Stability Investigation of the Non-linear Periodic Oscillations

By

Chihiro HAYASHI

Department of Electrical Engineering

(Received January, 1952)

SUMMARY

The stability of the non-linear periodic oscillations is discussed by solving a variational equation which characterizes small variations from the periodic states of equilibrium. This variational equation leads to a linear equation in which the coefficient is periodic in the time. If all solutions of this equation are bounded, then the oscillation is said to be stable, otherwise unstable. In order to establish the stability criterion, the characteristic exponents for the unbounded solutions are calculated by Whittaker's method. Then, the generalized stability condition is derived by comparing the said characteristic exponents with the damping of the system considered. Since the solutions of the variational equation have the form: $e^{\mu\tau}$ [$\sin(n\tau-\sigma)+\cdots$], our stability condition is secured not only for the unbounded solutions having the fundamental frequency ($n=1$), but also for the unbounded solutions with higher harmonic frequencies ($n=2, 3, 4, \cdots\cdots$). Hence the generalized stability condition obtained in this way is particularly effective in studying the oscillations in which the higher harmonics are excited. Finally our investigation is compared with one of the stability conditions derived by Mandelstam and Papalexi for the subharmonic oscillations.

In the appendix, the characteristic exponents are calculated at some length for the unbounded solutions of a variational equation in which the periodic coefficient involves sine series as well as cosine series.

1. Introduction

Non-linear oscillations governed by the differential equation:

$$\left.\begin{aligned} &\frac{d^2v}{d\tau^2}=F\left(v,\frac{dv}{d\tau},\tau\right), \\ \text{with}\quad & \\ &F\left(v,\frac{dv}{d\tau},\ \tau+T\right)=F\left(v,\frac{dv}{d\tau},\tau\right) \end{aligned}\right\} \qquad (1.1)$$

are not seldom encountered in several different kinds of physical problems. It has been pointed out by Trefftz (1) that if the solution of (1.1) is stable, it must finally lead to a periodic solution in which the period is equal to the period T of the external force, or as its least period equal to an integral multiple (different from unity) of T.* Consequently, a non-periodic oscillation, if established, must be unstable.

In the following lines we confine our attention to the periodic solutions which are essentially either harmonic or subharmonic even if higher harmonics may predominate. It is known from the theory of differential equations that (1.1) possesses such solutions $v(\tau)$ that are uniquely determined once the values of $v(0)$ and $(dv/d\tau)_{\tau=0}$, i.e. the initial conditions, are prescribed. It is, however, the distinctive character of non-linear oscillations that the various types of periodic solutions of (1.1) may exist corresponding to the different values of the initial conditions prescribed.

Contrary to many cases of linear differential equations, it is hardly possible to find the general solution of (1.1) for the given initial conditions. Moreover, since explicit solutions in terms of the elementary functions are not to be expected, the differential equation (1.1) is treated by various analytic approximation methods. As mentioned above, so long as we confine the problem to the periodic oscillations, our conventional method of solution is to assume for $v(\tau)$ a Fourier series development with undetermined coefficients, and then to fix them by the non-linear relations obtained by substituting the series into the original equation (1.1). It should, however, be noticed that this method of solution is merely to find out the periodic states of equilibrium which are not always sustained, but are only able to last out so long as they are stable. The circumstances under which this condition obtains are determined by a further stability investigation.

2. Stability of the periodic solutions

A state of equilibrium is said to be stable or unstable according to whether any variation from this state caused by a sufficiently small perturbation attenuates or not with the lapse of time. As a typical case of equation (1.1), we now consider the following non-linear differential equation:

$$\left.\begin{aligned} &\frac{d^2v}{d\tau^2}+f(v)\frac{dv}{d\tau}+g(v)=e(\tau)\,, \\ \text{with} \qquad &e(\tau+T)=e(\tau)\,. \end{aligned}\right\} \tag{2.1}$$

Let the periodic solution of (2.1) be expressed by

$$v(\tau)=\Phi(\tau)\,, \tag{2.2}$$

in which the period is equal to the period T of the external force, or as its least

* Corresponding to these two cases the terms "harmonic" and "subharmonic" oscillations are respectively applied.

period equal to an integral multiple of T. If the small variation from this periodic state is denoted by ξ, we obtain the following variational equation for ξ by the substitution of $v(\tau)+\xi$ in place of $v(\tau)$, i.e.,

$$\frac{d^2\xi}{d\tau^2}+f(v)\frac{d\xi}{d\tau}+\left[\frac{df}{d\tau}+\frac{dg}{dv}\right]\xi=0\,, \tag{2.3}$$

or

$$\frac{d^2\xi}{d\tau^2}+F(\tau)\frac{d\xi}{d\tau}+G(\tau)\xi=0\,, \tag{2.4}$$

in which $F(\tau)$ and $G(\tau)$ are periodic functions of τ, determined by the substitution of (2.2) into (2.3). Now, introducing a new variable η with the following relation:

$$\xi=\exp\left[-\frac{1}{2}\int F(\tau)d\tau\right]\cdot\eta\,, \tag{2.5}$$

equation (2.4) is transformed as follows:

$$\frac{d^2\eta}{d\tau^2}+\left[G(\tau)-\frac{1}{2}\frac{dF}{d\tau}-\frac{1}{4}\{F(\tau)\}^2\right]\eta=0\,. \tag{2.6}$$

This is a linear equation in which the coefficient of η is a periodic function of τ and may be developed into a Fourier series.

By Floquet's theorem (2) the general solution of (2.6) is given by

$$\eta=c_1e^{\mu\tau}\phi(\tau)+c_2e^{-\mu\tau}\psi(\tau),\quad c_1,\,c_2:\text{ constants}, \tag{2.7}$$

where μ is the characteristic exponent determined by the coefficients of the Fourier series in (2.6), and $\phi(\tau)$, $\psi(\tau)$ are periodic functions of τ in which the period is the same or twice as much as the period of the Fourier series.

Now we turn to the present stability investigation. As one readily sees from (2.5) and (2.7), the variation ξ tends to zero with the increase of τ if the real part of $-\frac{1}{2}F_0\pm\mu$ is negative, F_0 being the constant term in the series of $F(\tau)$, and the corresponding periodic state of equilibrium is stable. On the contrary, if the real part of $-\frac{1}{2}F_0\pm\mu$ is positive, the variation ξ diverges boundlessly with the increase of τ, and the corresponding periodic state is unstable. Hence, for establishing the stability criterion, it is necessary to evaluate the characteristic exponent μ in (2.7), and this will be discussed in the following section.

3. The stability problem for Hill's equation

As mentioned in the foregoing section, the variational equation associated with the stability of the periodic solution is reduced to the linear equation (2.6) with a periodic function of τ as its coefficient. The representative one of (2.6) is what is called a Hill's equation of the form:

$$\frac{d^2\eta}{d\tau^2}+\left(\theta_0+2\sum_{\nu=1}^{\infty}\theta_\nu\cos 2\nu\tau\right)\eta=0. \tag{3.1}$$

We shall now briefly discuss the solution of this equation by use of Whittaker's method of change of parameter.*

Substituting a solution:

$$\eta=e^{\mu\tau}\phi(\tau) \tag{3.2}$$

into (3.1), we obtain

$$\frac{d^2\phi}{d\tau^2}+2\mu\frac{d\phi}{d\tau}+\left[\theta_0+\mu^2+2\sum_{\nu=1}^{\infty}\theta_\nu\cos 2\nu\tau\right]\phi=0. \tag{3.3}$$

According to Whittaker the periodic function $\phi(\tau)$ in the n-th unstable region may be assumed to a first approximation in the form:

$$\phi(\tau)=\sin(n\tau-\sigma),\quad n=1,2,3,\cdots, \tag{3.4}$$

in which σ is a new parameter to be determined presently. Substituting (3.4) into (3.3) and equating to zero the coefficients of $\sin n\tau$ and $\cos n\tau$ respectively, we obtain

$$\left.\begin{aligned}2\mu n\sin\sigma+(\theta_0+\mu^2-n^2)\cos\sigma-\theta_n\cos\sigma&=0,\\ 2\mu n\cos\sigma-(\theta_0+\mu^2-n^2)\sin\sigma-\theta_n\sin\sigma&=0.\end{aligned}\right\} \tag{3.5}$$

Hence the characteristic exponent μ and the parameter σ are given by

$$\left.\begin{aligned}\mu&=\frac{\theta_n}{2n}\sin 2\sigma,\\ \text{with}\qquad \theta_0&=n^2+\theta_n\cos 2\sigma-\left(\frac{\theta_n}{2n}\right)^2\sin^2 2\sigma,\end{aligned}\right\} \tag{3.6}$$

from which we obtain, by eliminating σ,

$$\mu^2=-(\theta_0+n^2)\pm\sqrt{4n^2\theta_0+\theta_n^2}. \tag{3.7}$$

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* This method of solution was introduced by Whittaker (3) in obtaining the quasi-periodic solution in the neighbourhood of the characteristic functions (i.e. Mathieu functions) $ce_1(\tau)$ and $se_1(\tau)$ of Mathieu's equation—a special case of Hill's equation in which the periodic coefficient is a simple harmonic function of τ [i.e., $\theta_2=\theta_3=\cdots\cdots=0$ in (3.1)]. He assumed the solution in the form:

$$c_1e^{\mu\tau}[\sin(\tau-\sigma)+\cdots]+c_2e^{-\mu\tau}[\sin(\tau+\sigma)+\cdots],$$

where c_1 and c_2 are arbitrary constants and σ is a new parameter. This solution reduces to the Mathieu function $ce_1(\tau)$ for $\sigma=-\pi/2$, and to $se_1(\tau)$ for $\sigma=0$. Generally σ has a value lying between $-\pi/2$ and 0 for the unbounded solution in which the characteristic exponent μ may be considered as real.

Later, Young (4) has applied this method of solution to obtain the quasi-periodic solution in the neighbourhood of the Mathieu functions $ce_2(\tau)$, $se_2(\tau)$ and $ce_3(\tau)$, $se_3(\tau)$.

In our paper, we define the n-th unstable region in θ_0, θ_1-plane as associated with the unbounded solutions interposed between $ce_n(\tau)$ and $se_n(\tau)$, and apply the same term to the case of Hill's equation.

Since the characteristic exponent μ may be assumed to be purely imaginary or real according to whether the solution is bounded or not for all positive values of the time, we have $\mu^2>0$ for the unbounded solution [cf. equation (2.7)]. This condition is transformed by (3.7) to

$$\left.\begin{aligned}&(\theta_0-n^2+\theta_n)(\theta_0-n^2-\theta_n)<0\,,\\ \text{or}\quad &|\theta_n|>|\theta_0-n^2|\,.\end{aligned}\right\} \tag{3.8}$$

Since $\mu=0$ on the boundaries between the stable and unstable regions, the boundary lines of the n-th unstable region are given by

$$\theta_0 = n^2 \pm \theta_n\,, \tag{3.9}$$

which are also derived directly by putting $\sigma=-\pi/2$ and $\sigma=0$ in the second equation of (3.6).

It is apparent from these equations that the values of μ, σ and consequently the boundaries of the n-th unstable region are determined only by θ_0 and θ_n, and are not affected by other parameters. This is because we have confined our calculation to the first approximation. By the closer approximation, however, the remaining parameters will be related to them, as will be shown in the appendix where the variational equation considered contains the periodic coefficient involving sine series as well as cosine series.

4. Condition for the stability of the non-linear periodic oscillations

Following the preceding considerations, we shall derive the stability condition for the periodic oscillations governed by the following equation:

$$\frac{d^2v}{d\tau^2}+2\delta\frac{dv}{d\tau}+f(v)=e(\tau)\,, \tag{4.1}$$

in which 2δ is a constant damping coefficient, $f(v)$ a non-linear term and $e(\tau)$ a periodic external force. Let the variation of v be ξ. Then, corresponding to (2.5) and (2.6), we have

$$\xi = e^{-\delta\tau}\cdot\eta\,,$$

and

$$\frac{d^2\eta}{d\tau^2}+\left(\frac{df}{dv}-\delta^2\right)\eta=0\,. \tag{4.2}$$

Now, once the periodic state of equilibrium is determined (usually by applying either iteration or perturbation methods), the coefficient of η in the last equation may be developed into a Fourier series, so that (4.2) leads to

$$\frac{d^2\eta}{d\tau^2}+\left[\theta_0+2\sum_{\nu=1}^{\infty}\theta_\nu\cos(2\nu\tau-\varepsilon_\nu)\right]\eta=0\,. \tag{4.3}$$

According to the investigation in Section 2, the stability condition in this case is given by $\delta > |\mu|$, or substituting (3.7) we obtain*

$$(\theta_0 - n^2)^2 + 2(\theta_0 + n^2)\delta^2 + \delta^4 > \theta_n^2, \quad n = 1, 2, 3, \cdots. \tag{4.4}$$

This is the stability condition (to a first approximation**) for the n-th unstable region, so that, in order that the periodic state of equilibrium is stable, the condition (4.4) must be satisfied for all values of n simultaneously.

5. Some complementary remarks on the stability condition

In this section we shall compare the stability condition obtained in the foregoing section with the one hitherto reported, and explain the physical meaning of the instability in the n-th unstable region.

Stability investigations in non-linear oscillations are to be found in many physical and technical journals. Here the one reported by Mandelstam and Papalexi (5) with elegant form will be taken up for comparison. They have discussed the subharmonic oscillations in vacuum tube circuits governed by the following equation:

$$\frac{d^2v}{d\tau^2} + v = \lambda \cdot F\left(v, \frac{dv}{d\tau}\right) + B\cos\nu\tau, \quad \nu = 2, 3, 4, \cdots, \tag{5.1}$$

in which the parametric coefficient λ of the non-linear function $F(v, dv/d\tau)$ is sufficiently small. They have treated the problem by the perturbation method, and obtained the following periodic solution for the subharmonic oscillation of order $1/\nu$, i.e.,

$$v = x\sin\tau + y\cos\tau + w\cos\nu\tau, \quad w = \frac{B}{1-\nu^2}. \tag{5.2}$$

in which the amplitudes x and y are to be determined by the conditions:

$$\left.\begin{aligned} &\int_0^{2\pi} \psi(x, y, \tau)\sin\tau d\tau = 0, \\ &\int_0^{2\pi} \psi(x, y, \tau)\cos\tau d\tau = 0, \\ \text{where}\quad &\psi(x, y, \tau) = F(x\sin\tau + y\cos\tau + w\cos\nu\tau, \\ &\qquad x\cos\tau - y\sin\tau - \nu w\sin\nu\tau). \end{aligned}\right\} \tag{5.3}$$

This is quite the same relation as that obtained by substituting (5.2) into (5.1)

* So long as we confine our calculation to a first approximation, equation (3.7) may be applied in case of $\varepsilon_\nu \neq 0$.

** In the case when the stability condition of the higher order approximation is preferable, we should refer to the appendix for the closer evaluation of μ.

and equating to zero the coefficients of $\sin\tau$ and $\cos\tau$ respectively. Then, as for the stability condition, they have derived the following relation:

$$\begin{vmatrix} \int_0^{2\pi}\frac{\partial\psi}{\partial x}\sin\tau d\tau & \int_0^{2\pi}\frac{\partial\psi}{\partial x}\cos\tau d\tau \\ \int_0^{2\pi}\frac{\partial\psi}{\partial y}\sin\tau d\tau & \int_0^{2\pi}\frac{\partial\psi}{\partial y}\cos\tau d\tau \end{vmatrix} > 0, \tag{5.4}$$

which has been deduced from the consideration that the variations of the amplitudes x and y of the subharmonic oscillation tend to zero with the lapse of time τ.

On the other hand, in our preceding investigation, the differential equation which governs the oscillation is given by (4.1), in which the damping coefficient 2δ is constant, but not necessarily small. In order to investigate the subharmonic oscillations, the external force $e(\tau)$ in (4.1) may conveniently be expressed by $B\cos\nu\tau$, and then the periodic solution will be given by (5.2).

Now we shall proceed to show that the stability condition (5.4) is included in our equation (4.4) as one of the conditions corresponding to $n=1$. Since $v(\tau)$ is a periodic function of τ [cf. equation (5.2)], df/dv in (4.2) may be expanded in a Fourier series as:

$$\left.\begin{aligned} \frac{df}{dv} &= a_0 + a_1\cos 2\tau + a_2\cos 4\tau + \cdots \\ &\quad + b_1\sin 2\tau + b_2\sin 4\tau + \cdots, \\ \text{where}\quad a_0 &= \frac{1}{2\pi}\int_0^{2\pi}\frac{df}{dv}d\tau, \\ a_\nu &= \frac{1}{\pi}\int_0^{2\pi}\frac{df}{dv}\cos 2\nu\tau d\tau, (\nu = 1, 2, 3, \cdots), \\ b_\nu &= \frac{1}{\pi}\int_0^{2\pi}\frac{df}{dv}\sin 2\nu\tau d\tau, (\nu = 1, 2, 3, \cdots). \end{aligned}\right\} \tag{5.5}$$

Substituting this into (4.2), and comparing with (4.3), we obtain

$$\left.\begin{aligned} \theta_0 &= a_0 - \delta^2, \\ 2\theta_\nu &= \sqrt{a_\nu^2 + b_\nu^2}, \quad \varepsilon_\nu = \arctan\frac{b_\nu}{a_\nu}. \end{aligned}\right\} \tag{5.6}$$

Hence the stability condition (4.4) may be written as:

$$(a_0 - n^2)^2 + 4n^2\delta^2 > \frac{1}{4}(a_n^2 + b_n^2),$$

and further, substituting for a_0, a_n, and b_n their values as given by (5.5), we have

$$\left[\int_0^{2\pi}\left(\frac{df}{dv} - n^2\right)d\tau\right]^2 + 16n^2\pi^2\delta^2 > \left[\int_0^{2\pi}\frac{df}{dv}\cos 2n\tau\, d\tau\right]^2 + \left[\int_0^{2\pi}\frac{df}{dv}\sin 2n\tau\, d\tau\right]^2,$$

or

$$\int_0^{2\pi}\left(\frac{df}{dv}-n^2\right)\sin^2 n\tau\, d\tau\cdot\int_0^{2\pi}\left(\frac{df}{dv}-n^2\right)\cos^2 n\tau\, d\tau$$
$$-\left[\int_0^{2\pi}\left(\frac{df}{dv}-n^2\right)\sin n\tau\cos n\tau\, d\tau\right]^2+4n^2\pi^2\delta^2>0.$$

This stability condition may be rewritten in a form similar to that of (5.4) as:

$$\left.\begin{aligned}&\begin{vmatrix}\int_0^{2\pi}\Psi_x\sin n\tau\, d\tau & \int_0^{2\pi}\Psi_x\cos n\tau\, d\tau\\ \int_0^{2\pi}\Psi_y\sin n\tau\, d\tau & \int_0^{2\pi}\Psi_y\cos n\tau\, d\tau\end{vmatrix}>0,\\ &\text{in which}\\ &\Psi_x=\left(\frac{df}{dv}-n^2\right)\sin n\tau+2n\delta\cos n\tau,\\ &\Psi_y=\left(\frac{df}{dv}-n^2\right)\cos n\tau-2n\delta\sin n\tau.\end{aligned}\right\}\quad(5.7)$$

Now the stability condition (5.4) may be derived by putting $n=1$ in (5.7), i.e., upon comparing (5.1) with (4.1), we have

$$\psi(x, y, \tau)=F\left(v,\frac{dv}{d\tau}\right)=\frac{1}{\lambda}\left[v-f(v)-2\delta\frac{dv}{d\tau}\right],$$

and hence, by (5.2),

$$\frac{\partial\psi}{\partial x}=\frac{1}{\lambda}\left(1-\frac{df}{dv}\right)\sin\tau-\frac{2\delta}{\lambda}\cos\tau=-\frac{1}{\lambda}[\Psi_x]_{n=1},$$

$$\frac{\partial\psi}{\partial y}=\frac{1}{\lambda}\left(1-\frac{df}{dv}\right)\cos\tau+\frac{2\delta}{\lambda}\sin\tau=-\frac{1}{\lambda}[\Psi_y]_{n=1}.$$

The substitution of these two relations into (5.7) will yield (5.4).

Therefore, the stability condition given by Mandelstam and Papalexi offers no information for $n\geqq 2$. This is because they have discussed the problem by the perturbation method, assuming that the non-linearity expressed by λ in (5.1) is sufficiently small. Whereas in our investigation, the generalized stability condition (4.4) or (5.7) for the n-th unstable region will furnish the criterion to distinguish the stability for the n-th harmonic of the fundamental oscillation. This will be clear from the following consideration that an oscillation with n times the fundamental frequency is excited in the n-th unstable region, for the solution of the variational equation in this region has the form: $e^{\mu\tau}\cdot[\sin(n\tau-\sigma)+\text{higher order terms in }\theta_1,\theta_2,\cdots]$.

Our generalized condition is effective when we investigate the harmonic oscillation in which the higher harmonics are predominantly superposed with negative damping. Our condition is also effective in the case when we discuss the stability of the subharmonic oscillations;— e.g., in studying the subharmonic oscillation of order 1/3, we are frequently encountered with the self-excitation of the second harmonic of the subharmonic oscillation, i.e. the oscillation of order 2/3, which will no longer permit the continuation of the original subharmonic oscillation. Thus, the stability condition for the first unstable region is not sufficient in this case, and the instability above-mentioned may be detected by putting $n=2$ in (4.4) or (5.7).*

Acknowledgements

This is a part of the author's study on non-linear oscillations performed under the guidance of Dr. R. Torikai. The author is heartily grateful to his help and encouragement. The author's thanks are also due to Drs. T. Matsumoto and S. Tomotika for their valuable critisms and suggestions. In the preparation of this paper, the author is particularly indebted to Dr. Tomotika for checking the manuscript with much good advice of all kinds.

APPENDIX

Evaluation of the characteristic exponent for the variational equation (4.3)

As has been indicated in Section 4, the variational equation associated with the periodic oscillation is given by (4.3). If all the arguments ε_ν's are zero, equation (4.3) reduces to a Hill's equation. As far as we are aware, there has been no report on the unbounded solution of (4.3) in which the periodic coefficient involves sine series as well as cosine series. As noticed in the text, the closer evaluation of the characteristic exponent will be necessary in the case when the stability condition of the higher order approximation is desired. We shall, therefore, write down some expansions for the characteristic exponent μ by Whittaker's method for the following equation:

$$\frac{d^2\eta}{d\tau^2}+\left[\theta_0+2\sum_{\nu=1}^{4}\theta_\nu \cos(2\nu\tau-\varepsilon_\nu)\right]\eta=0.$$

(a) For the unbounded solution associated with the first unstable region:—

The characteristic exponent μ is given by the following expansion:

$$\mu=\frac{1}{2}\theta_1\sin 2\sigma+\frac{1}{8}\theta_1\theta_2\sin(2\sigma+2\varepsilon_1-\varepsilon_2)$$
$$+\frac{1}{24}\theta_2\theta_3\sin(2\sigma+\varepsilon_1+\varepsilon_2-\varepsilon_3)$$
$$+\frac{1}{48}\theta_3\theta_4\sin(2\sigma+\varepsilon_1+\varepsilon_3-\varepsilon_4)+\cdots,$$

* An application of the generalized stability condition to the subharmonic oscillation of order 1/3 will be reported in the following paper.

in which the parameter σ is to be determined by

$$\theta_0 = 1+\theta_1 \cos 2\sigma + \left(-\frac{1}{4} + \frac{1}{8}\cos 4\sigma\right)\theta_1^2 - \frac{1}{6}\theta_2^2$$
$$-\frac{1}{16}\theta_3^2 - \frac{1}{30}\theta_4^2 + \frac{1}{4}\theta_1\theta_2 \cos(2\sigma+2\varepsilon_1-\varepsilon_2)$$
$$+\frac{1}{12}\theta_2\theta_3 \cos(2\sigma+\varepsilon_1+\varepsilon_2-\varepsilon_3)$$
$$+\frac{1}{24}\theta_3\theta_4 \cos(2\sigma+\varepsilon_1+\varepsilon_3-\varepsilon_4)+\cdots.$$

(b) For the unbounded solution associated with the second unstable region :—

Similarly to the preceding case, the expansions for μ and θ_0 are given by

$$\mu = \frac{1}{4}\theta_2 \sin 2\sigma - \frac{1}{16}\theta_1^2 \sin(2\sigma-2\varepsilon_1+\varepsilon_2)$$
$$+\frac{1}{24}\theta_1\theta_3 \sin(2\sigma+\varepsilon_1+\varepsilon_2-\varepsilon_3)$$
$$+\frac{1}{64}\theta_2\theta_4 \sin(2\sigma+2\varepsilon_2-\varepsilon_4)+\cdots,$$

$$\theta_0 = 4+\theta_2 \cos 2\sigma + \left[\frac{1}{6} - \frac{1}{4}\cos(2\sigma-2\varepsilon_1+\varepsilon_2)\right]\theta_1^2$$
$$+\left(-\frac{1}{16}+\frac{1}{32}\cos 4\sigma\right)\theta_2^2 - \frac{1}{10}\theta_3^2 - \frac{1}{24}\theta_4^2$$
$$+\frac{1}{6}\theta_1\theta_3 \cos(2\sigma+\varepsilon_1+\varepsilon_2-\varepsilon_3)$$
$$+\frac{1}{16}\theta_2\theta_4 \cos(2\sigma+2\varepsilon_2-\varepsilon_4)+\cdots.$$

(c) For the unbounded solution associated with the third unstable region :—

$$\mu = \frac{1}{6}\theta_3 \sin 2\sigma - \frac{1}{24}\theta_1\theta_2 \sin(2\sigma-\varepsilon_1-\varepsilon_2+\varepsilon_3)$$
$$+\frac{1}{48}\theta_1\theta_4 \sin(2\sigma+\varepsilon_1+\varepsilon_3-\varepsilon_4)+\cdots,$$

$$\theta_0 = 9+\theta_3 \cos 2\sigma + \frac{1}{16}\theta_1^2 + \frac{1}{10}\theta_2^2 + \left(-\frac{1}{36}+\frac{1}{72}\cos 4\sigma\right)\theta_3^2$$
$$-\frac{1}{14}\theta_4^2 - \frac{1}{4}\theta_1\theta_2 \cos(2\sigma-\varepsilon_1-\varepsilon_2+\varepsilon_3)$$
$$+\frac{1}{8}\theta_1\theta_4 \cos(2\sigma+\varepsilon_1+\varepsilon_3-\varepsilon_4)+\cdots.$$

The periodic functions $\phi(\tau)$ and $\psi(\tau)$ in the solutions [cf. equation (2.7)] are obtained at the same time. They have, however, no direct concern to the stability problem, so that the lengthy expansions for $\phi(\tau)$ and $\psi(\tau)$ will be omitted here.

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