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By

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It is known that there exists laminar layer on the surface of bodies even in the case of turbulent flow. The author has once treated the problem of this laminar sub-layer (reference 1). Rotta has published his paper on the same problem in "Ingenieur-Archiv" (reference 2). His result coincides with the author's. Rotta has treated the effect of the rough surface, while the author had dealt with the effect of pressure gradient; therefore, in this paper, the result obtained by the author will be explained.

1. Thickness of Laminar Sub-Layer

Firstly, the flow along the smooth flat plate without pressure gradient or the flow in the straight smooth pipe will be described.

In the case of turbulent flow, the shearing stress τ is expressed as follows:

$$\tau = \rho\nu\frac{du}{dy}+\rho l^2\left(\frac{du}{dy}\right)^2,$$

where ν is the kinematic coefficient of viscosity, ρ the density of fluid, u the velocity at distance y from the surface, and l the mixing length of turbulent flow. Usually, it is assumed that l is proportional to y. Let $l=\kappa y$, then

$$\tau = \rho\nu\frac{du}{dy}+\rho\kappa^2y^2\left(\frac{du}{dy}\right)^2.$$

Considering velocity distribution in the neighbourhood of the surface, assume τ is constant and equal to the shearing stress τ_0 at the surface, and express the friction velocity, i.e., $\sqrt{\frac{\tau_0}{\rho}}$, by u^*, then

$$1 = \frac{d\mathrm{u}}{d\mathrm{y}}+\kappa^2\mathrm{y}^2\left(\frac{d\mathrm{u}}{d\mathrm{y}}\right)^2. \tag{1}$$

For the sake of simplicity, the following notations are used:

$$\frac{u}{u^*}=\mathrm{u}, \quad \frac{u^*y}{\nu}=\mathrm{y}, \quad \text{etc.}$$

By integrating eq. (1) with $\kappa=0.4$, the line *a* shown in Fig. 1 is obtained. Fig. 1 shows velocity distribution. $\frac{u}{u^*}$ and $\frac{u^*y}{\nu}$ are taken as coordinates, and the line *b* shows pure laminar flow

$$\mathrm{u}=\mathrm{y}.$$

On the other hand, it is found in the experiment that the following relation holds in the turbulent region:

$$\mathrm{u}=5.5+5.75\log_{10}\mathrm{y}, \qquad (2)$$

which is shown by the line *c* in Fig. 1. So it is observed that there is a large difference between lines *a* and *c*. Now, let

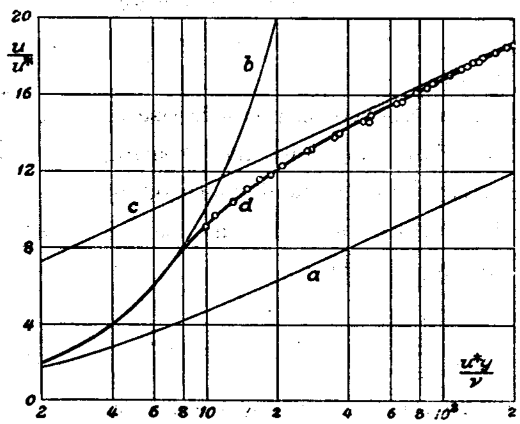

Fig. 1.

$$l=\kappa(y-y_0), \qquad (3)$$

where y_0 is a properly chosen distance, then

$$\frac{d\mathrm{u}}{d\mathrm{y}}+\kappa^2(\mathrm{y}-\mathrm{y}_0)^2\left(\frac{d\mathrm{u}}{d\mathrm{y}}\right)^2=1. \qquad (4)$$

By taking $\frac{u^*y_0}{\nu}=\mathrm{y}_0=6.8$ and integrating the above equation, the line *d* shown in Fig. 1 is obtained, which coincides exactly with the experimental result of Nikuradse (reference 3).

From the above calculation, it is concluded that there is a pure laminar region on the surface of the flat plate and pipe where velocity fluctuation may exist, but without correlation.

This result is the same with that obtained by Rotta except the constant $\mathrm{y}_0=6.8$.

2. Laminar Sub-Layer in the Case of Flow with Pressure Gradient

Relation between u and y is already well established in the case of the smooth straight pipe. It seems, however, that the same relation also holds in the region close to the surface of body even when the flow is accompanied by fair pressure rise and fall. Fig. 2 shows some examples which are calculated from experimental results obtained by Nikuradse in the case of convergent and divergent channels (reference 4) and by Fage and Falkner in the case of aerofoil (reference 5). Experimental results in these instances do not coincide satisfactorily with the line of eq. (2), viz. $u = 5.5 + 5.75 \log_{10} y$, which is shown by straight lines in Fig. 2. All points which show the measured velocities lie below the lines.

But in the present calculation, it is assumed that eq. (2) also holds.

When there is pressure rise or fall along the flow, the shearing stress τ in the neighbourhood of the surface of body can be expressed as follows:

$$\tau = \tau_0\left(1 + a_0\frac{y}{\delta}\right), \qquad (5)$$

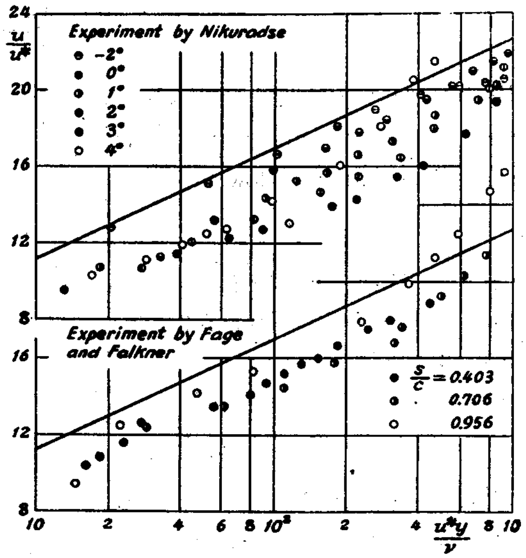

Fig. 2.

where τ is the shearing stress at y, and δ the thickness of boundary layer. Value of a_0 is theoretically given by the relation $a_0 = \frac{\delta}{\tau_0} \cdot \frac{dp}{dx}$, where p is the static pressure, and x the distance measured along the surface.

If Kármán's similarity hypothesis of turbulent flow (reference 6) is applied, then

$$l = \kappa(y - y_0)\left\{1 + a_0\frac{y - y_0}{4\delta}\right\}. \qquad (6)$$

By eq. (4)

$$\begin{aligned} y = {} & y_0 + \frac{1}{2}a_0 y_0^2 \\ & + \int_0^y \frac{1}{2l^2}\left\{1 + 4l^2(1 + a_0 y)\right\}^{\frac{1}{2}} d(y - y_0) \\ & - \int_0^y \frac{1}{2l^2} d(y - y_0), \end{aligned} \qquad (7)$$

where $a_0 = \frac{a_0 \nu}{u^* \delta}$, $l = \frac{u^* l}{\nu}$ and l is given by eq. (6). By integrating the above equation numerically and determining y_0 so that velocity reaches the value given by eq. (2) at $y = 40$ except in the case of $a_0 = -0.0362$, the following table is obtained:

$\frac{a_0 \nu}{u^* \delta}$	-0.0362	-0.02	0	0.02	0.04
$\frac{u^* y_0}{\nu}$	27.6	10	6.8	5.7	5.1

Hence, the laminar sub-layer is thin in the case of the flow with pressure rise, and thick in the case of pressure fall. As shown in Fig. 2, the result of experiment shows that the velocity distribution in the case of pressure rise lies below the line given by eq. (2), so it can be considered that the sub-layer is thinner than the calculated value given in the table. Anyway it can be concluded that the laminar sub-layer is thin when the pressure rises in the direction of flow.

3. Shearing Stress Distribution in the Laminar Sub-Layer

In paragraph 2, the distribution of shearing stress in the laminar sub-layer is assumed to be eq. (5). However, a_0 itself varies in the laminar part, or τ can be expressed by

$$\tau = \tau_0\left\{1+a_1\left(\frac{y}{\delta}\right)+a_2\left(\frac{y}{\delta}\right)^2+\cdots\cdots\right\}. \tag{8}$$

In the following pages, variation of a_1 will be considered. The equation of the two-dimensional steady flow is as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cdot\frac{dp}{dx}+\frac{1}{\rho}\cdot\frac{\partial\tau}{\partial y}. \tag{9}$$

At $y=0$, $u=v=0$; hence $\left(\frac{\partial\tau}{\partial y}\right)_{y=0}=\frac{dp}{dx}$. From this relation,

$$a_1 = \frac{\delta}{\tau_0}\cdot\frac{dp}{dx}$$

is obtained, and in paragraph 2 this value is taken as the theoretical value of a_0.

From eq. (8)

$$\frac{\partial u}{\partial y} = \frac{\tau_0}{\rho\nu}\left(1+a_1\frac{y}{\delta}+\cdots\cdots\right),$$

$$u = \frac{\tau_0 y}{\rho\nu}\left(1+a_1\frac{y}{2\delta}+\cdots\cdots\right).$$

By the equation of continuity

$$\begin{aligned} v &= -\int_0^y \frac{\partial u}{\partial x}dy \\ &= -\frac{y^2}{2\rho\nu}\cdot\frac{d\tau_0}{dx}\left(1+a_1\frac{y}{3\delta}+\cdots\right)-\frac{\tau_0 y^3}{6\rho\nu\delta}\cdot\frac{da_1}{dx} \\ &\quad +\frac{\tau_0 a_1 y^3}{6\rho\nu\delta^2}\cdot\frac{d\delta}{dx}+\cdots\cdots \end{aligned}$$

Hence

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= \frac{\tau_0 y^2}{\rho^2\nu^2}\left\{\frac{1}{2}\cdot\frac{d\tau_0}{dx}+\frac{y}{3}\cdot\frac{d^2p}{dx^2}+\cdots\right\} \\ &\approx \frac{\tau_0 y^2}{2\rho^2\nu^2}\cdot\frac{d\tau_0}{dx}. \end{aligned} \tag{10}$$

From eqs. (9) and (10)

$$\frac{1}{\rho}\cdot\frac{\partial\tau}{\partial y}=\frac{1}{\rho}\cdot\frac{dp}{dx}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y},$$

$$\frac{1}{\rho}\left\{\left(\frac{\partial\tau}{\partial y}\right)_{y=0}-\frac{\partial\tau}{\partial y}\right\}=-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}$$

$$\approx-\frac{\tau_0 y^2}{2\rho^2\nu^2}\cdot\frac{d\tau_0}{dx}, \qquad (11)$$

or

$$a_2\approx\frac{y\delta^2}{4\rho^2\nu^2}\cdot\frac{d\tau_0}{dx}.$$

Now, let

$$\tau_0=\xi\rho u_0^2\left(\frac{\nu}{u_0\theta}\right)^{\frac{1}{4}},$$

where θ is the momentum thickness, and u_0 the velocity outside the boundary layer, then

$$\frac{d\tau_0}{dx}=\frac{7\tau_0}{4u_0}\cdot\frac{du_0}{dx}-\frac{\tau_0}{4\theta}\cdot\frac{d\theta}{dx}.$$

On the other hand

$$\frac{d\theta}{dx}=-\frac{\theta}{u_0}\cdot\frac{du_0}{dx}(2+H)+\frac{\tau_0}{\rho u_0^2},$$

where H is the ratio between displacement thickness and momentum thickness. Hence

$$\frac{d\tau_0}{dx}=-\frac{\tau_0}{\rho u_0^2}\left\{\frac{9+H}{4}\cdot\frac{dp}{dx}+\frac{\tau_0}{4\theta}\right\}. \qquad (12)$$

Mean value of H is about 1.4 and $\delta\approx 8\theta$, so from eqs. (11) and (12) the direction of tangent line to the shearing stress distribution curve at the outside of the laminar sub-layer is given by

$$\frac{\delta}{\tau_0}\cdot\frac{\partial\tau}{\partial y}=\frac{\delta}{\tau_0}\left(\frac{\partial\tau}{\partial y}\right)_{y=0}-\frac{\mathrm{y}^2}{\mathrm{u}_0^2}(1.3a_1+1), \qquad (13)$$

where $\mathrm{u}_0=\frac{u_0}{u^*}$.

In the case of flow in the divergent channel in which turbulent flow is well developed,

$$\frac{d\tau_0}{dx}=-2\tau_0\frac{\alpha}{b},$$

where α is half the angle of divergence, and b is half the breadth of the channel. Hence,

$$\frac{b}{\tau_0}\cdot\frac{\partial\tau}{\partial y}=\frac{b}{\tau_0}\left(\frac{\partial\tau}{\partial y}\right)_{y=0}-\mathrm{y}^2\alpha, \qquad (14)$$

and

$$\left(\frac{\partial\tau}{\partial y}\right)_{y=0}=\frac{b}{\tau_0}\cdot\frac{dp}{dx}.$$

Fig. 3 shows the shearing stress distribution obtained by Nikuradse in his experiment already mentioned (reference 4). As the effects of viscosity and turbulent velocity fluctuation on the shearing stress are almost equal to each other at y=10, tangent line to the shearing stress distribution curve for the region of turbulent flow is calculated by eq. (14) by taking y=10, and shown in the accompanying table and in Fig. 3. The directions of these tangent lines coincide well with the experimental results.

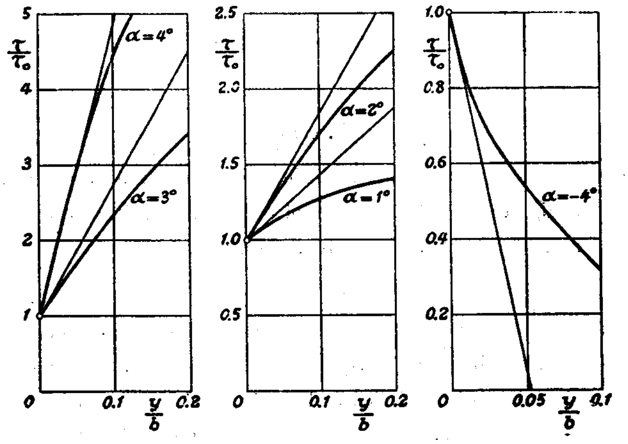

Fig. 3.

α°	4°	3°	2°	1°	0°	−2°	−4°	−8°
$\frac{b}{\tau_0}\cdot\frac{dp}{dx}$	45.9	23.1	11.9	6.12	−1.01	−14.8	−24.4	−43.9
$\frac{b}{\tau_0}\cdot\frac{dp}{dx}-y^2{}_x$	38.9	17.9	8.4	4.4	−1.01	−11.3	−19.2	−36.9

In eq. (5) in paragraph 2, it is assumed that $a_0=\frac{\delta}{\tau_0}\cdot\frac{dp}{dx}$, but as is shown above, $\frac{\partial\tau}{\partial y}$ varies in the laminar sub-layer, so correction must be made as to a_0. It seems more reasonable to take the value of $\frac{\partial\tau}{\partial y}$ at the outer boundary of the laminar part. Still, there is no change in the coclusion obtained in paragraph 2.

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