On Fundamental Equation of The Dynamical Behaviours of Nonlinear Visco-Elastic Bodies

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(Received March, 1954)

I. Introduction

Engineering materials generally possess different dynamical properties from that of perfect elastic bodies or ideal viscous fluids. In the case of perfect elastic body, the stress becomes invariable when the strain is applied on it, and in the case of ideal viscous fluid, it also becomes invariable when the rate of shear is given. The perfect elastic body follows the Hooke's law in the small deformation; and the ideal viscous fluid behaves as a simple Newtonian liquid. While many solids and liquids function respectively as purely elastic bodies and as the idealized Newtonian liquids, there are many materials that have dynamic properties which are inexplicable unless these two attributes are simultaneously considered. These materials are called the visco-elastic bodies, and the theory of visco-elasticity has been developed on the assumption that the dynamical behaviours can be expressed by the behaviours of the dynamic models shown in figs. 1 and 2. In this theory the time element is explicitly introduced in the stress-strain relation by an excellent means. However, there exist many a phenomena in various materials which can not be explained by this linear visco-elastic theory. To develope the theory explicable of these phenomena, therefore, is an important

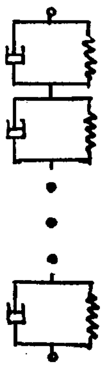

Fig. 1.

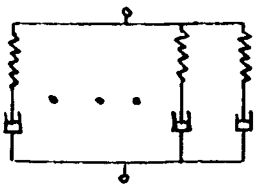

Fig. 2 (a)

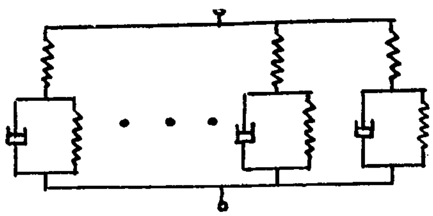

Fig. 2 (b)

problem of the day.

Many materials possess hysteresis effect and the treatment of this quality is conducted quite indistinctly. For example, this effect is contended to be explicable by the theory of linear visco-elasticity, or it has merely been stated that the effect is simply a phenomenon when the stress-strain curves are drawn. The authors wish to point out the fact that it is necessary to consider pure hysteresis effect caused by the internal solid friction mechanism together with the hysteresis effect produced by visco-elasticity, and proposed a new fundamental equation and a model thereof[1]. For example, the dynamic modulus or dynamic internal friction of a vulcanized rubber containing carbon black depends not only upon the frequency but also on the amplitude and, moreover, this amplitude dependency is remarkable when the amplitude is small. The frequency dependency is explained by the visco-elastic theory, but the amplitude dependency can only be explained by our theory. In this paper, the authors intend to further generalize the theory and present a general view of it.

II. Deduction of the fundumental equation of the strain relaxation type.

We consider the dynamical behaviours of the non-linear visco-elastic body of a unit cube under a condition in which the dynamic behaviours are represented by the model shown in fig. 3. The mechanisms designated by E and η are the springs and the dashpots respectively, and the mechanisms designated by s_1, s_2, ..., s_m are the solid friction mechanisms each of which begins to slip when the stress reaches the points of $s_1, s_2, \dots, s_m$, respectively. n_i elements, of which the slipping stresses are equal to s_i, comprise the ith group. We assume that the groups are arranged as follows:

$$s_1 < s_2 < \cdots < s_m \qquad (1)$$

and we scrutinize the jth element of ith group. ε_{ij}, which is the strain of this element, is given as

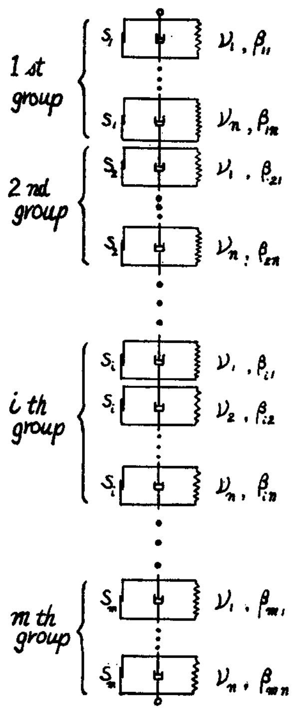

Fig 3 (a)

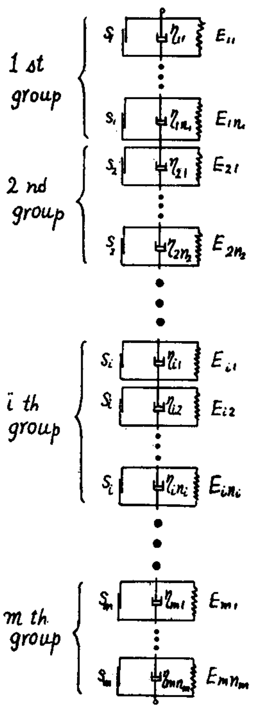

Fig. 3 (b)

follows:

$$\left.\begin{array}{lll}\text{for} & \sigma \leqq s_i & \varepsilon_{ij} = 0, \\ \text{for} & \sigma > s_i & \sigma = E_{ij}\varepsilon_{ij} + \eta_{ij}\dfrac{d\varepsilon_{ij}}{dt} + s_i.\end{array}\right\} \tag{2}$$

However, these relations hold true only when the stress is applied by a non-decreasing function of t, i.e. $d\sigma/dt \geqq 0$. In the case $\sigma(t)$ is a non-increasing function and we put $\sigma' = -\sigma$ and $\varepsilon' = -\varepsilon$, the (2) relation is obtained between σ' and ε'_{ij}. Further, when the stress is given by an arbitrary function of t, the relation (2) is applicable until the sign of $d\sigma/dt$ makes initial change. Integrating (2), we have

$$\left.\begin{array}{lll}\text{for} & \sigma(t) \leqq s_i & \varepsilon_{ij} = 0, \\ \text{for} & \sigma(t) > s_i & \varepsilon_{ij} = \beta_{ij}\displaystyle\int_{t_i}^{t} e^{-\nu_{ij}(t-\tau)}\{\sigma(\tau) - s_i\}\, d\tau,\end{array}\right\} \tag{3}$$

where

$$\beta_{ij} = 1/\eta_{ij}, \qquad \nu_{ij} = E_{ij}/\eta_{ij}, \tag{4}$$

and t_i is the time that satisfies $\sigma(t_i) = s_i$ (i.e. $t_i = \sigma^{-1}(s_i)$). ν_{ij} is the reciprocal of the retardation time which characterizes the deformation process of this element.

Since the number of elements is equal to $\sum_{i=1}^{m} n_i$, the number of ν_{ij}'s is also equal to $\sum n_i$. It is possible to have the same retardation mechanism existing among these different groups and, by reducing these, we assume that there are n kinds of retardation times of different values. (Such a value of n, of course, satisfies the relation of $\max_i n_i \leqq n \leqq \sum n_i$.) Now we denote these ν_{ij}'s of n kindes as $\nu_1, \nu_2, \ldots, \nu_n$ from the smallest value, consecutively, and assume that each group consists of n elements as shown in fig. 3(a). Though $\nu_{i1}, \nu_{i2}, \ldots, \nu_{in_i}$ equal to any one of these ν_j's; ν_j's of the number $(n-n_i)$ do not have corresponding element among $\nu_{i1}, \nu_{i2}, \ldots, \nu_{in_i}$, therefore, β_{ij}'s corresponding to these ν_j's of the number $(n-n_i)$ must be put as zero. Thus, we redefine β_{ij} of the model with its arrangement adjusted deliberatatively and formally as shown in fig. 3(a), then the total strain is given as follows;

$$\varepsilon(t) = \sum_{j=1}^{n} \sum_{i=1}^{k} \beta_{ij} \int_{t_i}^{t} e^{-\nu_j(t-\tau)}\{\sigma(\tau) - s_i\}\, d\tau, \tag{5}$$

where

$$s_k < \sigma(t) \leqq s_{k+1}. \tag{6}$$

Now putting

$$\beta_{ij} \equiv F(\nu_j, s_i)(\nu_j - \nu_{j-1})(s_i - s_{i-1}) \equiv F(\nu_j, s_i)\Delta\nu_j \Delta s_i, \tag{7}$$

and assuming

$$n \to \infty, \qquad m \to \infty,$$
$$\max_j \Delta\nu_j \to 0, \quad \max_i \Delta s_i \to 0,$$

the discontinuous system goes over to the continuous one, whose total strain is given by the following integral

$$\varepsilon(t) = \int_0^{\sigma} ds \int_0^{\infty} F(\nu, s)\, d\nu \int_{t(s)}^{t} e^{-\nu(t-\tau)} \{\sigma(\tau) - s\}\, d\tau, \qquad (8)$$

where $t(s)$ is the value satisfying $s = \sigma(t)$ (i.e. $t(s) = \sigma^{-1}(s)$). Equation (8) is the fundamental equation at the time stress is applied for the first time. In other words, it is a fundamental equation in the virgin stage.

We now consider the case in which, after the stress increases up to σ_0 as an arbitrary non-decreasing function of t, it is kept at constant σ_0 for a sufficiently long time to allow the after effect to disappear (an infinite duration is idealistic) and then the stress decreases as an arbitrary non-increasing function. Putting

$$\sigma_0 - \sigma = \sigma', \qquad \varepsilon_0 - \varepsilon = \varepsilon',$$

where ε_0 is the strain at the time the decrease commences, we have the following corresponding to the equation (2):

$$\left.\begin{array}{lll} \text{for} & \sigma' \leqq 2s_i & \varepsilon_{ij}' = 0, \\ \text{for} & \sigma' > 2s_i & \sigma' = E_{ij}\varepsilon'_{ij} + \eta_{ij}\dfrac{d\varepsilon'_{ij}}{dt} + 2s_i. \end{array}\right\} \qquad (9)$$

Since the above is possible, consequently we have the following corresponding to the equation (8):

$$\varepsilon'(t) = \int_0^{\sigma'/2} ds \int_0^{\infty} F(\nu, s)\, d\nu \int_{t(s)}^{t} e^{-\nu(t-\tau)} \{\sigma'(t) - 2s\}\, d\tau, \qquad (10)_a$$

where $$t(s) = \sigma'^{-1}(2s).$$

Eq. $(10)_a$ consummates only in the interval $0 \leqq \sigma' \leqq 2\sigma_0$ (i.e. $|\sigma| \leqq |\sigma_0|$). Out side of this interval, eq. (8) prevails.

Even when the sign of loading velocity $d\sigma/dt$ changes after several repetitions of loading and unloading, eq. $(10)_a$ is established for the new co-ordinate, shown in fig. 4, if the stress is kept at constance for sufficiently long period. However, it holds true only under the condition that the absolute value of the stress, measured from the initial origin, is less than the maximum absolute value of the stress ever imposed, and when the former exceeds the latter, eq. (8) applies. Eq. $(10)_a$ is the fundamental equation showing the dynamical properties of this model in the non-virgin state.

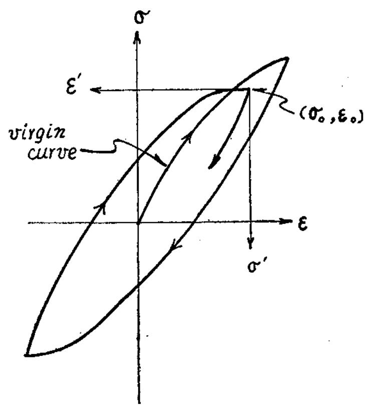

Fig. 4.

Permuting the order of integration, we have

$$\varepsilon(t)=\int_0^\infty d\nu\int_T^t e^{-\nu(t-\tau)}d\tau\int_0^{\sigma(\tau)/2}F(\nu,s)\{\sigma(\tau)-2s\}ds\,, \tag{10$_b$}$$

where the symbol $'$ is omitted; and T is the value satisfying

$$\left.\begin{array}{lll}\text{for} & t\leqq T & \sigma(t)=0\,,\\ \text{for} & t>T & \sigma(t)>0\,.\end{array}\right\} \tag{11}$$

Putting

$$1/\nu\equiv\lambda\,,\qquad F(1/\lambda,s)/\lambda\equiv F_1(\lambda,s)\,, \tag{12}$$

and changing the order of integration, we have

$$\varepsilon(t)=\int_0^\infty d\lambda\int_0^{\sigma/2}F_1(\lambda,s)ds\int_{t(s)}^t\frac{1}{\lambda}e^{-\frac{t-\tau}{\lambda}}\{\sigma(\tau)-2s\}d\tau \tag{10$_c$}$$

Eqs. $(10)_b$ and $(10)_c$ may also be used as the fundamental equations.

III. Reduction of the fundamental equation

(A) In the case in which the distribution function takes the special form of

$$\left.\begin{array}{l}\lim\limits_{\delta\to0}\int_0^\delta F(\nu,s)ds=\Phi(\nu)\,,\\ \text{for}\quad s>\delta\quad F(\nu,s)=0,\end{array}\right\} \tag{a}$$

eq. $(10)_b$ is written

$$\varepsilon(t)=\int_0^\infty\Phi(\nu)d\nu\int_T^t e^{-\nu(t-\tau)}\sigma(\tau)d\tau\,, \tag{13}$$

which is the fundamental formula representing the mechanical behaviour of the linear visco-elastic body, of which an analogous model is indicated in fig. 1. If $F(\nu,s)$ does not have such an ideal form but distributes only in the neighbourhood of the very small value of s as shown in fig. 5, its dynamical behaviour, as a matter of course, resembles approximately that of the linear visco-elastic body.

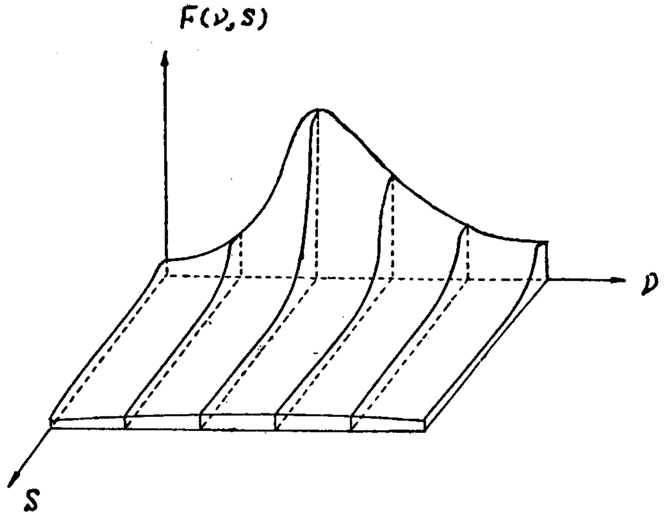

Fig. 5.

(B) In the case

$$\left.\begin{array}{l}\lim_{\delta\to 0}\int_0^{\delta} F(\nu, s)d\nu = f(s)\,, \\ \text{for}\quad \nu > \delta \quad F(\nu, s) = 0\,,\end{array}\right\} \tag{b}$$

the eq. $(10)_a$ becomes

$$\varepsilon(t) = \int_0^{\sigma/2} f(s)ds \int_{t(s)}^{t} \{\sigma(\tau) - 2s\} d\tau\,. \tag{14$_a$}$$

Changing the order of integration, we get

$$\varepsilon(t) = \int_T^t d\tau \int_0^{\sigma(\tau)/2} f(s)\{\sigma(\tau) - 2s\} ds\,. \tag{14$_b$}$$

Differentiating the above, we have

$$\frac{d\varepsilon}{dt} = \int_0^{\sigma/2} f(s)\{\sigma(t) - 2s\} ds\,. \tag{14$_c$}$$

Eqs. (14) are the fundamental formula representing the mechanical behaviour given by the model shown in fig. 6, and when σ is the shearing stress and ε is the shearing strain, it is suitable to apply it as the fundamental equations expressing the dynamical behaviour of the non-Newtonian liquids. From eq. (14) we obtain $1/\eta$ as follows:

$$1/\eta = \frac{d\varepsilon}{dt}\Big/\sigma = \int_0^{\sigma/2} f(s)\left(1 - \frac{2s}{\sigma}\right) ds\,. \tag{15}$$

That is, the slope of the flow curve is the function of the stress, and we know it gives non-Newtonian viscosity. Since

$$\frac{d}{d\sigma}(1/\eta) = \int_0^{\sigma/2} f(s) 2\frac{s}{\sigma^2} ds \geq 0\,, \tag{16}$$

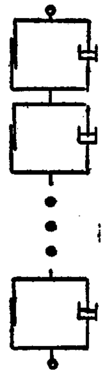

Fig. 6.

the flow curve is always concave upwards.

(C) In the case

$$\left.\begin{array}{l}\lim_{\delta\to 0}\int_0^{\delta} F_1(\lambda, s)d\lambda = \varphi(s)\,, \\ \text{for}\quad \lambda > \delta \quad F_1(\lambda, s) = 0\,,\end{array}\right\} \tag{c}$$

If $\sigma(t)$ is continuous, eq. $(10)_c$ is written as follows:

$$\varepsilon(t) = \int_0^{\sigma/2} \varphi(s)\{\sigma(t) - 2s\} ds\,. \tag{17}$$

This is the fundamental equation expressing the dynamical behaviour of the model shown in fig. 7, which the authors proposed at the outset.[1] In the body whose behaviour is represented by this equation, the after effect does not exist at all and only the pure hysteresis characteristic appears; i. e. if the stress is given, the strain is deter-

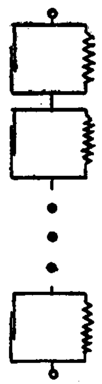

Fig. 7.

mined independently of the time. But the strain varies according to the sign of $d\sigma/dt$.
(D) In the case $F(\nu, s)$ has the distribution only on a certain curve, which, for convenience's sake, we denote as $\nu = \nu(s)$, is lying on the ν–s plane and, further, this distribution is given by δ-function as in the case mentioned above, it becomes the equation which is given as the general fundamental equation[1] previously proposed by the authors.
(E) in the case the distribution function is given by:

$$\left.\begin{aligned} &\lim_{\substack{\Delta\to 0\\ \delta\to 0}}\int_0^{\Delta} d\lambda \int_0^{\delta} F_1(\lambda, s)ds = 1/E\,, \\ &\text{for}\quad \lambda > \Delta \quad \text{or} \quad s > \delta \;\; ; \; F_1(\lambda, s) = 0\,, \end{aligned}\right\} \tag{d}$$

we obtain

$$\varepsilon = \frac{1}{E}\sigma\,, \tag{18}$$

which is the fundamental equation for ideal elastic bodies.
(F) Placing the distribution function as:

$$\left.\begin{aligned} &\lim_{\substack{\Delta\to 0\\ \delta\to 0}}\int_0^{\Delta} d\nu \int_0^{\delta} F(\nu, s)ds = 1/\eta\,, \\ &\text{for}\quad \nu > \Delta \quad \text{or} \quad s > \delta \;\; ; \; F(\nu, s) = 0\,, \end{aligned}\right\} \tag{e}$$

we obtain

$$\varepsilon = \frac{1}{\eta}\int_0^t \sigma(\tau)d\tau\,, \tag{19}$$

and by differentiating,

$$\eta\frac{d\varepsilon}{dt} = \sigma\,, \tag{19}$$

which is the fundamental equation for ideal viscous fluids.

As explained above, the various cases, which have been hitherto treated, are reduced as the special cases of the generalized equations (10). Summarizing the the above discussions in a diagram, we have fig. 8. The illustration shows that if $F(\nu, s)$ or $F_1(\lambda, s)$ distributes only at the point or on the

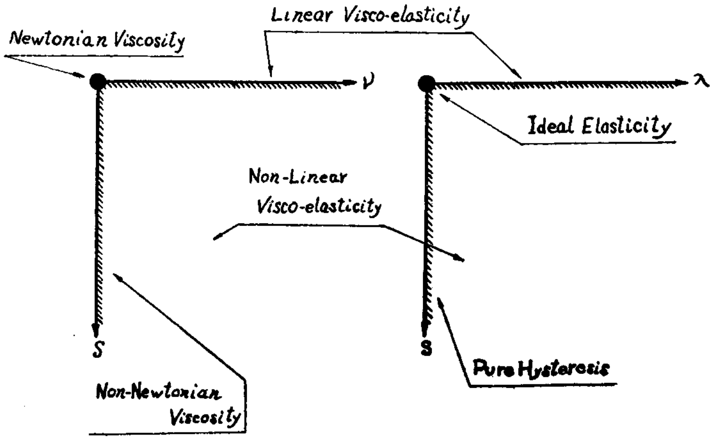

Fig. 8.

line indicated by a circle or a hutching, respectively, the behaviour of the material takes the respective characteristics which are indicated by the arrows.

In real materials, strictly speaking, their behaviours may follow the general equations (10) or more complicated equation; however, there are many materials which suffice to consider to follow the reduced equation approximately or any of the combinations of them. For example, we know that the vulcanized rubber stocks containing carbon black may be handled as a combination of the characteristics of (E), (A) and (C).

IV. Examples

(i) We shall consider the case in which the velocity of loading is constant. Since

$$\left.\begin{array}{lll} \text{for} & \tau<0 & \sigma(\tau)=0\,, \\ \text{for} & \tau>0 & \sigma(\tau)=v\tau\,, \end{array}\right\}$$

and

$$t(s)=2s/v\,, \tag{20}$$

integrating eq. (10) under these condition, we obtain

$$\varepsilon=\int_0^{\sigma/2}ds\int_0^\infty F(\nu,s)\frac{1}{\nu}\Big[\Big(\sigma-2s\Big)-\frac{v}{\nu}\Big\{1-e^{-\nu\left(t-\frac{2s}{v}\right)}\Big\}\Big]d\nu\,. \tag{21}$$

(ii) We shall consider the case in which the load is kept at constant after the loading is applied at a constant rate. In this case, since the stress is given by:

$$\left.\begin{array}{lll} \text{for} & \tau<0 & \sigma(\tau)=0, \\ \text{for} & 0<\tau<t_1 & \sigma(\tau)=v\tau\,, \\ \text{for} & \tau>t_1 & \sigma(\tau)=vt_1\equiv\sigma_1\,, \end{array}\right\}$$

eq. (10) becomes

$$\begin{aligned}\varepsilon=&\int_0^{\sigma_1/2}ds\int_0^\infty F_1(\lambda,s)\Big\{(\sigma_1-2s)-v\lambda\Big(1-e^{-\frac{t_1-\frac{2s}{v}}{\lambda}}\Big)\Big\}d\lambda \\ &+\int_0^{\sigma_1/2}(\sigma_1-2s)ds\int_0^\infty F_1(\lambda,s)\Big(1-e^{-\frac{t-t_1}{\lambda}}\Big)d\lambda\,.\end{aligned} \tag{22}$$

(iii) We shall consider the case of the creep test. For this case, we may place as follows in eq. (22),

$$v\to\infty\,,\quad t_1\to0\,,\quad vt_1\to\sigma_1\ \text{(finite value)}$$

then, we obtain

$$\varepsilon=\int_0^{\sigma_1/2}(\sigma_1-2s)ds\int_0^\infty F_1(\lambda,s)\Big(1-e^{-\frac{t}{\lambda}}\Big)d\lambda\,. \tag{23}$$

Differentiating twice with respect to σ_1, we get

$$\frac{d^2\varepsilon}{d\sigma_1{}^2}=\int_0^\infty F_1\Big(\lambda,\frac{\sigma_1}{2}\Big)\Big(1-e^{-\frac{t}{\lambda}}\Big)d\lambda\,. \tag{24}$$

When $d^2\varepsilon/d\sigma_1^2$ is experimentally obtained, we can determine the distribution function by solving this integral equation.

V. Deduction of the fundamental equation of the stress relaxation type

We have considered heretofore the fundamental eqation of the strain relaxation type and now we shall treat the model of the stress relaxation type shown in fig. 9. We assume

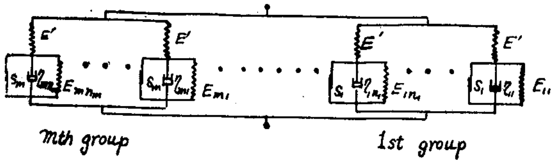

Fig. 9.

$$s_1 < s_2 < \cdots < s_m ,$$

and deal with the jth element of ith group at first. Putting the stress shared by this element as σ_{ij}, we have

$$\text{for} \quad \sigma_{ij} \leqq s_i \quad \sigma_{ij} = E'\varepsilon ,$$
$$\text{for} \quad \sigma_{ij} > s_i \quad \frac{d\varepsilon}{dt} + \frac{E_{ij}}{\eta_{ij}}\varepsilon = \frac{1}{E'}\frac{d\sigma_{ij}}{dt} + \frac{E'+E_{ij}}{E'\eta_{ij}}\sigma_{ij} - \frac{1}{\eta_{ij}}s_i . \tag{25}$$

This equation, however, hold true only when the strain is given by non-decreasing function of t. Integrating (25), we obtain

$$\left.\begin{aligned} &\text{for} \quad \sigma_{ij} \leqq s_i \quad \sigma_{ij} = E'\varepsilon , \\ &\text{for} \quad \sigma_{ij} > s_i \quad \sigma_{ij} = E'\varepsilon - \alpha_{ij}\int_{t_i}^{t} e^{-\mu_{ij}(t-\tau)}\left\{\varepsilon(\tau) - \frac{s_i}{E'}\right\}d\tau , \end{aligned}\right\} \tag{26}$$

where

$$\alpha_{ij} = E'^2/\eta_{ij} , \quad \mu_{ij} = (E'+E_{ij})/\eta_{ij} , \tag{27}$$

and t_i is the time satisfying $\sigma_{ij} = s_i$. μ_{ij} is the reciprocal of the relaxation time of this element. Since E' of each element is equal to the other E', we get the following by changing equation (26)

$$\left.\begin{aligned} &\text{for} \quad \varepsilon \leqq x_i \quad \sigma_{ij} = 0 , \\ &\text{for} \quad \varepsilon > x_i \quad \sigma_{ij} = E'\varepsilon - \alpha_{ij}\int_{t_i}^{t} e^{-\mu_{ij}(t-\tau)}\{\varepsilon(\tau) - x_i\}d\tau , \end{aligned}\right\} \tag{28}$$

where

$$x_i \equiv s_i/E', \qquad t_i = \varepsilon^{-1}(x_i) . \tag{29}$$

Rearranging the model formally as in the case of the strain relaxation type and redefining μ_j and α_{ij}, the stress received by the model and the whole body is:

$$\sigma = mnE'\varepsilon - \sum_{j=1}^{n}\sum_{i=1}^{k}\alpha_{ij}\int_{t_i}^{t} e^{-\mu_j(t-\tau)}\{\varepsilon(\tau) - x_i\}d\tau . \tag{30}$$

However, k is given by the following:

$$x_k < \varepsilon(t) \leqq x_{k+1}\,. \tag{31}$$

Putting

$$\alpha_{ij} \equiv G(\mu_j, x_i)(\mu_j - \mu_{j-1})(x_i - x_{i-1}) \equiv G(\mu_j, x_i)\Delta\mu_j \Delta x_i\,, \tag{32}$$

and letting this discontinuous system go over the continuous one, eq. (30) becomes

$$\sigma(t) = E\varepsilon - \int_0^\infty d\mu \int_0^\varepsilon G(\mu, x) dx \int_{t(x)}^t e^{-\mu(t-\tau)}\{\varepsilon(\tau) - x\} d\tau\,, \tag{33}$$

where $\qquad t(x) = \varepsilon^{-1}(x), \qquad E = mnE'\,.$

Eq. (33) is the fundamental equation representing dynamical behaviours of this model in the virgin state. In the case of non-virgin state the following equation holds:

$$\sigma(t) = E\varepsilon - \int_0^\infty d\mu \int_0^{\varepsilon/2} G(\mu, x) dx \int_{t(x)}^t e^{-\mu(t-\tau)}\{\varepsilon(\tau) - 2x\} d\tau\,, \tag{34$_a$}$$

where $\qquad t(x) = \varepsilon^{-1}(2x)\,.$

Changing the order of integration, we get

$$\sigma(t) = E\varepsilon - \int_0^\infty d\mu \int_T^t d\tau \int_0^{\varepsilon(\tau)/2} G(\mu, x) e^{-\mu(t-\tau)}\{\varepsilon(\tau) - 2x\} dx\,. \tag{34$_b$}$$

Or putting

$$1/\mu \equiv \kappa\,, \qquad G(1/\kappa, x)/\kappa \equiv G_1(\kappa, x)\,,$$

we have

$$\sigma(t) = E\varepsilon - \int_0^\infty d\kappa \int_0^{\varepsilon/2} G_1(\kappa, x) dx \int_{t(x)}^t \frac{1}{\kappa} e^{-\frac{t-\tau}{\kappa}}\{\varepsilon(\tau) - 2x\} d\tau\,. \tag{34$_c$}$$

These equations are established when the same condition received by the strain in the case of the strain relaxation type satisfies the stress.

Integrating eq. (34) by parts, using the condition $\varepsilon(t)_{t=t(x)} = 0$, we obtain the following:

$$\begin{aligned}\sigma(t) = E\varepsilon(t) &- \int_0^\infty d\mu \int_0^{\varepsilon/2} \frac{G(\mu, x)}{\mu}\{\varepsilon(t) - 2x\} dx \\ &+ \int_0^\infty d\mu \int_0^{\varepsilon/2} \frac{G(\mu, x)}{\mu} dx \int_{t(x)}^t e^{-\mu(t-\tau)} \frac{d\varepsilon(\tau)}{d\tau} d\tau\,.\end{aligned} \tag{35}$$

In the eq. (35), 1st and 2nd terms depend upon the present strain and the sign of $d\varepsilon/dt$ only, but it has no relation to histor of the strain. In other words, they give the hysteretic characteristic. On the other hand, the 3rd term depends upon the history of the rate of strain (i.e. the history of the strain), and gives the non-linear visco-elasticity. If $d\varepsilon/dt$ is equal to zero, the 3rd term becomes zero. Therefore, 1st and 2nd terms are the terms of pure hysteresis. The distribution function $G(\mu, x)$ is introduced into the 2nd and the 3rd term in the same form. Therefore, the fact that eq. (35) can be separated into the pure hysteresis and the visco-elasticity, essentially differs from the fact that the fundamental equation of certain material (for example, rubber containing carbon black) is the combination of (13) and (17).

VI. Reduction of the fundamental equation of the stress relaxation type

(A) In the case the distribution function is

$$\left.\begin{aligned} &\lim_{\delta\to 0}\int_0^{\delta} G(\mu, x)dx = \Psi(\mu)\,, \\ &\text{for}\quad x>\delta\quad G(\mu, x)=0\,, \end{aligned}\right\} \tag{f}$$

eq. (34) becomes

$$\sigma = E\varepsilon - \int_0^{\infty}\Psi(\mu)d\mu\int_T^t e^{-\mu(t-\tau)}\varepsilon(\tau)d\tau\,, \tag{36}$$

which is the fundamental equation of linear visco-elastic bodies and gives the behaviour of the model shown in fig. 2(b).

(B) If following relations are satisfied,

$$\Psi(\mu)/\mu \equiv \Psi_0(\mu), \qquad E \equiv \int_0^{\infty}\Psi_0(\mu)d\mu\,, \tag{g}$$

integrating by parts, we obtain

$$\sigma = \int_0^{\infty}\Psi_0(\mu)\,d\mu\int_T^t e^{-\mu(t-\tau)}\frac{d\varepsilon(\tau)}{d\tau}d\tau\,, \tag{37}$$

which is the fundamental equation of linear visco-elastic bodies corresponding to the model shown in fig. 1(a). $E(\tau)$ or $G(\tau)$ which is used usually as the distribution function of relaxation time is equal to $\Psi_0(1/\tau)/\tau^2$.

(C) In the case the distribution function is given by

$$\left.\begin{aligned} &\lim_{\delta\to 0}\int_0^{\delta} G_1(\kappa, x)d\kappa = \psi(x)\,, \\ &\text{for}\quad \kappa>\delta\quad G_1(\kappa, x)=0\,, \end{aligned}\right\} \tag{h}$$

and if $\varepsilon(\tau)$ is continuous, we have

$$\sigma = E\varepsilon - \int_0^{\varepsilon/2}\psi(x)\{\varepsilon(t)-2x\}dx\,, \tag{38}$$

which is the fundamental equation of the dynamic behaviour of the model shown in fig. 10 and if the strain is given, the stress is determined at once. However, it depends upon the sign of $d\varepsilon/dt$, that is, the eq. (38) gives the pure hysteretic characteristics.

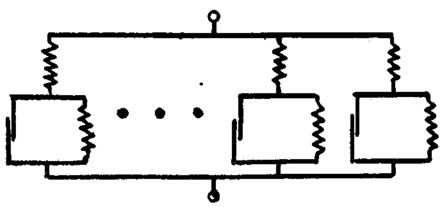

Fig. 10.

VII. Conclusion

The generalized fundamental equations above-mentioned are so complicated that they may not be applicable except in simple problems.

It is, however, very interesting that the characteristics, such as the pure elasticity, Newtonian viscosity, linear visco-elasticity, pure hysteresis and non-Newtonian viscosity which have actually been found hitherto in various materials, are led from these generalized fundamental equations as special cases. Also the various distribution functions brought forth in this paper will make it possible to treat quantitatively the non-linearity which has heretofore been treated only qualitatively.

The authors treated the equations in correspondence with the respective models for the purpose of facility of understanding, but it is not always necessary, so long as the dynamic behaviour of a material follows the fundamental equation. There is no doubt that, unless the effect of the internal solid friction exists in the material even indirectly, the material may not follow the fundamental equation, and for the materials, for which fundamental equation can be recognized to be appropriate, the mechanisms causing the solid frictional process will somehow be found by microscopic investigations.

Acknowledgement

The authors would wish to express their hearty appreciation to Professor Shūjirō Kunii for his interest and useful suggestions in the progress of this work.

Reference

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