Effects of Anisotropy of Media on the Self-potential Curves of Electrical Prospecting

By

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1. Introduction

When we study theoretically the surface potential due to some electrical sources in electrical prospecting, we usually pick up the medium in which the orebody lies as the homogeneous and isotropic one. However, it is evident that this assumption is too far from the real circumstances, for the fields we try to prospect are generally heterogeneous and anisotropic. Therefore in interpreting the field data obtained by the electrical prospecting, it is very important to study the effects of anisotropy of media on the surface potential curves. As for the resistivity method of electrical prospecting this problem has already been treated by some authors, for instance, by Schlumberger and Leonardon [1], by Slichter [2], by Müller [3] and by Pirson [4].

The present author applied the problem for the case of the spontaneous polarization method of electrical prospecting and will report here the results obtained for the case where the polarized orebody of the rod type lies in the anisotropic medium.

2. Theoretical Considerations

First of all, let us consider the simple case, that is, the medium in which the orebody lies is composed of a single anisotropic stratum.

Defining the resistivity parallel to the surface plane as ρ_h and the resistivity perpendicular to it as ρ_v, we can denote the anisotropy coefficient a as follows:

$$a = \rho_h / \rho_v. \tag{1}$$

Let us take a rectangular co-ordinate system (xyz-system) of which the xy-plane is parallel to the surface plane and z-axis is perpendicular to it, and assume an electrical point source C is given in this medium. Then the current densities through the planes perpendicular to the axes ox, oy and oz are given by the following expressions:

$$\left.\begin{aligned} i_x &= -\frac{1}{\rho_h}\cdot\frac{\partial V_1}{\partial x}, \\ i_y &= -\frac{1}{\rho_h}\cdot\frac{\partial V_1}{\partial y}, \\ i_z &= -\frac{1}{\rho_v}\cdot\frac{\partial V_1}{\partial z}, \end{aligned}\right\} \tag{2}$$

where V_1 is the potential of any point in this medium.

Putting (2) in the following expression, the condition of the conservation of current,

$$\frac{\partial i_x}{\partial x}+\frac{\partial i_y}{\partial y}+\frac{\partial i_z}{\partial z} = 0, \tag{3}$$

we can easily obtain the following result,

$$\frac{1}{\rho_h}\left(\frac{\partial^2 V_1}{\partial x^2}+\frac{\partial^2 V_1}{\partial y^2}\right)+\frac{1}{\rho_v}\cdot\frac{\partial^2 V_1}{\partial z^2} = 0. \tag{4}$$

Now let us take another rectangular co-ordinate system ($\xi\eta\varsigma$-system) which has the following relations against the old one,

$$\left.\begin{aligned} \xi &= \sqrt{a}\, x, \\ \eta &= \sqrt{a}\, y, \\ \varsigma &= z. \end{aligned}\right\} \tag{5}$$

By using the relation (5), we can write (4) as follows:

$$\frac{\partial^2 V_1}{\partial \xi^2}+\frac{\partial^2 V_1}{\partial \eta^2}+\frac{\partial^2 V_1}{\partial \varsigma^2} = 0. \tag{6}$$

Therefore, with respect to this new co-ordinate system ($\xi\eta\varsigma$), the potential V_1 satisfies the Laplace's equation for the homogeneous and isotropic medium. And the resistivity ρ' for this new space becomes as follows:

$$\rho' = \rho_h. \tag{7}$$

(Refer to Appendix 1.)

Thus we can treat the anisotropic problem as the isotropic one by using the relations (5) and (7).

Hence, when two point sources which represent the polarized orebody of the rod type are given at points $A(a_1, 0, d_1)$ and $B(a_2, 0, d_2)$ in the anisotropic medium (xyz-space), we can consider the corresponding points $A'(\sqrt{a}\, a_1, 0, d_1)$ and $B'(\sqrt{a}\, a_2, 0, d_2)$ in the isotropic medium ($\xi\eta\varsigma$-space) by using the relations (5) and (7) and the surface potentials due to these sources can easily be calculated. Namely, the potential V of any point $P(\xi, \eta, 0)$ on the surface is given by the following equation:

$$V=-\frac{\rho' I}{2\pi}\left[\frac{1}{\sqrt{(\xi-\sqrt{\alpha}a_1)^2+\eta^2+d_1^2}}-\frac{1}{\sqrt{(\xi-\sqrt{\alpha}a_2)^2+\eta^2+d_2^2}}\right]. \tag{8}$$

And also by using (5), equation (8) can be written as

$$V=-\frac{\rho_h I}{2\pi}\left[\frac{1}{\sqrt{\alpha(x-a_1)^2+\alpha y^2+d_1^2}}-\frac{1}{\sqrt{\alpha(x-a_2)^2+\alpha y^2+d_2^2}}\right]. \tag{9}$$

Next, let us treat the case of two-horizontal layered and anisotropic strata.

If we assume the resistivity of each layer parallel to the surface plane as ρ_{1h}, ρ_{2h} and those of perpendicular to it as ρ_{1v}, ρ_{2v} respectively, the anisotropy coefficients of each layer α_1 and α_2 can be represented as follows:

$$\left.\begin{aligned}\alpha_1 &= \rho_{1h}/\rho_{1v},\\ \alpha_2 &= \rho_{2h}/\rho_{2v}.\end{aligned}\right\} \tag{10}$$

Now by using the same procedure mentioned above, we can convert the anisotropic layers (xyz-space) to two isotropic layers ($\xi'\eta'\varsigma'$-space and $\xi''\eta''\varsigma''$-space) separately. Namely the upper anisotropic layer can be transformed into an isotropic layer by

$$\left.\begin{aligned}\xi' &= \sqrt{\alpha_1}\,x,\\ \eta' &= \sqrt{\alpha_1}\,y,\\ \varsigma' &= z,\end{aligned}\right\} \tag{11}$$

and

$$\rho' = \rho_{1h}, \tag{12}$$

where ρ' is the resistivity of the new isotropic medium. Similarly the lower anisotropic layer can be transformed into an isotropic layer by

$$\left.\begin{aligned}\xi'' &= \sqrt{\alpha_2}\,x,\\ \eta'' &= \sqrt{\alpha_2}\,y,\\ \varsigma'' &= z,\end{aligned}\right\} \tag{13}$$

and

$$\rho'' = \rho_{2h}, \tag{14}$$

where ρ'' is the resistivity of the new isotropic medium.

Again it can be easily proved that the Laplace's equation is still satisfied in the space ($\xi'''\eta'''\varsigma'''$) which is obtained by the following transformation:

$$\left.\begin{aligned}\xi''' &= \sqrt{\frac{\alpha_1}{\alpha_2}}\,\xi'' = \sqrt{\alpha_1}\,x = \xi',\\ \eta''' &= \sqrt{\frac{\alpha_1}{\alpha_2}}\,\eta'' = \sqrt{\alpha_1}\,y = \eta',\\ \varsigma''' &= \sqrt{\frac{\alpha_1}{\alpha_2}}\,\varsigma'' = \sqrt{\frac{\alpha_1}{\alpha_2}}\,z,\end{aligned}\right\} \tag{15}$$

and by this transformation the resistivity of the medium also changes to the following value,

$$\rho''' = \sqrt{\frac{\alpha_1}{\alpha_2}}\rho'' = \sqrt{\frac{\alpha_1}{\alpha_2}}\rho_{2h}. \tag{16}$$

Therefore we can also treat the anisotropic problem as an isotropic one by using the above transformations and the formula for calculating the potential which satisfies the boundary conditions of these new isotropic spaces also satisfies the boundary conditions of the corresponding anisotropic media. (Refer to Appendix 2.)

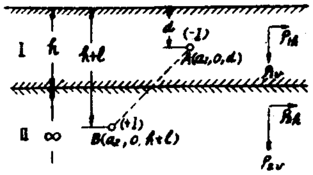

Fig. 1

For instance, when two point sources which represent the polarized orebody of the rod type are given in the anisotropic media at $A(a_1, 0, d)$ and $B(a_2, 0, h+l)$ as shown in Fig. 1, we can easily find the co-ordinates of these sources in the isotropic media by using equations (11) and (15). Then the formula for calculating the potential at any point $P(\xi, \eta, 0)$ on the surface is given by

$$V = -\frac{\rho' I}{2\pi}\Bigg[\sum_{n=0}^{\infty}\frac{k^n}{\sqrt{(\xi-\sqrt{\alpha_1}\,a_1)^2+\eta^2+(2nh+d)^2}}$$
$$+\sum_{n=1}^{\infty}\frac{k^n}{\sqrt{(\xi-\sqrt{\alpha_1}\,a_1)^2+\eta^2+(2nh-d)^2}}$$
$$-(1+k)\sum_{n=0}^{\infty}\frac{k^n}{\sqrt{(\xi-\sqrt{\alpha_1}\,a_2)^2+\eta^2+(\overline{2n+1}\;h+\sqrt{\frac{\alpha_1}{\alpha_2}}\,l)^2}}\Bigg], \tag{17}$$

where h is the thickness of the upper layer and

$$k = \frac{\rho'''-\rho'}{\rho'''+\rho'}. \tag{18}$$

By using the equations (11), (12), (15) and (16), the above formula can also be written as

$$V = -\frac{\rho_{1h} I}{2\pi}\Bigg[\sum_{n=0}^{\infty}\frac{k^n}{\sqrt{\alpha_1(x-a_1)^2+\alpha_1 y^2+(2nh+d)^2}}$$
$$+\sum_{n=1}^{\infty}\frac{k^n}{\sqrt{\alpha_1(x-a_1)^2+\alpha_1 y^2+(2nh-d)^2}}$$
$$-(1+k)\sum_{n=0}^{\infty}\frac{k^n}{\sqrt{\alpha_1(x-a_2)^2+\alpha_1 y^2+(\overline{2n+1}\;h+\sqrt{\frac{\alpha_1}{\alpha_2}}\,l)^2}}\Bigg], \tag{19}$$

where
$$k = \frac{\rho''' - \rho'}{\rho''' + \rho'} = \frac{\sqrt{\frac{a_1}{a_2}}\,\rho_{2h} - \rho_{1h}}{\sqrt{\frac{a_1}{a_2}}\,\rho_{2h} + \rho_{1h}}. \tag{20}$$

(17) or (19) is the formula for the case where two electrical point sources lie over the anisotropic layers and similarly we can also find the formula for the case where two sources together in the upper layer or in the lower layer.

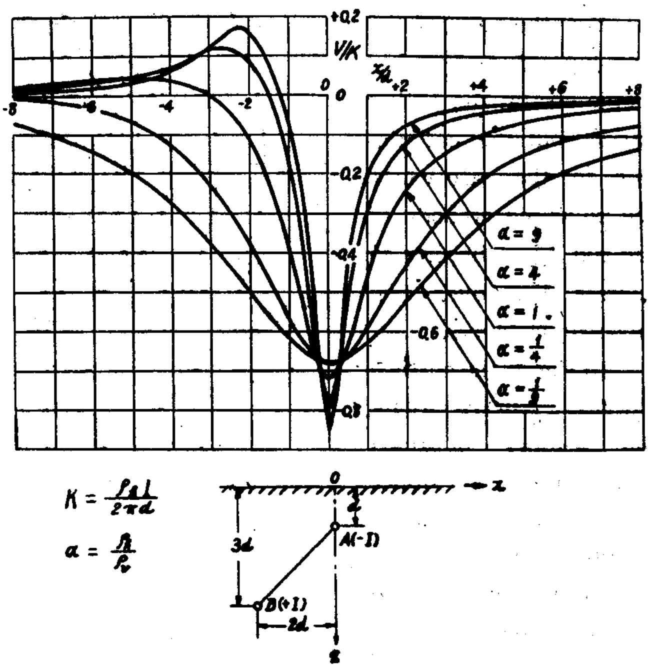

Fig. 2

3. Examples of Numerical Calculations and Consideration of the Results

Numerical calculations can be carried out by using formula (9) or (19).

For the case of a single anisotropic stratum, the author computed the surface potential of the polarized orebody of the rod type dipping 45 degrees, α being 9, 4, 1, $\frac{1}{4}$ and $\frac{1}{9}$. In Fig. 2 the surface potential curves along the x-axis are shown and in Figs. 3 and 4 the equipotential lines obtained on the surface,

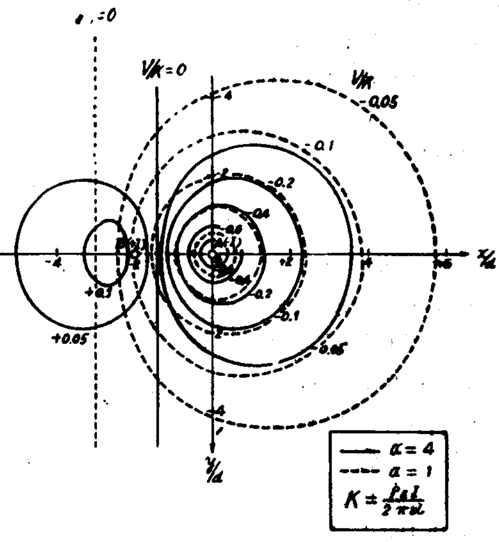

Fig. 3

when α equals to 4 and to $\frac{1}{4}$ respectively, are shown comparing with an isotropic case.

For the case of the two-layered and anisotropic strata, the author carried out the calculation for the case where the polarized orebody of the rod type dipping 45 degrees lies over two anisotropic layers.

In Fig. 5 the surface potential curves along the x-axis, obtained when either the upper layer or the lower layer is isotropic and the other is anisotropic, are shown. The curves along the x-axis, obtained when both layers are anisotropic and α_1 equals to α_2, are shown in Fig. 6; and those obtained when α_1 does not equal to α_2 are shown in Fig. 7.

And it must be noticed that all these results are obtained for the same electrical sources, that is, for the same polarized orebody.

By comparing these results with the isotropic case we can summarize

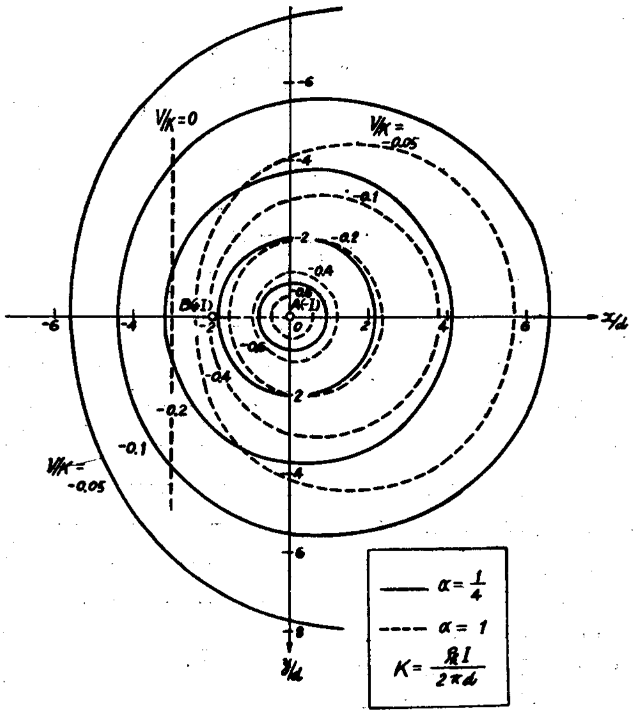

Fig. 4

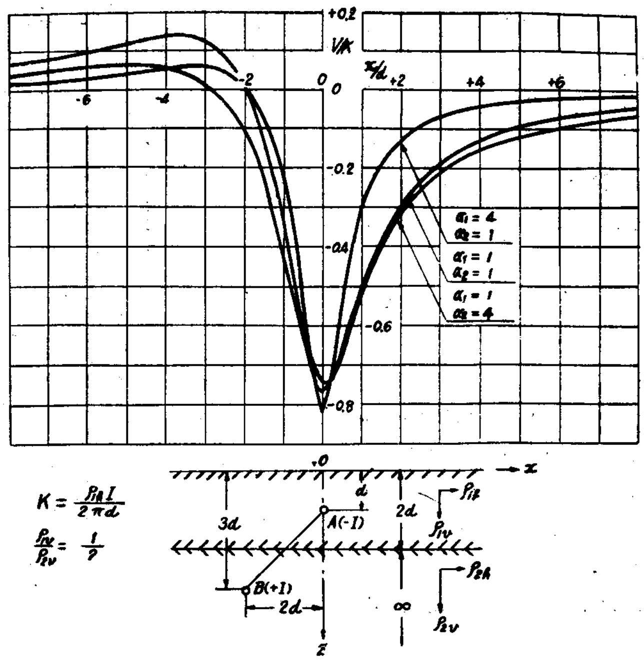

Fig. 5

the effects of anisotropy on the potential curves as follows:

a) When α is greater than 1, in other words, when the current can pass through the bedding plane more easily than along it, the range of the negative potential on the equipotential lines becomes narrower, but the value of the negative potential at the negative centre becomes a little larger.

b) On the other hand, when α is less than 1, in other words, when the current can pass along the bedding plane more easily than through it, the range of the negative potential on the equipotential lines becomes wider, but the value of the negative potential at the negative centre becomes a little smaller.

c) The effects of anisotropy on the position of the negative centre is negligibly small.

d) For the case of the layered and anisotropic media, the effects

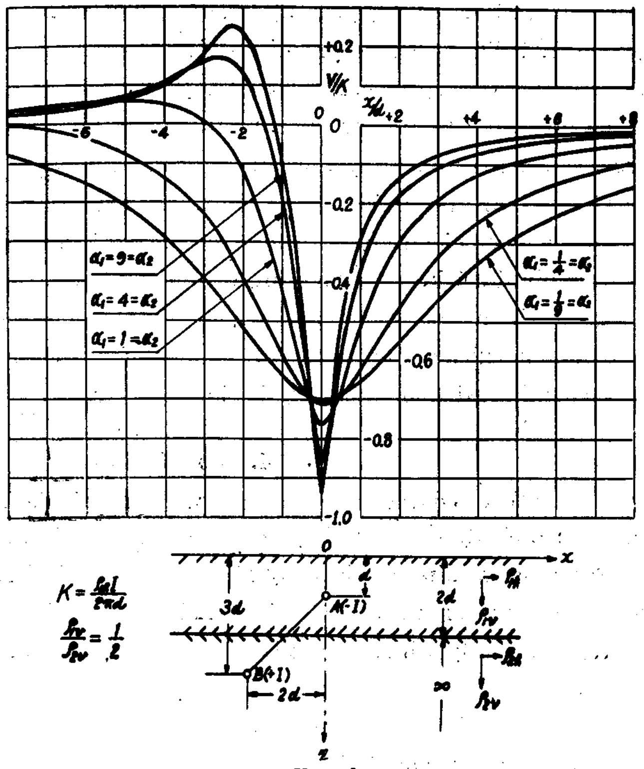

Fig. 6

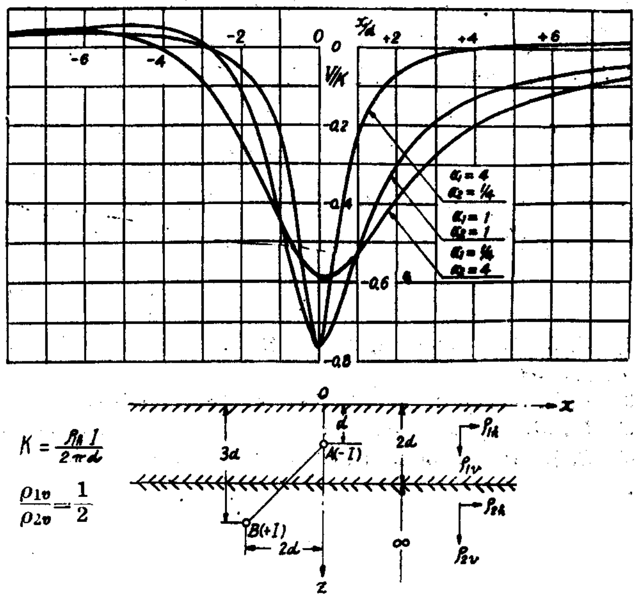

Fig. 7

of layers and anisotropy are appeared together in the curves. However, when ρ_{1v} does not differ so much from ρ_{2v}, as these examples, the effects of anisotropy are remarkably larger than those of layers.

e) If either the upper layer or the lower layer is anisotropic, the effects are larger when the upper layer is anisotropic.

f) If both the upper layer and the lower layer are anisotropic and the anisotropy coefficients of both layers are equal, the curves obtained are similar to those obtained in the single anisotropic stratum.

g) If both the upper layer and the lower layer are anisotropic and the anisotropy coefficients of both layers are different, the effects of the upper layer are predominated than those of the lower layer.

4. Conclusions

The author reduced here the formula for calculating surface potential of the polarized orebody of the rod type in the anisotropic media.

From the results obtained by numerical calculations it can be concluded that the anisotropy of media has the considerable effects on the surface potential curves, especially when the anisotropy coefficients are much larger or smaller than 1.

In practice it is supposed that the anisotropy coefficients are not so much larger or smaller than 1, but also it is reported that in stratified structures the electric current tends to pass along the bedding plane more easily than through it. Hence, we must be careful in interpreting the field data of the spontaneous polarization method obtained in the anisotropic stratified media for the effects of anisotropy.

Acknowledgement

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Appendix 1

Let us consider the following transformation by which the anisotropic xyz-space, of which horizontal resistivity is ρ_h and vertical resistivity is ρ_v, changes to the isotropic $\xi\eta\varsigma$-space of which resistivity is ρ',

$$\left.\begin{aligned} \xi &= ax, \\ \eta &= ay, \\ \varsigma &= cz. \end{aligned}\right\} \quad (1)$$

As shown in Fig. 1 if we consider an infinitely long electrical line source of which the intensity of current per unit length is J, the current density i_r at the point (x, y) separated from the line source with r in this xyz-space is given by the following equation;

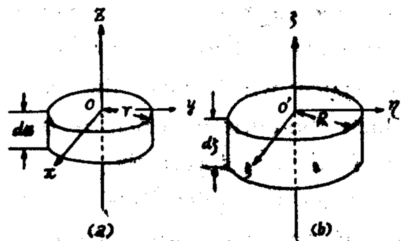

Fig. 1

$$i_r = \frac{Jdz}{2\pi r} \quad ; \quad r = \sqrt{x^2+y^2}, \quad (2)$$

while the current density i_R' at the corresponding point in the $\xi\eta\varsigma$-space is also given by

$$i_R' = \frac{Jd\varsigma}{2\pi R} \quad ; \quad R = \sqrt{\xi^2+\eta^2}. \quad (3)$$

However, by equation (1) the following relation can be obtained.

$$R = a\sqrt{x^2+y^2} = ar. \quad (4)$$

Substituting (4) to (3) and comparing the result with (2), we can easily obtain the following relation,

$$i_R' = \frac{Jcdz}{2\pi ar} = i_r\frac{c}{a},$$

hence,

$$\frac{i_r}{i_R'} = \frac{a}{c}. \quad (5)$$

On the other hand, if we represent the electrical potential at any point in these two spaces as V, the current densities at above points can be shown as follows:

$$\left.\begin{aligned} i_r &= -\frac{1}{\rho_h}\cdot\frac{\partial V}{\partial r}, \\ i_R' &= -\frac{1}{\rho'}\cdot\frac{\partial V}{\partial R}. \end{aligned}\right\} \quad (6)$$

Substituting (6) to (5) and considering the relation

$$\frac{\partial V}{\partial R} = \frac{1}{a}\cdot\frac{\partial V}{\partial r},$$

we can obtain the following relation in the last,

$$\rho' = \frac{1}{c}\rho_h. \tag{7}$$

Next as shown in Fig. 2, let us consider the electrical plane sources of infinitely wide in both the xyz-space and the $\xi\eta\varsigma$-space which pass the origin and of which the current densities per unit area are j and j' respectively.

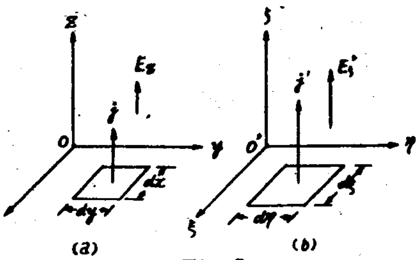

Fig. 2

Then the intensities of the electrical field E_z and E_ς' at any corresponding points in both spaces are given by

$$\left.\begin{aligned} E_z &= 2\pi j\rho_v, \\ E_\varsigma' &= 2\pi j'\rho'. \end{aligned}\right\} \tag{8}$$

But there exists another relation

$$jdxdy = j'd\xi d\eta,$$

and by the relation of (1) it follows

$$j' = \frac{1}{a^2}j,$$

hence, substituting this to (8) we obtain as a result

$$\frac{E_\varsigma'}{E_z} = \frac{1}{a^2}\cdot\frac{\rho'}{\rho_v}. \tag{9}$$

On the other hand,

$$\left.\begin{aligned} E_z &= -\frac{\partial V}{\partial z}, \\ E_\varsigma' &= -\frac{\partial V}{\partial \varsigma} = -\frac{1}{c}\cdot\frac{\partial V}{\partial z}, \end{aligned}\right\}$$

then

$$\frac{E_\varsigma'}{E_z} = \frac{1}{c}. \tag{10}$$

From (9) and (10) we can easily obtain

$$\rho' = a^2 c\rho_v. \tag{11}$$

Consequently, when a and c take the following value as shown at equation (2) in the original paper

$$\left.\begin{aligned} a &= \sqrt{\alpha}, \\ c &= 1, \\ \alpha &= \frac{\rho_h}{\rho_v}, \end{aligned}\right\}$$

both (7) and (11) show the same value of resistivity ρ' as

$$\rho' = \rho_h. \tag{12}$$

Appendix 2

As shown in original paper, we can treat the anisotropic problem as an isotropic one by using suitable transformation of co-ordinate axis. Consequently here let us prove that the formula for calculating the potential which satisfies the boundary conditions of the transformed isotropic spaces also satisfies the boundary conditions of the original anisotropic spaces.

As has already shown we can transform the anisotropic xyz-space, of which the horizontal resistivities in the upper layer and the lower layer are ρ_{1h} and ρ_{2h} respectively and the vertical ones in respective layers are ρ_{1v} and ρ_{2v}, into the isotropic $\xi\eta\varsigma$-space of which the resistivities in the upper and the lower layers are ρ' and ρ''; and also the equations of transformation of axis in the upper layer are

$$\left.\begin{aligned} \xi &= \sqrt{\alpha_1}\,x, \\ \eta &= \sqrt{\alpha_1}\,y, \\ \varsigma &= z, \end{aligned}\right\} \tag{1}$$

and $$\rho' = \rho_{1h}, \tag{2}$$

and those of in the lower layer are

$$\left.\begin{aligned} \xi &= \sqrt{\alpha_1}\,x, \\ \eta &= \sqrt{\alpha_1}\,y, \\ \varsigma &= \sqrt{\frac{\alpha_1}{\alpha_2}}\,z, \end{aligned}\right\} \tag{3}$$

and $$\rho'' = \rho_{2h}; \tag{4}$$

where $$\left.\begin{aligned} \alpha_1 &= \frac{\rho_{1h}}{\rho_{1v}}, \\ \alpha_2 &= \frac{\rho_{2h}}{\rho_{2v}}. \end{aligned}\right\} \tag{5}$$

(1) Let us first consider the case where an electrical point source lies in the upper layer.

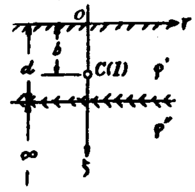

Fig. 3

Now let us consider with regard to the transformed $\xi\eta\varsigma$-space. As shown in Fig. 3 if we take a cylindrical co-ordinate system of which origin lies directly over the point source on the surface plane and ς-axis takes the downward direction as positive, then the potential V_1 at any point in the upper layer and V_2 in the lower layer which satisfy the boundary conditions in this space are given by

$$V_1=\frac{\rho' I}{4\pi}\int_0^\infty (e^{\lambda b}+e^{-\lambda b})\left\{e^{-\lambda\varsigma}+k\frac{e^{-2\lambda d}(e^{\lambda\varsigma}+e^{-\lambda\varsigma})}{1-k\,e^{-2\lambda d}}\right\} J_0(\lambda r)d\lambda, \quad (0\leq\varsigma\leq d) \tag{6}$$

and

$$V_2=\frac{\rho' I}{4\pi}\int_0^\infty e^{-\lambda\varsigma}(e^{\lambda b}+e^{-\lambda b})\left\{1+k\frac{1+e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\right\} J_0(\lambda r)d\lambda, \quad (d\leq\varsigma) \tag{7}$$

where

$$k=\frac{\rho''-\rho'}{\rho''+\rho'}. \tag{8}$$

From equations (1) and (3) the relation between the cylindrical co-ordinate (r, φ, ς) in the $\xi\eta\varsigma$-space and the corresponding one (R, Φ, z) in the xyz-space are shown as

$$\left.\begin{aligned} r &= \sqrt{a_1}\,R, \\ \varphi &= \Phi, \\ \varsigma &= z, \qquad (d\geq z\geq 0) \\ \text{or}\quad \varsigma &= \sqrt{\frac{a_1}{a_2}}\,z'+d. \quad (z'\geq 0) \end{aligned}\right\} \tag{9}$$

Then by using (9) we can transform the equations (6) and (7) with respect to $(R, \Phi, z$ or $z')$, obtaining the following results,

$$V_1=\frac{\rho' I}{4\pi}\int_0^\infty (e^{\lambda b}+e^{-\lambda b})\left\{e^{-\lambda z}+k\frac{e^{-2\lambda d}(e^{\lambda z}+e^{-\lambda z})}{1-k\,e^{-2\lambda d}}\right\} J_0(\lambda R)d\lambda, \quad (d\geq z\geq 0) \tag{10}$$

and

$$V_2=\frac{\rho' I}{4\pi}\int_0^\infty e^{-\lambda\sqrt{\frac{\alpha_1}{\alpha_2}}z'}\cdot e^{-\lambda d}(e^{\lambda b}+e^{-\lambda b})\left\{1+k\frac{1+e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\right\} J_0(\lambda R)d\lambda. \quad (z'\geq 0) \tag{11}$$

The boundary conditions in the xyz-space are

$$V_{1(z=d)} = V_{2(z'=0)}, \tag{12}$$

and

$$\frac{1}{\rho_{1v}}\left(\frac{\partial V_1}{\partial z}\right)_{z=d} = \frac{1}{\rho_{2v}}\left(\frac{\partial V_2}{\partial z'}\right)_{z'=0}. \tag{13}$$

From equations (10) and (11) we can easily obtain

$$V_{1(z=d)}=\frac{\rho' I}{4\pi}\int_0^\infty (e^{\lambda b}+e^{-\lambda b})\cdot e^{-\lambda d}\cdot\left(1+k\,\frac{1+e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\right)J_0(\lambda R)d\lambda, \tag{14}$$

and

$$V_{2(z'=0)}=\frac{\rho' I}{4\pi}\int_0^\infty (e^{\lambda b}+e^{-\lambda b})\cdot e^{-\lambda d}\cdot\left(1+k\,\frac{1+e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\right)J_0(\lambda R)d\lambda. \tag{15}$$

Hence equations (10) and (11) satisfy the first boundary condition, that is, equation (12).

On the other hand, by using equations (2), (4), (5) and (8) we can get the following relation from equation (11).

$$\begin{aligned}&\frac{\rho'}{\rho_{1v}}(e^{\lambda b}+e^{-\lambda b})\left\{-\lambda\, e^{-\lambda d}+k\,\frac{e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}(\lambda\, e^{\lambda d}-\lambda\, e^{-\lambda d})\right\}\\ &=(e^{\lambda b}+e^{-\lambda b})\cdot\lambda\cdot e^{-\lambda d}\,\frac{\alpha_1(k-1)}{1-k\,e^{-2\lambda d}}\\ &=(e^{\lambda b}+e^{-\lambda b})\cdot\lambda\cdot e^{-\lambda d}\cdot\frac{-2\rho_{1h}\cdot\alpha_1\cdot\sqrt{\alpha_2}}{(1-k\,e^{-2\lambda d})(\rho_{2h}\sqrt{\alpha_1}+\rho_{1h}\sqrt{\alpha_2})}.\end{aligned} \tag{16}$$

Similarly from equation (12) we also get

$$\begin{aligned}&\frac{\rho'}{\rho_{2v}}\cdot e^{-\lambda d}\left(-\lambda\sqrt{\frac{\alpha_1}{\alpha_2}}\right)\cdot e^{0}\cdot(e^{\lambda b}+e^{-\lambda b})\left(1+k\,\frac{1+e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\right)\\ &=(e^{\lambda b}+e^{-\lambda b})\cdot\lambda\cdot e^{-\lambda d}\cdot\frac{-2\rho_{1h}\cdot\alpha_1\cdot\sqrt{\alpha_2}}{(1-k\,e^{-2\lambda d})(\rho_{2h}\sqrt{\alpha_1}+\rho_{1h}\sqrt{\alpha_2})}.\end{aligned} \tag{17}$$

From equations (16) and (17) we can prove

$$\frac{1}{\rho_{1v}}\left(\frac{\partial V_1}{\partial z}\right)_{z=d} = \frac{1}{\rho_{2v}}\left(\frac{\partial V_2}{\partial z'}\right)_{z'=0}.$$

Hence equations (10) and (11) also satisfy the second boundary condition, that is, equation (13).

(2) Next we have to treat the case where an electrical point source lies in the lower layer.

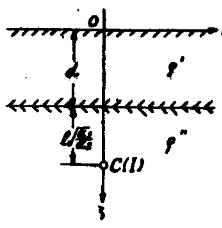

Fig. 4

Taking the cylindrical co-ordinate system and assuming the co-ordinate of the point source as $\left(0,\ 0,\ d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)$ as shown in Fig. 4, the potential V_1 at any point in the upper layer and V_2 in the lower layer which satisfy the boundary conditions in the $\xi\eta\varsigma$-space are given by

$$V_1=\frac{\rho'' I}{4\pi}\Big[\int_0^\infty \Big\{e^{\lambda\left(\varsigma-d-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}+e^{-\lambda\left(\varsigma+d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\Big\}J_0(\lambda r)d\lambda$$

$$-k\int_0^\infty \frac{1-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\cdot(e^{\lambda\varsigma}+e^{-\lambda\varsigma})\,J_0(\lambda r)d\lambda\Big], \quad (18)$$

$$(d\geq\varsigma\geq 0)$$

and

$$V_2=\frac{\rho'' I}{4\pi}\Big[\int_0^\infty \Big\{e^{\lambda\left(\varsigma-d-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}+e^{-\lambda\left(\varsigma+d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\Big\}J_0(\lambda r)d\lambda$$

$$-k\int_0^\infty \frac{e^{2\lambda d}-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\cdot e^{-\lambda\varsigma}\,J_0(\lambda r)d\lambda\Big]. \quad (19)$$

$$(\varsigma\geq d)$$

By using equation (9) the above equations can be shown with respect to the co-ordinate $(R,\ \Phi,\ z)$ and the results are as follows:

$$V_1=\frac{\rho'' I}{4\pi}\Big[\int_0^\infty \Big\{e^{\lambda\left(z-d-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}+e^{-\lambda\left(z+d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\Big\}J_0(\lambda R)d\lambda$$

$$-k\int_0^\infty \frac{1-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\cdot(e^{\lambda z}+e^{-\lambda z})\,J_0(\lambda R)d\lambda\Big], \quad (20)$$

and

$$V_2=\frac{\rho'' I}{4\pi}\Big[\int_0^\infty \Big\{e^{\lambda'\left(\sqrt{\frac{\alpha_1}{\alpha_2}}z'-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}+e^{-\lambda\left(2d+\sqrt{\frac{\alpha_1}{\alpha_2}}z'+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\Big\}J_0(\lambda R)d\lambda$$

$$-k\int_0^\infty \frac{e^{2\lambda d}-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\cdot e^{-\lambda\left(d+\sqrt{\frac{\alpha_1}{\alpha_2}}z'\right)}J_0(\lambda R)d\lambda\Big]. \quad (21)$$

The boundary conditions in the xyz-space are

$$V_{1\,(z=d)} = V_{2\,(z'=0)}, \quad (22)$$

and

$$\frac{1}{\rho_{1v}}\left(\frac{\partial V_1}{\partial z}\right)_{z=d} = \frac{1}{\rho_{2v}}\left(\frac{\partial V_2}{\partial z'}\right)_{z'=0}. \quad (23)$$

From equations (20) and (21) we get

$$V_{1\,(z=d)}=\frac{\rho'' I}{4\pi}\Big[\int_0^\infty \Big\{e^{\lambda\left(-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}+e^{-\lambda\left(2d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\Big\}J_0(\lambda R)d\lambda$$

$$-k\int_0^\infty \frac{e^{2\lambda d}-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda d}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}J_0(\lambda R)d\lambda\Big], \quad (24)$$

and

$$V_{2\,(z'=0)}=\frac{\rho''I}{4\pi}\left[\int_0^\infty\left\{e^{\lambda\left(-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}+e^{-\lambda\left(2d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\right\}J_0(\lambda R)d\lambda\right.$$

$$\left.-k\int_0^\infty\frac{e^{2\lambda d}-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda d}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}J_0(\lambda R)d\lambda\right]. \tag{25}$$

Hence equations (20) and (21) satisfy the first boundary condition, that is, equation (22).

On the other hand, from equations (20) and (21) the following relations can be obtained;

$$\frac{\rho''}{\rho_{1v}}\left\{\lambda e^{\lambda\left(-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}-\lambda e^{-\lambda\left(2d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}-k\frac{1-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\cdot(\lambda e^{\lambda d}-\lambda e^{-\lambda d})\right.$$

$$=\lambda e^{-\lambda l\sqrt{\frac{\alpha_1}{\alpha_2}}}(1-e^{-2\lambda d})\frac{1}{1-k\,e^{-2\lambda d}}\cdot\frac{2\rho_{2h}\cdot a_1\sqrt{a_1}}{\rho_{2h}\sqrt{a_1}+\rho_{1h}\sqrt{a_2}}, \tag{26}$$

and

$$\frac{\rho''}{\rho_{2v}}\left\{\lambda\sqrt{\frac{a_1}{a_2}}\,e^{\lambda\left(-l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}-\lambda\sqrt{\frac{a_1}{a_2}}\,e^{-\lambda\left(2d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}-k\frac{e^{2\lambda d}-e^{-2\lambda d}}{1-k\,e^{-2\lambda d}}\cdot e^{-\lambda\left(d+l\sqrt{\frac{\alpha_1}{\alpha_2}}\right)}\left(-\lambda\sqrt{\frac{a_1}{a_2}}\,e^{-\lambda d}\right)\right.$$

$$=\lambda e^{-\lambda l\sqrt{\frac{\alpha_1}{\alpha_2}}}(1-e^{-2\lambda d})\frac{1}{1-k\,e^{-2\lambda d}}\cdot\frac{2\rho_{2h}\cdot a_1\sqrt{a_1}}{\rho_{2h}\sqrt{a_1}+\rho_{1h}\sqrt{a_2}}. \tag{27}$$

From equations (26) and (27) we can prove

$$\frac{1}{\rho_{1h}}\left(\frac{\partial V_1}{\partial z}\right)_{z=d}=\frac{1}{\rho_{2v}}\left(\frac{\partial V_2}{\partial z'}\right)_{z'=0}.$$

Hence the potential V_1 and V_2 in equations (20) and (21) also satisfy the second boundary condition, that is, equation (23).