Subharmonic Oscillations in Non-linear Systems

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(Received July, 1952)

SUMMARY

This paper deals with the subharmonic oscillations which occur in systems with non-linear restoring force. It is first investigated that the order of the subharmonics has the close connection to the form of the non-linear characteristics. Then the subharmonic oscillation of order 1/3, i.e., the oscillation whose fundamental frequency is one-third that of the applied force, is particularly investigated for the cases in which the non-linear characteristics are expressed by (1) cubic and (2) quintic functions. In both cases the stability problem of the periodic solutions is discussed in detail following the stability criterion given previously by the present author.[1] The analysis has revealed that in the latter case (2) the second higher-harmonc of the subharmonic, i.e., the oscillation of order 2/3 causes the collapse of the original subharmonic oscillation under certain circumstances.

1. Introduction

Subharmonic oscillations whose frequency is a fraction $1/\nu$ ($\nu=2, 3, 4, \cdots$) of that of the applied forc϶ may not seldom occur in non-linear systems. We consider the differential equation

$$\frac{d^2v}{d\tau^2}+2\delta\frac{dv}{d\tau}+f(v)=B\cos\nu\tau\,, \tag{1.1}$$

in which 2δ is a constant damping coefficient and $f(v)$ is a term characterizing the non-linear restoring force. It will be noticed that, since the period of the applied force is $2\pi/\nu$, the subharmonic oscillation of order $1/\nu$ has the period 2π and may be expressed by a linear combination of $\sin\tau$ and $\cos\tau$.

In the following lines we shall first investigate the relationship between the non-linear characteristics expressed by the term $f(v)$ and the order $1/\nu$ of the subharmonic oscillations. Then the subharmonic oscillation of order 1/3

[1] C. Hayashi, Memoirs of the Faculty of Eng. Kyoto Univ. **14**, 92 (1952).

is particularly investigated with special attention directed to the stability of the periodic oscillation. (The discussion throughout the present paper is confined to the steady states of oscillations. A brief discussion on the transient states has previously been reported by the author.[2])

2. Relationship between the non-linear characteristics and the order of subharmonic oscillations

In order to investigate this we shall consider the following polynomial for the restoring force $f(v)$; i.e.,

$$f(v) = c_1 v + c_2 v^2 + c_3 v^3 + \cdots, \tag{2.1}$$

where $c_1, c_2, c_3, \cdots$ are constants determined by the non-linear characteristics and subjected, without loss of generality, to the condition

$$c_1 + c_2 + c_3 + \cdots = 1. \tag{2.2}$$

In so far as we deal with the steady states of oscillations, the periodic solution of equation (1.1) may be approximated by

$$v = v_0 + x \sin\tau + y\cos\tau + w\cos\nu\tau, \tag{2.3}$$

in which only the constant term v_0, the subharmonic oscillation $x\sin\tau + y\cos\tau$, and the oscillation having the applied frequency $w\cos\nu\tau$ are considered on account of their prime importance.* Following Mandelstam and Papalexi,[3] the amplitude w may further be approximated by

$$w = \frac{1}{1-\nu^2} B. \tag{2.4}$$

This approximation is legitimate in the case when the non-linearity is small. But, as will be shown later, the relation (2.4) is a fairly good approximation even when the departure from linearity is large.

Substituting (2.3) into (1.1), and equating to zero the coefficients of $\sin\tau$ and $\cos\tau$ respectively, we have the following cases according to the form of the non-linear characteristics (2.1).

Case 1. When the non-linearity is given by $f(v) = c_1 v + c_3 v^3$

Under such a case that the non-linearity is symmetrical, i.e., $f(v)$ is odd in v,

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[2] C. Hayashi, Memoirs of the Faculty of Eng. Kyoto Univ. **13**, 180 (1951).

* It is tacitly assumed that the damping coefficient 2δ is not so large that the term containing $\sin\nu\tau$ may be discarded in equation (2.3).

[3] L. Mandelstam and N. Papalexi, Z. f. Phys. **73**, 227 (1932).

the constant term v_0 in equation (2.3) is usually discarded, and, for $\nu=2, 4, 5, \cdots$, the substitution above-mentioned leads to

$$\left.\begin{aligned} \left[1-\frac{3}{4}(x^2+y^2)-\frac{3}{2}w^2\right]x+ky &= 0, \\ \left[1-\frac{3}{4}(x^2+y^2)-\frac{3}{2}w^2\right]y-kx &= 0, \end{aligned}\right\} \qquad (2.5)$$

where $k=2\delta/c_3$. These simultaneous equations of x and y have no roots other than $x=y=0$. Hence the subharmonic oscillations of orders 1/2, 1/4, 1/5, $\cdots$ cannot occur in this case. However, as will be investigated in the following section, real roots of x and y which are not simultaneously zero may be obtained in the case of $\nu=3$, and some of them are maintained in stable states.† Hence it is concluded that the subharmonic oscillation of order 1/3 may occur in the case when the non-linear term c_3v^3 is contained in equation (2.1).

Case 2. When the non-linearity is given by $f(v)=c_1v+c_2v^2+c_3v^3$

The non-linearity is unsymmetrical and the constant term v_0 in equation (2.3) must be considered. The subharmonic oscillation of order 1/2 occurs predominantly in this case. It takes a considerable length to analyse the subharmonic oscillation of order 1/2, so that the detailed discussion will be deferred to another paper.‡

Case 3. When the non-linearity is given by $f(v)=c_1v+c_5v^5$

Although the non-linear term c_3v^3 is absent in this case, the subharmonic oscillation of order 1/3 is maintained. The investigation will be given in Section 4. The subharmonic oscillation of order 1/5 is also maintained in this case.

From the foregoing considerations, it may be deduced in general that the presence of the term $c_\nu v^\nu$ in equation (2.1) is the sufficient condition for the occurrence of the subharmonic oscillation of order ν. However, this is not the necessary condition as one sees in Case 3. It may also be concluded that the subharmonic oscillation of order ν does not occur when the highest degree of the power of the non-linear terms in equation (2.1) is less than ν.

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† These real roots represent the states of equilibrium which are not always sustained, but are only able to last so long as they are stable.

‡ It is only mentioned here that, by a closer investigation in which the term v_0 is taken into account, the subharmonic oscillation of order 1/2 may narrowly take place even in the case when $c_2=0$ in equation (2.1). This result is also verified by experiments, though the range of occurrence is considerably constricted. Therefore, strictly speaking, the conclusion in case 1 should be modified in this respect.

3. Subharmonic oscillation of order 1/3 with the non-linear characteristic: $u=c_1v+c_3v^3$

We shall hereafter investigate the subharmonic oscillation of order 1/3 in detail.

(1) Non-dissipative case

Putting

$$\left.\begin{aligned} &\nu = 3\,, \quad k = 0\,, \\ &c_2 = c_4 = c_5 = \cdots = 0\,, \end{aligned}\right\}$$

in equations (1.1) and (2.1), and remembering the condition (2.2), we have the following equation for the subharmonic oscillation of order 1/3; i.e.,

$$\frac{d^2v}{d\tau^2}+v = c_3(v-v^3)+B\cos 3\tau\,. \tag{3.1}$$

By the use of equation (2,4), the periodic solution will be given by

$$\left.\begin{aligned} &v = x\sin\tau+y\cos\tau+w\cos 3\tau\,, \\ \text{with} \quad & w = \frac{1}{1-3^2}B = -\frac{1}{8}B\,. \end{aligned}\right\} \tag{3.2}$$

Substituting (3.2) into (3.1), and equating to zero the coefficients of $\sin\tau$ and $\cos\tau$ respectively, we have

$$\left.\begin{aligned} x\left[1-\frac{3}{4}(x^2+y^2)-\frac{3}{2}w^2\right] &= -\frac{3w}{4}\cdot 2xy\,, \\ y\left[1-\frac{3}{4}(x^2+y^2)-\frac{3}{2}w^2\right] &= -\frac{3w}{4}(x^2-y^2)\,. \end{aligned}\right\} \tag{3.3}$$

Multiplying the first equation by y, the second by x, and subtracting the products so formed, we obtain

$$x = 0 \quad \text{or} \quad x = \pm\sqrt{3}\,y\,.$$

For each value of x we have two pairs of roots, hence we obtain six pairs of equilibrium states in all. But we are sufficed only with the investigation of the equilibrium states in which $x=0$, because, as will be shown later, the other states have the same amplitude, but differ in phase by $(2/3)\pi$ or $(4/3)\pi$ radians in τ, viz., one or two complete cycles of the applied force. This is a plausible result since we are concerned with the subharmonic oscillation of order 1/3.

For $x=0$, assuming $y\neq 0$ in the second equation of (3.3), we obtain

$$y^2 + yw + 2w^2 - \frac{4}{3} = 0 . \qquad (3.4)$$

This equation shows the relationship between the amplitude w (which is here assumed to be proportional to the applied force B) and the amplitude y of the subharmonic oscillation, and is plotted by a part of an ellipse in Fig. 1.

We are now to determine the stability of the equilibrium states given by equation (3.4). The condition for the stability of the periodic solution is obtained by examining whether any variation from this state caused by a sufficiently small perturbation attenuates or not with the lapse of time. If this variation is denoted by ξ, then with $v(\tau)+\xi$ for $v(\tau)$ in equation (3.1), we obtain the following variational equation

$$\frac{d^2\xi}{d\tau^2} + (c_1 + 3c_3 v^2)\xi = 0 .$$

Substituting into this the equilibrium states represented by

$$v = y \cos \tau + w \cos 3\tau ,$$

the following equation of Hill's type is derived; i.e.,

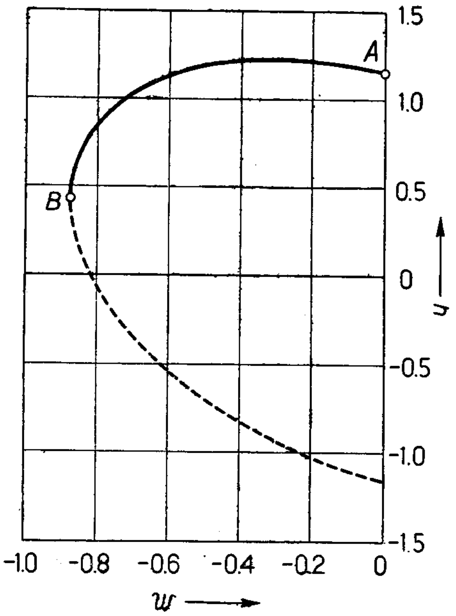

FIG. 1. Amplitude characteristic of 1/3-harmonic oscillation (non-linearity by cubic function).

$$\left.\begin{aligned} &\frac{d^2\xi}{d\tau^2} + \left[\theta_0 + 2\theta_1 \cos 2\tau + 2\theta_2 \cos 4\tau + 2\theta_3 \cos 6\tau\right]\xi = 0 , \\ &\text{where} \\ &\qquad \theta_0 = c_1 + \frac{3}{2} c_3 (y^2 + w^2) , \\ &\qquad \theta_1 = \frac{3}{4} c_3 (y^2 + 2yw) , \\ &\qquad \theta_2 = \frac{3}{2} c_3 yw , \\ &\qquad \theta_3 = \frac{3}{4} c_3 w^2 . \end{aligned}\right\} \qquad (3.5)$$

Since w and y are determined by equations (3.2) and (3.4) respectively, the parameters θ's in equations (3.5) may readily be calculated.

According to the previous investigation,[1] the stability conditions of the first approximation are given by

$$\left.\begin{aligned}&(\theta_0-n^2+\theta_n)(\theta_0-n^2-\theta_n)>0\,,\\ \text{or}\quad &|\theta_n|<|\theta_0-n^2|\,,\quad n=1,2,3,\cdots.\end{aligned}\right\}\tag{3.6}$$

The condition for $n=1$ examines the stability against building up of the unstable oscillation having the same frequency as that of the subharmonic. By the use of equations (3.4) and (3.5) this condition leads to

$$2y>-w\,.\tag{3.7}$$

Hence the equilibrium states represented by the dotted-line curve in Fig. 1 are unstable and do not exist actually. Fig. 2 shows the trajectories of $\theta_n\,(n=1,2,3)$

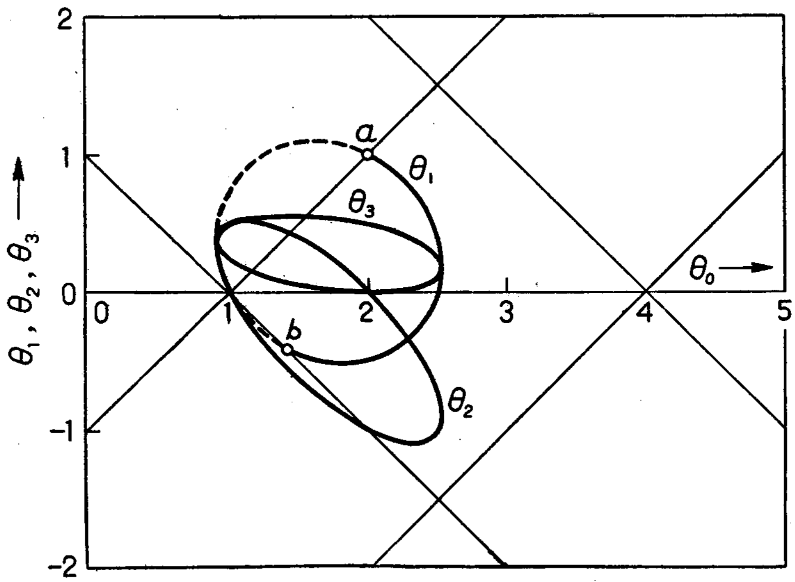

FIG. 2. Trajectories of θ's in (3.5) with varying w.

which are drawn by varying the value of w (or B) for the limiting case of $c_1=0$ and $c_3=1$. As expected from the stability condition for $n=1$, θ_1 enters into the first unstable region in the dotted-line interval ab. At the critical points a and b,

$$2y=-w\,,$$

and these points correspond respectively to A and B in Fig. 1.

It is clear from Fig. 2 that neither θ_2 nor θ_3 enters into their corresponding unstable regions. Thus we see that, since the conditions (3.6) for $n=2$ and $n=3$ are satisfied, the only condition (3.7) is sufficient to determine the stability in the present case.* Hence, in the end, we see that the part of the curve (in Fig. 1) which lies between A and B represents the stable oscillations.

(2) Dissipative case

When the damping is considered in non-linear systems, the fundamental equation takes the form

$$\frac{d^2v}{d\tau^2}+2\delta\frac{dv}{d\tau}+c_1v+c_3v^3 = B\cos 3\tau, \tag{3.8}$$

and the periodic solution is expressed by

$$\left.\begin{aligned} v &= x\sin\tau+y\cos\tau+w\cos 3\tau, \\ \text{with} \qquad w &= -\frac{B}{8}. \end{aligned}\right\} \tag{3.9}$$

Substituting (3.9) into (3.8), and equating to zero the coefficients of $\sin\tau$ and $\cos\tau$ respectively, we have

$$\left.\begin{aligned} Ax+ky &= -\frac{3w}{4}\cdot 2xy, \\ Ay-kx &= -\frac{3w}{4}(x^2-y^2), \\ \text{where} \qquad A &= 1-\frac{3}{4}(x^2+y^2)-\frac{3}{2}w^2, \\ k &= \frac{2\delta}{c_3}. \end{aligned}\right\} \tag{3.10}$$

Squaring and adding the first two equations, we obtain

$$A^2+k^2 = \left(\frac{3w}{4}\right)^2 r^2, \quad r^2 = x^2+y^2, \tag{3.11}$$

or, by putting $r^2=R$ and $w^2=W$, this leads to

$$\frac{9}{16}R^2+\left(\frac{27}{16}W-\frac{3}{2}\right)R+\left(\frac{9}{4}W^2-3W+k^2+1\right)=0. \tag{3.12}$$

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* If $\theta_m(m\geqq 2)$ enters into the m-th unstable region, the stability conditions (3.6) are not satisfied for $n=m$, and the unstable oscillation of order $m/3$ will take place. If this oscillation grows up predominantly, the original subharmonic oscillation of order 1/3 may no longer be maintained. An example for such a case ($m=2$) will be given later in Section 4.

Hence the relationship between W and R, and consequently the amplitude r of the subharmonic oscillation are determined and depicted in Fig. 3 a, b for various values of k.

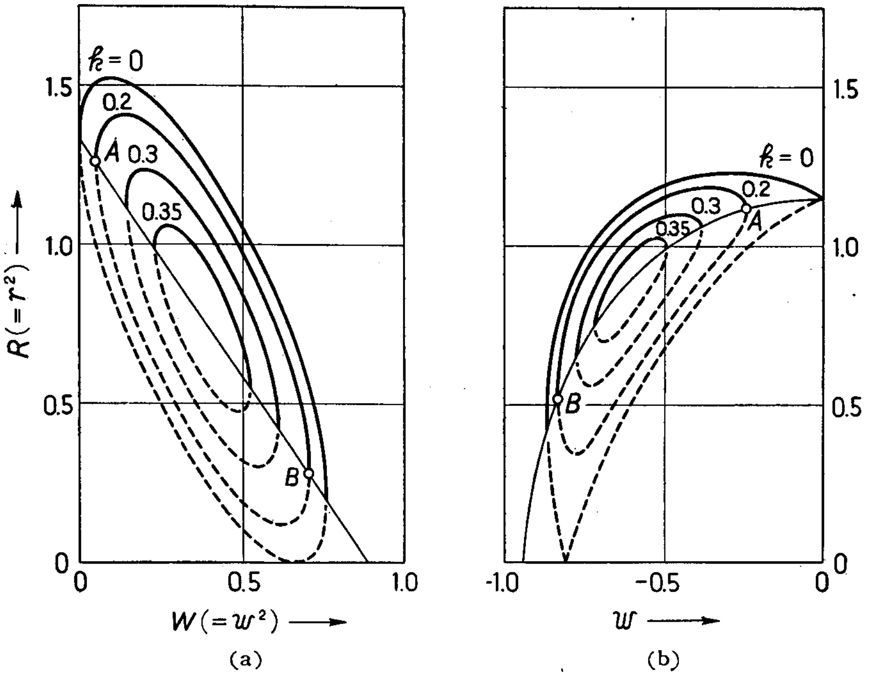

FIG. 3a. Relationship between W and R in (3.12). b. Amplitude characteristic of 1/3-harmonic oscillation.

The components x, y of the amplitude r may readily be obtained as follows. By equations (3.10) and (3.11), we have

$$x^3-\frac{3}{4}Rx-\frac{kR}{3w}=0,$$

and

$$y^3-\frac{3}{4}Ry-\frac{AR}{3w}=0.$$

Hence the components x, y are given by

$$\left.\begin{aligned} x &= -r\cos\theta, \quad -r\cos(\theta+120°), \quad -r\cos(\theta+240°), \\ y &= -r\sin\theta, \quad -r\sin(\theta+120°), \quad -r\sin(\theta+240°), \\ &\text{with} \quad \cos 3\theta = -\frac{4k}{3wr}. \end{aligned}\right\} \tag{3.13}$$

The stability problem may be treated in the same manner as in the preceding case. The equation which characterizes the small variation from the periodic states of equilibrium is

$$\frac{d^2\xi}{d\tau^2}+2\delta\frac{d\xi}{d\tau}+(c_1+3c_3v^2)\xi=0\,.$$

By the well-known transformation $\xi=e^{-\delta\tau}\cdot\eta$, this leads to

$$\frac{d^2\eta}{d\tau^2}+\left[c_1-\delta^2+3c_3v^2\right]\eta=0\,.$$

Substituting the periodic solution (3.9), we obtain the following Hill's equation

$$\left.\begin{aligned}
&\frac{d^2\eta}{d\tau^2}+\Big[\theta_0+2\theta_1\cos(2\tau-\varepsilon_1)+2\theta_2\cos(4\tau-\varepsilon_2)\\
&\qquad\qquad+2\theta_3\cos(6\tau-\varepsilon_3)\Big]\eta=0\,,\\
&\text{where}\\
&\theta_0=c_1-\delta^2+\frac{3}{2}c_3(x^2+y^2+w^2)\,,\\
&\theta_n^2=\theta_{ns}^2+\theta_{nc}^2\,,\quad \varepsilon_n=\arctan\frac{\theta_{ns}}{\theta_{nc}}\,,\quad n=1,2,3\,,\\
&\theta_{1s}=\frac{3}{2}c_3x(y-w)\,,\quad \theta_{1c}=\frac{3}{4}c_3(-x^2+y^2+2yw)\,,\\
&\theta_{2s}=\frac{3}{2}c_3xw\,,\qquad\quad \theta_{2c}=\frac{3}{2}c_3yw\,,\\
&\theta_{3s}=0\,,\qquad\qquad\quad \theta_{3c}=\frac{3}{4}c_3w^2\,.
\end{aligned}\right\}\tag{3.14}$$

The stability conditions in dissipative systems are given by [1]

$$(\theta_0-n^2)^2+2(\theta_0+n^2)\delta^2+\delta^4>\theta_n^2\,,\quad n=1,2,3,\cdots.\tag{3.15}$$

The condition for $n=1$ is obtained by substituting θ_0 and θ_1 in (3.14) into (3.15), and further by virtue of (3.10) and (3.12), we ultimately find

$$R+\frac{3}{2}W-\frac{4}{3}>0\,.\tag{3.16}$$

Hence the equilibrium states represented by the dotted-line curves in Fig. 3 are unstable and do not actually exist. Fig. 4 shows the trajectories of $\theta_n(n=1,2,3)$ which are drawn by varying the value of w (or B) for the case of $c_1=0$, $c_3=1$,

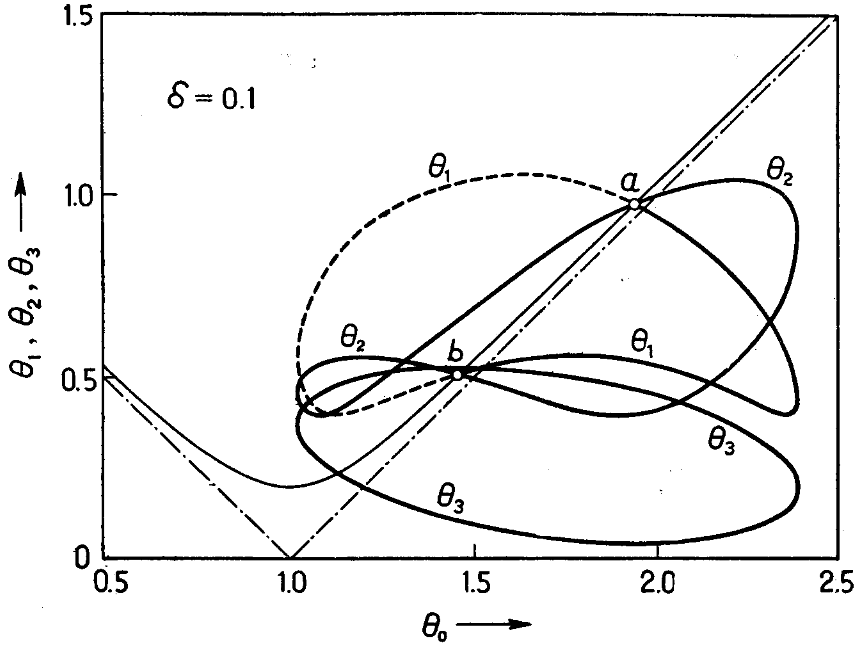

FIG. 4. Trajectories of θ's in (3.14) with varying w.

and $\delta=0.1$. As illustrated in the figure, the boundary curve of the first unstable region is given by a hyperbola,* and θ_1 enters into this region in the dotted-line interval ab. At the critical points a and b, we have

$$R+\frac{3}{2}W-\frac{4}{3}=0,$$

and these points correspond respectively to A and B in Fig. 3. It is also obvious from Fig. 4 that neither θ_2 nor θ_3 enters into their corresponding unstable regions. Hence, in the end, the condition (3.16) for $n=1$ is sufficient to determine the stability of equilibrium states in the case when the non-linearity is characterized by a cubic function.

(3) Some remarks on the approximation in the foregoing analysis

As shown in Figs. 1 or 3, the stable range of equilibrium states is interposed between the critical points A and B at which w has its limiting values. Since w is proportional to B by equations (3.2) or (3.9), these stability limits take place when the applied force also has its limiting values. This is a plausible result from the physical point of view.

* In the case of $\delta=0$, this hyperbola is reduced to the straight lines (drawn by chain lines in Fig. 4) which intersect the abscissa at the point (1, 0), [cf. Fig. 2].

However, as mentioned before, the approximation (2.4) is only legitimate so long as the non-linearity is small, viz., $c_3 \ll c_1$ in the foregoing analysis. Thus, it might seem improper to apply this assumption to the case of $c_1=0$, $c_3=1$ in Figs. 2 or 4. But it will be explained in the following that the relation (2.4) may be applied with a fairly good approximation even when the non-linearity is predominant, and also that the stability limits above-mentioned are given by the condition that the applied force B has its limiting values.

For the sake of simplicity, we consider the non-dissipative case, and putting $c_1=0$, $c_3=1$ in equation (3.1), we have

$$\frac{d^2v}{d\tau^2}+v^3 = B\cos 3\tau .$$

Substituting the periodic solution

$$v = y\cos\tau + w\cos 3\tau ,$$

and equating to zero the coefficients of $\cos\tau$ and $\cos 3\tau$ respectively, we obtain

$$\left.\begin{aligned} & y^2+yw+2w^2-\frac{4}{3}=0 , \\ & \frac{1}{4}y^3+\frac{3}{2}y^2w+\frac{3}{4}w^3-9w = B . \end{aligned}\right\} \quad (3.17)$$

Fig. 5 is obtained by plotting (3.17). Thus we see that the approximation $w=-B/8$ may be applied without serious error. It is also noticed that, since B is no longer proportional to w, the limiting values of B and w do not take place simultaneously.

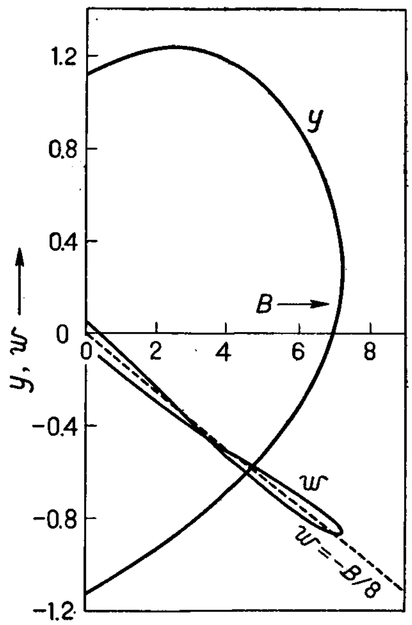

FIG. 5. Relationship between B and y, w in (3.17).

We shall next show that the stability limits are given by the limiting values of B, in other words, the parameters θ's in Hill's equation (3.5) take the characteristic values when $B=0$ or maximum.* For these limiting values of B, y and w are calculated by (3.17), and the parameters θ's are found from (3.5). They are shown in Table I below.

* In the case when the non-dissipative system is considered, the characteristic exponent in the solution of Hill's equation becomes zero at the stability limits.

Table I.† Values of y, w, and θ's for $B=0$ and $B=$Max.

B	y	w	θ_0	θ_1	θ_2	θ_3
0	1.127	0.0510	1.9092	1.0392	0.0862	0.0019
Max.=7.205	0.284	−0.8655	1.2443	−0.3078	−0.3681	0.5619

† For the case in which the non-linear characteristic is give by a cubic function.

The boundaries of the first unstable region are given by equations (3.6), namely,

$$\theta_0 = 1 \pm \theta_1 , \tag{3.18}$$

and, substituting the values of θ_1 in Table I, we obtain

$$\theta_0 = 2.0392 \quad \text{for} \quad B = 0 ,$$

$$\theta_0 = 1.3078 \quad \text{for} \quad B = \text{Max.}$$

These values differ from those given in Table I by 6.8% and 5.1% respectively. This is due to the deficiency of approximation in equation (3.18). Hence, for the accurate values of y, w calculated by equations (3.17), we have to apply the closer approximation given, for example, by the following development[4]

$$\begin{aligned}\theta_0 = 1 &\pm \theta_1 - \frac{1}{8}\theta_1^2 - \frac{1}{6}\theta_2^2 - \frac{1}{16}\theta_3^2 \pm \frac{1}{4}\theta_1\theta_2 \pm \frac{1}{12}\theta_2\theta_3 \\ &\mp \frac{1}{64}\theta_1^3 + \frac{1}{48}\theta_1^2\theta_2 \pm \frac{5}{192}\theta_1^2\theta_3 \pm \frac{1}{48}\theta_2^2\theta_3 \\ &\mp \frac{1}{144}\theta_1\theta_2^2 \mp \frac{1}{2304}\theta_1\theta_3^2 - \frac{13}{288}\theta_1\theta_2\theta_3 + \cdots .\end{aligned} \tag{3.19}$$

Now, substituting the values of $\theta_1 \sim \theta_3$ in Table I into equation (3.19), we obtain

$$\theta_0 = 1.9038 \quad \text{for} \quad B = 0 ,$$

$$\theta_0 = 1.2352 \quad \text{for} \quad B = \text{Max.},$$

which differ from the values given in Table I by 0.28% and 0.73% respectively. Thus the discrepancies are considerably reduced, and we may conclude that the stability limits are given by the condition that the applied force B has its limiting values.*

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[4] E. L. Ince, Monthly Notices of the Roy. Astron. Soc. **75**, 436 (1915).

* A more complete discussion on the stability limits will be reported later.

4. Subharmonic oscillation of order 1/3 with the non-linear characteristic: $u=c_1v+c_5v^5$

For the brevity of calculation, we consider the non-dissipative case only. Putting

$$\left.\begin{aligned} &\nu=3, \quad \delta=0, \\ &c_2=c_3=c_4=c_6=\cdots=0, \end{aligned}\right\}$$

in equations (1.1) and (2.1), and remembering the condition (2.2), we have

$$\frac{d^2v}{d\tau^2}+v=c_5(v-v^5)+B\cos 3\tau. \quad (4.1)$$

Substituting the periodic solution

$$v=y\cos\tau+w\cos 3\tau, \quad w=-B/8,$$

into (4.1), and equating to zero the coefficient of $\cos\tau$, we obtain

$$y^4+\frac{5}{2}y^3w+6y^2w^2+3yw^3+3w^4-\frac{8}{5}=0. \quad (4.2)$$

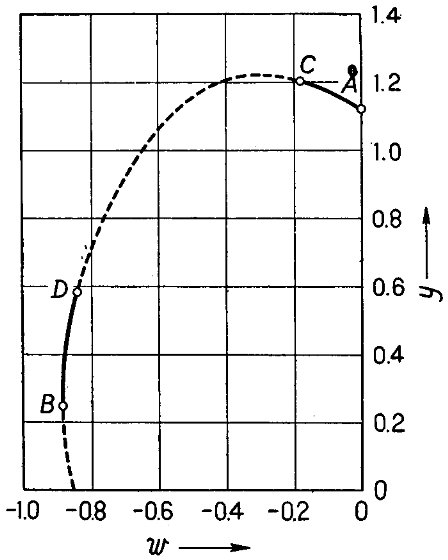

FIG. 6. Amplitude characteristic of 1/3-harmonic oscillation (non-linearity by quintic function).

The relationship between y and w is illustrated in Fig. 6. The negative part of y is omitted in the figure, because, as will be shown later, the equilibrium states in this part are unstable.

The stability problem may be treated in the same manner as in the preceding section, and the variational equation leads to the following Hill's equation

$$\left.\begin{aligned} &\frac{d^2\xi}{d\tau^2}+\left[\theta_0+2\sum_{s=1}^{6}\theta_s\cos 2s\tau\right]\xi=0, \\ &\text{where} \\ &\theta_0=c_1+\frac{5}{8}c_5(3y^4+4y^3w+12y^2w^2+3w^4), \\ &\theta_1=\frac{5}{4}c_5(y^4+3y^3w+3y^2w^2+3yw^3), \\ &\theta_2=\frac{5}{16}c_5(y^4+12y^3w+6y^2w^2+12yw^3), \\ &\theta_3=\frac{5}{4}c_5(y^3w+3y^2w^2+w^4), \\ &\theta_4=\frac{5}{8}c_5(3y^2w^2+2yw^3), \\ &\theta_5=\frac{5}{4}c_5yw^3, \\ &\theta_6=\frac{5}{16}c_5w^4. \end{aligned}\right\} \quad (4.3)$$

The stability condition for $n=1$ is obtained by substituting θ_0 and θ_1 into (3.6), and further by virtue of (4.2), we ultimately find

$$y^3+\frac{15}{8}y^2w+3yw^2+\frac{3}{4}w^3>0, \quad \text{for} \quad w<0. \tag{4.4}$$

Referring to Fig. 6, the equilibrium states in the range between A and B satisfy the condition (4.4). The stability limits A and B are given by the conditions that $w=0$ and $w=\text{Max.}$ respectively.

Fig. 7 shows the trajectories of θ_n ($n=1, 2, 3$) which are drawn by varying the value of w (or B) for the limiting case of $c_1=0$ and $c_5=1$. As expected

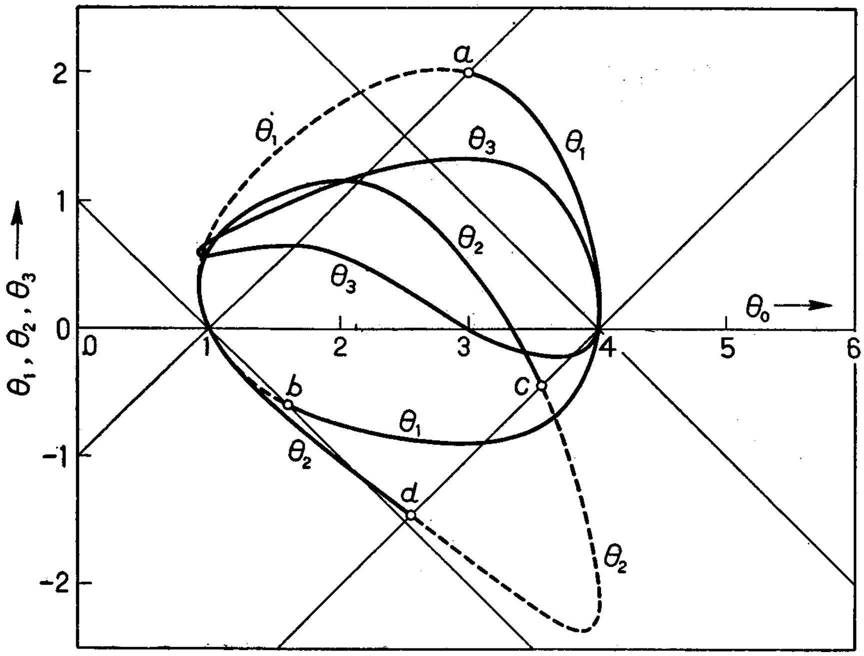

FIG. 7. Trajectories of θ's in (4.3) with varying w.

from the stability condition for $n=1$, θ_1 enters into the first unstable region in the dotted-line interval ab. We see moreover that θ_2 enters into the second unstable region in the dotted-line interval cd. Hence the stability condition for $n=2$ is no more satisfied in the interval cd, and the oscillation of order 2/3 will be excited, disturbing the continuation of the original subharmonic oscillation.

As mentioned above, the curves in Fig. 7 are drawn for the case of $c_1=0$ and $c_5=1$. With increasing c_1 (or decreasing c_5), however, these curves move towards the point (1, 0), as one sees by the expressions for θ's in equations (4.3). Hence, as the departure from linearity is reduced, the interval cd in the second unstable region contracts and finally disappears. One sees also that in no case the parameters $\theta_3 \sim \theta_6$ enter into their corresponding unstable regions. Hence, in the end, it may be concluded that the stability conditions for $n=2$ as well as for $n=1$ must be considered in the case when the non-linearity is characterized by a quintic function. The points A, B, C, D in Fig. 6 correspond respectively to the critical points a, b, c, d in Fig. 7, so that the subharmonic oscillation of order 1/3 is maintained only in the intervals AC and BD.

We shall now discuss the approximation in the foregoing analysis as we have done at the end of the preceding section. Let the differential equation be given by

$$\frac{d^2v}{d\tau^2}+v^5 = B\cos 3\tau .$$

Substituting the periodic solution

$$v = y\cos\tau + w\cos 3\tau ,$$

and equating to zero the coefficients of $\cos\tau$ and $\cos 3\tau$ respectively, we obtain

$$\left.\begin{aligned} &y^4+\frac{5}{2}y^3w+6y^2w^2+3yw^3+3w^4-\frac{8}{5}=0, \\ &\frac{5}{16}y^5+\frac{15}{8}y^4w+\frac{15}{8}y^3w^2+\frac{15}{4}y^2w^3+\frac{5}{8}w^5-9w = B. \end{aligned}\right\} \tag{4.5}$$

Fig. 8 is obtained by plotting (4.5). Thus we see that the approximation $w=-B/8$ may also be applied without serious error.

For $B=0$ and $B=$Max., the values of y, w and the parameters $\theta_0 \sim \theta_3$ in equations (4.3) are calculated and shown in Table II below.

Substituting the values of θ_1 into equation (3.18), we have

$$\theta_0 = 3.0327 \quad \text{for} \quad B=0,$$
$$\theta_0 = 1.3385 \quad \text{for} \quad B=\text{Max.}$$

These values differ from those given in Table II by 9.1% and 5.8% respectively. Applying again the closer approximation (3.19) instead of (3.18), we obtain

$$\theta_0 = 2.7234 \quad \text{for} \quad B=0,$$
$$\theta_0 = 1.2415 \quad \text{for} \quad B=\text{Max.}$$

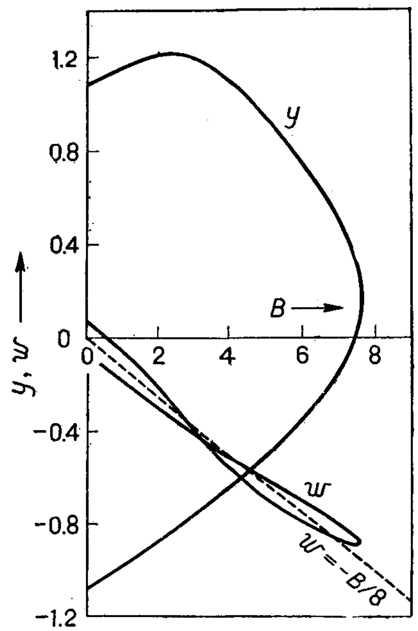

FIG. 8. Relationship between B and y, w in (4.5).

Table II.† Values of y, w, and θ's for $B=0$ and $B=$Max.

B	y	w	θ_0	θ_1	θ_2	θ_3
0	1.0759	0.0716	2.7795	2.0327	0.7655	0.1337
Max.=7.546	0.1542	−0.8818	1.2651	−0.3385	−0.3736	0.8210

† For the case in which the non-linear characteristic is given by a quintic function.

The discrepancies are reduced down to 1.9% and 2.0% respectively, and we may conclude that the stability limits of the first unstable region are given by the condition that the applied force B has its limiting values.

We have so far discussed the stability limits of the first unstable region. A similar investigation may be carried out for the second unstable region. But in this case the stability limits have no particular relation to the applied force, and besides, as one will see in a later experiment, the small parasitic oscillation of order 2/3 can coexist with the original subharmonic oscillation in the neighborhood of the stability limits. Hence the points c, d in Fig. 7, however accurately they might be determined, would not represent the exact critical points at which the original subharmonic oscillation is interrupted. Therefore, further investigation into the stability limits of the second unstable region is not so important and is omitted here.

5. Experimental considerations

In the present section we shall compare the theoretical results (in the preceding sections) with some experiments conducted for an electrical oscillatory circuit containing a saturable iron-core inductance and a capacitance. As shown previously by the author,[2] the circuit equation has the form of (1.1) when an alternating voltage (60 c.p.s. in our case) is applied to the circuit. With appropriately prescribed initial conditions, the subharmonic oscillation of order 1/3, i.e., of 20 c.p.s., may easily be started in the circuit.

Now, making use of a transformer core inductance as the non-linear element, we have first determined the region in which the subharmonic oscillation of order 1/3 is sustained. In Fig. 9a this region is depicted by hatched lines. The appearance of the vacant part inside the sustaining region is an arresting feature and was previously reported by the present author[5] and others.[6] But no theoretical consideration was given at that time.

From the preceding analysis, however, it will be deduced that the vacant

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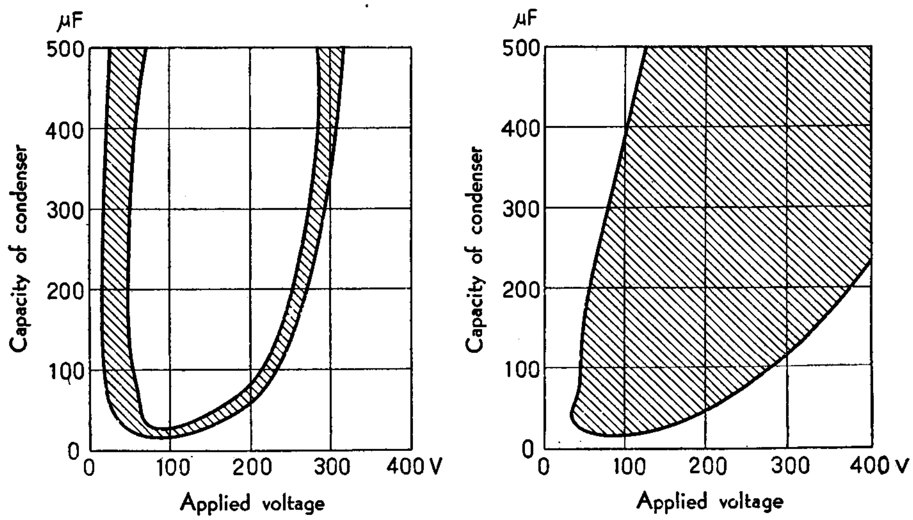
[5] C. Hayashi, Mitsubishi Denki 18, 128 (1942), [in Japanese].
[6] For instance, J. D. McCrumm, Trans. Amer. Inst. Elec. Eng. 60, 533 (1941).



(a) Magnetization curve by (5.1). (b) Magnetization curve by (5.2).

FIG. 9. Sustaining regions of 1/3-harmonic oscillation.

part above-mentioned corresponds to the unstable regions of order $n \geqq 2$, because the non-linear characteristic of the ordinary transformer core is expressed by

$$f(v) = c_1 v + c_3 v^3 + c_5 v^5 + c_7 v^7 + \cdots, * \tag{5.1}$$

in which the coefficients c_5, $c_7, \cdots$ predominate over c_1, c_3. If we use a core whose characteristic is expressed by

$$f(v) = c_1 v + c_3 v^3, \tag{5.2}$$

the vacant part inside the sustaining region shall be eliminated. Such a core is not available in practice, but we can obtain the characteristic (5.2) by connecting a number of inductance coils in series and adjusting the length of the air-gap which is interposed

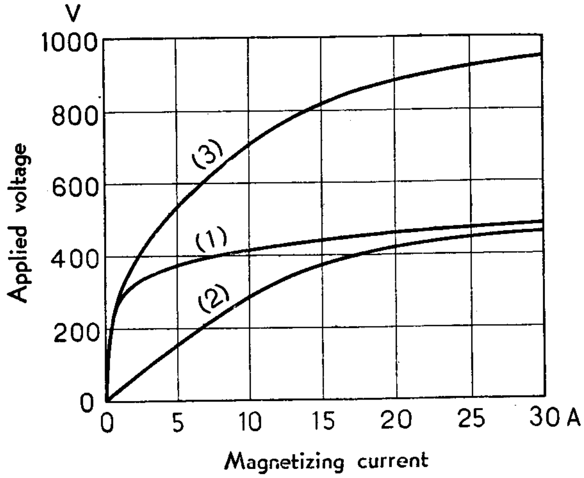
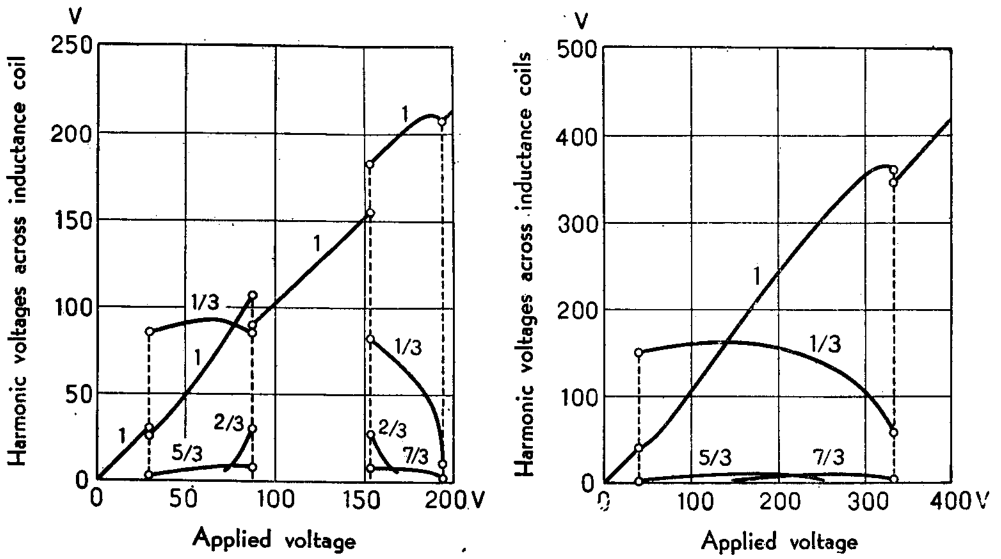


FIG. 10. Combined characteristic approximated to a cubic curve: (1) without air-gap, (2) with air-gap, (3) combined characteristic of (1) and (2).

* Physically, equation (5.1) represents the magnetization curve of the core, i.e., the relationship between the magnetic flux v and the magnetizing current $f(v)$.

in each core. Fig. 10 shows an example in which two cores are used, one with air-gap and the other without. The resultant characteristic shows a fairly good approximation to equation (5.2). By making use of this composite inductance, the sustaining region of the subharmonic oscillation of order 1/3 is determined and plotted in Fig. 9b. We see that the unstable oscillations corresponding to the unstable regions of order $n \geqq 2$ are completely excluded, and so the experimental verification is quite satisfactory.

We have further measured the harmonic contents in those oscillations with a heterodyne harmonic analyser. Fig. 11 a, b shows the result for the cases in



(a) Magnetization curve by (5.1). (b) Magnetization curve by (5.2).

FIG. 11. Harmonic analysis of oscillations.

which the non-linearities are given by equations (5.1) and (5.2) respectively. In Fig. 11a, we observe the higher harmonics of orders 2/3, 5/3, 7/3, ⋯, among which the oscillation of order 2/3 is significant, because it is this oscillation (related to the second unstable region) that grows up rapidly and interrupts the original subharmonic oscillation (see Figs. 7 and 9a). Whereas, in Fig. 11b, no such obstructive oscillation is observed, and the subharmonic oscillation of order 1/3 is sustained in the whole region (see Figs. 2 and 9b).

Finally it is added that the subharmonic oscillation of order 1/5 can occur when the non-linearity is given by equation (5.1), but this oscillation is by no means observed when the non-linearity is given by equation (5.2). These results also agree with the investigation in Section 2.

Acknowledgments

The author wishes particularly to acknowledge his indebtedness to Dr. R. Torikai at whose instigation this study was started. The author's thanks are also due to Drs. T. Matsumoto and S. Tomotika for their valuable criticisms and suggestions, and to Mr. H. Takatsuki for his assistance in carrying out the calculations. In publishing this paper, the author was fortunate in having the help of Dr. Tomotika.