*, (Fluid Mechanics for Hydraulic Engineers), Maruzen, 1955, (in Japanese).
4) Massé, P., Ressaut et ligne d'eau dans les cours d'eau à pente variable, ***Rev. gén. Hydraulique***, Nos. 19–20, Avril, 1938.
5) Jaeger, C., *Engineering Fluid Mechanics*, Blackie, London, 1956.
6) Escoffier, F. F., Transition Profiles in Non-Unform Channels, Jour. Hydraulics Division, *Proc. ASCE*, No. HY 3, June, 1956.
7) Ishihara, Y., K. Ezaki and F. Makino, Surface Profiles of Flow near Estuaries, *Proc. 3rd Conference for Hydraulic Research*, JSCE, May, 1958, (in Japanese).
8) Iwasa, Y., Boundary Layer Growth of Open Channel Flows and Associated Probelms, *Proc. 3rd Conference for Hydraulic Research*, JSCE, May, 1958, (in Japanese).
9) Iwasa, Y., Theoretical Study of Hydraulic Behaviours of Boundary Characteristics to Channel Transitions and Controls in Divergent or Convergent Channels, *Trans. JSCE*, No. 59, Nov., 1958, (in Japanese).
10) Ishihara, T., and Y. Iwasa, Contribution of Transitional Characteristics of Flow Produced by Control Structures to Discharge Measurement, *Bulletin of Dis. Prev. Res. Inst.*, Kyoto University, (in preparation).