Theory of Two and Three Phase Generators

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Contents.

Chapter I.

Two Phase Generator.

§ 1. Fundamental equations.

Let

a and b = resistance and total inductance of the armature circuit,

a_1 and b_1 = those of the field circuit,

c = maximum mutual inductance between the field and armature winding,

$d = cI_f\omega$, where I_f = the exciting current,

x and y = the armature currents,

z = alternating current in the field winding.

Then if the field poles be non-salient and laminated and the field and armature windings be so arranged that the distribution of the magnetic flux circumferentially along the air gap is sinusoidal, the fundamental equations will be

$$ax + b\frac{dx}{dt} + c\frac{d}{dt}(z\cos\alpha) = d\sin\alpha \quad \text{.........(1)}$$

$$ay + b\frac{dy}{dt} + c\frac{d}{dt}(z\sin\alpha) = -d\cos\alpha \quad \text{.........(2)}$$

$$a_1 z + b_1\frac{dz}{dt} + c\frac{d}{dt}(x\cos\alpha + y\sin\alpha) = 0 \quad \text{.........(3)}$$

that is

$$a\xi + b\frac{d\xi}{dt} + c\frac{d\zeta}{dt}\cos\alpha = -\frac{d}{\omega}\cos\alpha \quad \text{.........(4)}$$

$$a\eta + b\frac{d\eta}{dt} + c\frac{d\zeta}{dt}\sin\alpha = -\frac{d}{\omega}\sin\alpha \quad \text{.........(5)}$$

$$a_1\zeta + b_1\frac{d\zeta}{dt} + c\left(\frac{d\xi}{dt}\cos\alpha + \frac{d\eta}{dt}\sin\alpha\right) = 0 \quad \text{.........(6)}$$

where

$$\xi = \int x\,dt + k_1, \quad \eta = \int y\,dt + k_2 \quad \text{and} \quad \zeta = \int z\,dt + k_3$$

so that

$$x = \frac{d\xi}{dt}, \quad y = \frac{d\eta}{dt} \quad \text{and} \quad z = \frac{d\zeta}{dt}$$

Now, from eqs. (4) and (5) we have

$$b\left(\frac{d\xi}{dt}\cos\alpha + \frac{d\eta}{dt}\sin\alpha\right) = -\left(\frac{d}{\omega} + c\frac{d\zeta}{dt}\right) - a\left(\xi\cos\alpha + \eta\sin\alpha\right)$$

so that eq. (6) gives

$$a_1\zeta + b_1\frac{d\zeta}{dt} - \frac{c}{b}\left[\frac{d}{\omega} + c\frac{d\zeta}{dt} + a\left(\xi\cos\alpha + \eta\sin\alpha\right)\right] = 0$$

that is

$$(bb_1 - c^2)\frac{d\zeta}{dt} + a_1 b\zeta - ca\left(\xi\cos\alpha + \eta\sin\alpha\right) - \frac{cd}{\omega} = 0$$

so that

$$(bb_1 - c^2)\frac{d^2\zeta}{dt^2} + a_1 b\frac{d\zeta}{dt} - ca\left[\frac{d\xi}{dt}\cos\alpha + \frac{d\eta}{dt}\sin\alpha - \omega\left(\xi\sin\alpha - \eta\cos\alpha\right)\right] = 0$$

that is

$$(bb_1 - c^2)\frac{d^2\zeta}{dt^2} + a_1 b\frac{d\zeta}{dt} + a\left(a_1\zeta + b_1\frac{d\zeta}{dt}\right) + ca\omega\left(\xi\sin\alpha - \eta\cos\alpha\right) = 0$$

that is

$$(bb_1 - c^2)\frac{d^2\zeta}{dt^2} + (ab_1 + a_1 b)\frac{d\zeta}{dt} + aa_1\zeta + ca\omega\left(\xi\sin\alpha - \eta\cos\alpha\right) = 0 \quad \ldots(8)$$

so that

$$(bb_1 - c^2)\frac{d^3\zeta}{dt^3} + (ab_1 + a_1 b)\frac{d^2\zeta}{dt^2} + aa_1\frac{d\zeta}{dt} + ca\omega\left(\frac{d\xi}{dt}\sin\alpha - \frac{d\eta}{dt}\cos\alpha\right)$$
$$+ ca\omega^2\left(\xi\cos\alpha + \eta\sin\alpha\right) = 0$$

But from eqs. (4) and (5) we have

$$b\left(\frac{d\xi}{dt}\sin\alpha-\frac{d\eta}{dt}\cos\alpha\right)=-a(\xi\sin\alpha-\eta\cos\alpha)$$

Therefore

$$(bb_1-c^2)\frac{d^3\zeta}{dt^3}+(ab_1+a_1b)\frac{d^2\zeta}{dt^2}+aa_1\frac{d\zeta}{dt}-\frac{ca^2\omega}{b}(\xi\sin\alpha-\eta\cos\alpha)$$
$$+ca\omega^2(\xi\cos\alpha+\eta\sin\alpha)=0$$

which from (7) and (8) becomes

$$b(bb_1-c^2)\frac{d^3\zeta}{dt^3}+b(ab_1+a_1b)\frac{d^2\zeta}{dt^2}+aa_1b\frac{d\zeta}{dt}$$
$$+a\left[(bb_1-c^2)\frac{d^2\zeta}{dt^2}+(ab_1+a_1b)\frac{d\zeta}{dt}+aa_1\zeta\right]$$
$$+b\omega^2\left[(bb_1-c^2)\frac{d\zeta}{dt}+a_1b\zeta-\frac{cd}{\omega}\right]=0$$

that is

$$b(bb_1-c^2)\frac{d^3\zeta}{dt^3}+(2abb_1+a_1b^2-ac^2)\frac{d^2\zeta}{dt^2}+[2aa_1b+a^2b_1+b(bb_1-c^2)\omega^2]\frac{d\zeta}{dt}$$
$$+a_1(a^2+b^2\omega^2)\zeta+bcd\omega=0$$

so that

$$b(bb_1-c^2)\frac{d^3z}{dt^3}+(2abb_1+a_1b^2-ac^2)\frac{d^2z}{dt^2}+[2aa_1b+a^2b_1+b(bb_1-c^2)\omega^2]\frac{dz}{dt}$$
$$+a_1(a^2+b^2\omega^2)z=0 \quad \text{...........(9)}$$

§ 2. Solution of the fundamental equations.

Equation (9) solves to

$$z=A\epsilon^{\alpha t}+B\epsilon^{\beta t}+C\epsilon^{\gamma t}$$

where A, B and C are the arbitrary constants and α, β and γ are the roots of

$$b(bb_1-c^2)\lambda^3+(2abb_1+a_1b^2-ac^2)\lambda^2+[2aa_1b_1+a^2b_1+b(bb_1-c^2)\omega^2]\lambda$$
$$+a_1(a^2+b^2\omega^2)=0$$

and if $\beta = \mu + j\nu$ and $\gamma = \mu - j\nu$, then

$$z = A\epsilon^{at} + D\epsilon^{\mu t}\sin(\nu t + \psi)$$

where $D = 2\sqrt{BC}$ and $\psi = \tan^{-1}\dfrac{B+C}{j(B-C)}$.

But referring to the theory of distortionless alternators, two of the three roots α, β and γ are usually imaginary. Therefore we shall put $z = A\epsilon^{at} + D\epsilon^{\mu t}\sin(\nu t + \psi)$ for solving x and y. Eq. (4) thus becomes

$$a\xi + b\frac{d\xi}{dt} + Ac\epsilon^{at}\cos\omega t + \frac{1}{2}Dc\epsilon^{\mu t}\left[\sin(\overline{\nu+\omega}t + \psi) + \sin(\overline{\nu-\omega}t + \psi)\right] = -\frac{d}{\omega}\cos\omega t$$

that is

$$\frac{d\xi}{dt} + \frac{a}{b}\xi + \frac{Ac}{b}\cdot\epsilon^{at}\cos\omega t + \frac{1}{2}\frac{Dc}{c}\cdot\epsilon^{\mu t}\left[\sin(\overline{\nu+\omega}t + \psi) + \sin(\overline{\nu-\omega}t + \psi)\right] = -\frac{d}{\omega}\cos\omega t$$

which solves to

$$\xi\epsilon^{\int\frac{a}{b}dt} = -\frac{d}{b\omega}\cdot\int\epsilon^{\frac{a}{b}t}\cos\omega t\,dt - \frac{Ac}{b}\int\epsilon^{\left(\frac{a}{b}+\alpha\right)t}\cos\omega t\,dt$$
$$-\frac{1}{2}\cdot\frac{Dc}{b}\int\epsilon^{\left(\frac{a}{b}+\mu\right)t}\left[\sin(\overline{\nu+\omega}t + \psi) + \sin(\overline{\nu-\omega}t + \psi)\right]dt + k$$

that is

$$\xi = \frac{-d}{\omega\sqrt{a^2+b^2\omega^2}}\cos\left(\omega t - \tan^{-1}\frac{b\omega}{a}\right) - \frac{Ac}{\sqrt{(a+b\alpha)^2+b^2\omega^2}}\epsilon^{at}\cos\left(\omega t - \tan^{-1}\frac{b\omega}{a+b\alpha}\right)$$
$$-\frac{1}{2}\frac{Dc}{\sqrt{(a+b\mu)^2+b^2(\nu+\omega)^2}}\epsilon^{\mu t}\sin\left(\overline{\nu+\omega}t + \psi - \tan^{-1}\frac{b(\nu+\omega)}{a+b\mu}\right)$$
$$-\frac{1}{2}\frac{Dc}{\sqrt{(a+b\mu)^2+b^2(\nu-\omega)^2}}\epsilon^{\mu t}\sin\left(\overline{\nu-\omega}t + \psi - \tan^{-1}\frac{b(\nu-\omega)}{a+b\mu}\right)$$
$$+k\epsilon^{-\frac{a}{b}t}$$

that is

$$x=\frac{d}{\sqrt{a^2+b^2\omega^2}}\sin\left(\omega t-\tan^{-1}\frac{b\omega}{a}\right)$$

$$+\frac{Ac\sqrt{\alpha^2+\omega^2}}{\sqrt{(a+b\alpha)^2+b^2\omega^2}}\epsilon^{\alpha t}\sin\left(\omega t-\tan^{-1}\frac{b\omega}{a+b\alpha}-\tan^{-1}\frac{\alpha}{\omega}\right)$$

$$-\frac{1}{2}\frac{Dc\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a+b\mu)^2+b^2(\nu+\omega)^2}}\epsilon^{\mu t}$$

$$\sin\left(\overline{\nu+\omega}t+\psi-\tan^{-1}\frac{b(\nu+\omega)}{a+b\mu}+\tan^{-1}\frac{\nu+\omega}{\mu}\right)$$

$$-\frac{1}{2}\frac{Dc\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a+b\mu)^2+b^2(\nu-\omega)^2}}\epsilon^{\mu t}$$

$$\sin\left(\overline{\nu-\omega}t+\psi-\tan^{-1}\frac{b(\nu-\omega)}{a+b\mu}+\tan^{-1}\frac{\nu-\omega}{\mu}\right)$$

$$-\frac{a}{b}k\epsilon^{-\frac{a}{b}t}$$

that is

$$x=X\sqrt{2}\sin(\omega t-\phi_x)+A_1\epsilon^{\alpha t}\sin(\omega t-\theta_\alpha)-D_1\epsilon^{\mu t}\sin(\overline{\nu+\omega}t+\psi+\psi_1)$$

$$-D_2\epsilon^{\mu t}\sin(\overline{\nu-\omega}t+\psi+\psi_2)+k_1\epsilon^{-\frac{a}{b}t}$$

where

$$X\sqrt{2}=\frac{d}{\sqrt{a^2+b^2\omega^2}} \qquad \phi_x=\tan^{-1}\frac{b\omega}{a}$$

$$A_1=A\frac{c\sqrt{\alpha^2+\omega^2}}{\sqrt{(a+b\alpha)^2+b^2\omega^2}} \qquad \theta_\alpha=\tan^{-1}\frac{a\alpha+b(\alpha^2+\omega^2)}{a\omega}$$

$$D_1=D\frac{\frac{1}{2}c\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a+b\mu)^2+b^2(\nu+\omega)^2}} \qquad \psi_1=\tan^{-1}\frac{a(\nu+\omega)}{(a+b\mu)\mu+b(\nu+\omega)^2}$$

$$D_2=D\frac{\frac{1}{2}c\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a+b\mu)^2+b^2(\nu-\omega)^2}} \qquad \psi_2=\tan^{-1}\frac{a(\nu-\omega)}{(a+b\mu)\mu+b(\nu-\omega)^2}$$

Next from eq. (5) we have

$$\frac{d\eta}{dt} + \frac{a}{b}\eta + \frac{Ac}{b}\epsilon^{at}\sin\omega t - \frac{1}{2}\frac{Dc}{b}\epsilon^{\mu t}[\cos(\overline{\nu+\omega}t+\psi) - \cos(\overline{\nu-\omega}t+\psi)]$$
$$= -\frac{d}{\omega}\sin\omega t$$

which similarly as before solves to

$$\eta = \frac{-d}{\omega\sqrt{a^2+b^2\omega^2}}\sin\left(\omega t - \tan^{-1}\frac{b\omega}{a}\right)$$
$$-\frac{Ac}{\sqrt{(a+b\alpha)^2+b^2\omega^2}}\epsilon^{at}\sin\left(\omega t - \tan^{-1}\frac{b\omega}{a+b\alpha}\right)$$
$$+\frac{1}{2}\frac{Dc}{\sqrt{(a+b\mu)^2+b^2(\nu+\omega)^2}}\epsilon^{\mu t}\cos\left(\overline{\nu+\omega}t+\psi-\tan^{-1}\frac{b(\nu+\omega)}{a+b\mu}\right)$$
$$-\frac{1}{2}\frac{Dc}{\sqrt{(a+b\mu)^2+b^2(\nu-\omega)^2}}\epsilon^{\mu t}\cos\left(\overline{\nu-\omega}t+\psi-\tan^{-1}\frac{b(\nu-\omega)}{a+b\mu}\right)$$
$$+k'\epsilon^{-\frac{a}{b}t}$$

so that

$$y = \frac{-d}{\sqrt{a^2+b^2\omega^2}}\cos\left(\omega t - \tan^{-1}\frac{b\omega}{a}\right)$$
$$-\frac{Ac\sqrt{\alpha^2+\omega^2}}{\sqrt{(a+b\alpha)^2+b^2\omega^2}}\epsilon^{at}\cos\left(\omega t - \tan^{-1}\frac{b\omega}{a+b\alpha} - \tan^{-1}\frac{\alpha}{\omega}\right)$$
$$+\frac{1}{2}\frac{Dc\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a+b\mu)^2+b^2(\nu+\omega)^2}}\epsilon^{\mu t}$$
$$\cos\left(\overline{\nu+\omega}t+\psi-\tan^{-1}\frac{b(\nu+\omega)}{a+b\mu}+\tan^{-1}\frac{\nu+\omega}{\mu}\right)$$
$$-\frac{1}{2}\frac{Dc\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a+b\mu)^2+b^2(\nu-\omega)^2}}\epsilon^{\mu t}$$
$$\cos\left(\overline{\nu-\omega}t+\psi-\tan^{-1}\frac{b(\nu-\omega)}{a+b\mu}+\tan^{-1}\frac{\nu-\omega}{\mu}\right)$$
$$-\frac{a}{b}k'\epsilon^{-\frac{a}{b}t}$$

that is

$$y = Y\sqrt{2}\sin(\omega t - \phi_y) - A_1\epsilon^{\alpha t}\cos(\omega t - \theta_a) + D_1\epsilon^{\mu t}\cos(\overline{\nu + \omega}t + \psi + \psi_1)$$
$$- D_2\epsilon^{\mu t}\cos(\overline{\nu - \omega}t + \psi + \psi_2) + k_1'\epsilon^{-\frac{a}{b}t}$$

where $Y\sqrt{2} = X\sqrt{2}$, $\phi_y = \phi_x + \frac{\pi}{2}$ and others are all the same as those of x.

Now putting these solutions of x, y and z in the fundamental equation of z, we have

$$a_1[A\epsilon^{\alpha t} + D\epsilon^{\mu t}\sin(\nu t + \psi)] + b_1\frac{d}{dt}[A\epsilon^{\alpha t} + D\epsilon^{\mu t}\sin(\nu t + \psi)]$$
$$+ c\frac{d}{dt}\{[A_1\epsilon^{\alpha t}\sin(\omega t - \theta_a) - D_1\epsilon^{\mu t}\sin(\overline{\nu + \omega}t + \psi + \psi_1)$$
$$- D_2\epsilon^{\mu t}\sin(\overline{\nu - \omega}t + \psi + \psi_2) + k_1\epsilon^{-\frac{a}{b}t}]\cos\omega t$$
$$-[A_1\epsilon^{\alpha t}\cos(\omega t - \theta_a) - D_1\epsilon^{\mu t}\cos(\overline{\nu + \omega}t + \psi + \psi_1)$$
$$+ D_2\epsilon^{\mu t}\cos(\overline{\nu - \omega}t + \psi + \psi_2) - k_1'\epsilon^{\frac{a}{b}t}]\sin\omega t\} = 0$$

that is

$$[(a_1 + b_1\alpha)A - cA_1\alpha\sin\theta_a]\epsilon^{\alpha t} + [(a_1 + b_1\mu)D\sin(\nu t + \psi) + b_1\nu D\cos(\nu t + \psi)]\epsilon^{\mu t}$$
$$- D_1 c[\mu\sin(\nu t + \psi + \psi_1) + \nu\cos(\nu t + \psi + \psi_1)]\epsilon^{\mu t}$$
$$- D_2 c[\mu\sin(\nu t + \psi + \psi_2) + \nu\cos(\nu t + \psi + \psi_2)]\epsilon^{\mu t}$$
$$- c\frac{d}{dt}\left[\epsilon^{-\frac{a}{b}t}.\sqrt{k_1^2 + k_1'^2}\sin\left(\omega t + \tan^{-1}\frac{k_1}{k_1'}\right)\right] = 0$$

so that the conditions sufficient for $k_1 = k_1' = 0$ are

$$(a_1 + b_1\alpha)A - cA_1\alpha\sin\theta_a = 0$$
$$(a_1 + b_1\mu)D - c\mu(D_1\cos\psi_1 + D_2\cos\psi_2) + c\nu(D_1\sin\psi_1 + D_2\sin\psi_2) = 0$$

and

$$b_1\nu D - c\mu(D_1\sin\psi_1 + D_2\sin\psi_2) - c\nu(D_1\cos\psi_1 + D_2\cos\psi_2) = 0$$

But referring to the theory of distortionless alternators we know, by interchanging the letters a, b and a_1, b_1, that all these conditions are satisfied. Therefore, we have $k_1 = k_1' = 0$.

§ 3. Sudden short-circuit currents.

Now $\frac{a_1}{b_1\omega} \doteqdot 0$ usually and at short-circuit $\frac{a_i}{b_i\omega} \doteqdot 0$ too; so that, if σ be not so small, then referring to the theory of distortionless alternators, we have

$$\alpha = -\frac{p}{\sigma}\omega, \quad \mu = -\frac{1}{2}(1+\sigma)\frac{q}{\sigma}\omega \quad \text{and} \quad \nu = \omega$$

where
$$p = \frac{a_1}{b_1\omega} \quad \text{and} \quad q = \frac{a_i}{b_i\omega}$$

so that
$$\frac{\alpha}{\omega} \doteqdot \frac{\mu}{\omega} \doteqdot 0$$

accordingly

$$X\sqrt{2} = Y\sqrt{2} \doteqdot \frac{d}{b_i\omega}, \quad \phi_x \doteqdot \frac{\pi}{2}, \quad \phi_y \doteqdot \pi$$

$$A_1 = \frac{Ac\sqrt{\left(\frac{\alpha}{\omega}\right)^2+1}}{b_i\sqrt{\left(\frac{a_i}{b_i\omega}+\frac{\alpha}{\omega}\right)^2+1}} \doteqdot \frac{Ac}{b_i} \qquad \theta_a = \tan^{-1}\frac{\frac{a_i}{b_i\omega}\cdot\frac{\alpha}{\omega}+\left(\frac{\alpha}{\omega}\right)^2+1}{\frac{a_i}{b_i\omega}} \doteqdot \frac{\pi}{2}$$

$$D_1 = \frac{\frac{1}{2}Dc\sqrt{\left(\frac{\mu}{\omega}\right)^2+\left(\frac{\nu}{\omega}+1\right)^2}}{b_i\sqrt{\left(\frac{a_i}{b_i\omega}+\frac{\mu}{\omega}\right)^2+\left(\frac{\nu}{\omega}+1\right)^2}} \doteqdot \frac{1}{2}\cdot\frac{Dc}{b_i}$$

$$\psi_1 = \tan^{-1}\frac{\frac{a_i}{b_i\omega}\cdot\left(\frac{\nu}{\omega}+1\right)}{\left(\frac{a_i}{b_i\omega}+\frac{\mu}{\omega}\right)\frac{\mu}{\omega}+\left(\frac{\nu}{\omega}+1\right)^2} \doteqdot 0$$

$$D_2=\frac{1}{2}Dc\frac{\mu}{a_i+b_i\mu}=\frac{1}{2}\cdot\frac{Dc}{b_i}\cdot\frac{-\frac{1}{2}(1+\sigma)\frac{q}{\sigma}}{\frac{a_i}{b_i\omega}-\frac{1}{2}(1+\sigma)\frac{q}{\sigma}}=\frac{1}{2}\cdot\frac{Dc}{b_i}\frac{1+\sigma}{1-\sigma}\qquad \psi_2=0$$

Therefore

$$x_s=\frac{d}{b_i\omega}\sin\left(\omega t-\frac{\pi}{2}\right)+\frac{Ac}{b_i}\epsilon^{at}\sin\left(\omega t-\frac{\pi}{2}\right)-\frac{1}{2}\cdot\frac{Dc}{b_i}\epsilon^{\mu t}\sin(2\omega t+\psi)$$
$$-\frac{1}{2}\cdot\frac{Dc}{b_i}\frac{1+\sigma}{1-\sigma}\epsilon^{\mu t}\sin\psi$$

$$=\frac{-d}{b_i\omega}\cos\omega t-\frac{Ac}{b_i}\epsilon^{at}\cos\omega t-\frac{1}{2}\cdot\frac{c}{b_i}(D\cos\psi)\epsilon^{\mu t}\sin 2\omega t$$
$$-\frac{1}{2}\cdot\frac{c}{b_i}(D\sin\psi)\epsilon^{\mu t}\left(\frac{1+\sigma}{1-\sigma}+\cos 2\omega t\right)$$

$$y_s=\frac{d}{b_i\omega}\sin(\omega t-\pi)-\frac{Ac}{b_i}\epsilon^{at}\cos\left(\omega t-\frac{\pi}{2}\right)+\frac{1}{2}\cdot\frac{Dc}{b_i}\epsilon^{\mu t}\cos(2\omega t+\psi)$$
$$-\frac{1}{2}\cdot\frac{Dc}{b_i}\frac{1+\sigma}{1-\sigma}\epsilon^{\mu t}\cos\psi$$

$$=\frac{-d}{b_i\omega}\sin\omega t-\frac{Ac}{b_i}\epsilon^{at}\sin\omega t-\frac{1}{2}\cdot\frac{c}{b_i}(D\cos\psi)\epsilon^{\mu t}\left(\frac{1+\sigma}{1-\sigma}-\cos 2\omega t\right)$$
$$-\frac{1}{2}\cdot\frac{c}{b_i}\epsilon^{\mu t}(D\sin\psi)\sin 2\omega t$$

$$z_s=A\epsilon^{at}+(D\cos\psi)\epsilon^{\mu t}\sin\omega t+(D\sin\psi)\epsilon^{\mu t}\cos\omega t$$

so that putting $D\cos\psi=E$, $D\sin\psi=F$ and $\frac{1+\sigma}{1-\sigma}=k$, we have

$$x_s=\frac{-d}{b_i\omega}\cos\omega t-\frac{Ac}{b_i}\epsilon^{at}\cos\omega t-\frac{1}{2}\cdot\frac{Ec}{b_i}\epsilon^{\mu t}\sin 2\omega t-\frac{1}{2}\cdot\frac{Fc}{b_i}\epsilon^{\mu t}(k+\cos 2\omega t)$$

$$y_s=\frac{-d}{b_i\omega}\sin\omega t-\frac{Ac}{b_i}\epsilon^{at}\sin\omega t-\frac{1}{2}\cdot\frac{Ec}{b_i}\epsilon^{\mu t}(k-\cos 2\omega t)-\frac{1}{2}\cdot\frac{Fc}{b_i}\epsilon^{\mu t}\sin 2\omega t$$

$$z_s=A\epsilon^{at}+E\epsilon^{\mu t}\sin\omega t+F\epsilon^{\mu t}\cos\omega t$$

Now to determine the arbitrary constants A, E and F, suppose that the alternator runs at no load before short-circuit and at $t=t_0$ the short-circuit takes place suddenly in two phases. Then putting $\omega t_0=\beta$, we have

$$A\epsilon^{at_0}\cos\beta+\tfrac{1}{2}E\epsilon^{\mu t_0}\sin 2\beta+\tfrac{1}{2}F\epsilon^{\mu t_0}(k+\cos 2\beta)=-\frac{d}{c\omega}\cos\beta\text{.........}(1)$$

$$A\epsilon^{at_0}\sin\beta+\tfrac{1}{2}E\epsilon^{\mu t_0}(k-\cos 2\beta)+\tfrac{1}{2}F\epsilon^{\mu t_0}\sin 2\beta=-\frac{d}{c\omega}\sin\beta\text{.........}(2)$$

$$A\epsilon^{at_0}+E\epsilon^{\mu t_0}\sin\beta+F\epsilon^{\mu t_0}\cos\beta=0\text{ ....}(3)$$

so that from $(3)\times\cos\beta-(1)$ we have

$$F\epsilon^{\mu t_0}=\frac{2d}{c\omega(1-k)}\cos\beta=\frac{-d(1-\sigma)}{c\omega\sigma}\cos\beta$$

and from $(2)\times\sin\beta-(1)$

$$E\epsilon^{\mu t_0}=\frac{2d}{c\omega(1-k)}\sin\beta=\frac{-d(1-\sigma)}{c\omega\sigma}\sin\beta$$

accordingly from (3)

$$A\epsilon^{at_0}=\frac{-2d}{c\omega(1-k)}=\frac{d(1-\sigma)}{c\omega\sigma}$$

Thus when a two phase generator running at no load is suddenly short-circuited at $t=t_0$, we have the expressions of the field and armature currents as follows:

$$x_s=\frac{-d}{b_i\omega}\cos\omega t-\frac{d}{b_i\omega}\cdot\frac{1-\sigma}{\sigma}\epsilon^{a(t-t_0)}\cos\omega t+\frac{d}{b_i\omega}\cdot\frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos(2\omega t-\omega t_0)$$
$$+\frac{d}{b_i\omega}\cdot\frac{1+\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\omega t_0$$

$$=\frac{-d}{b_i\omega}\cdot\left[\cos\omega t+\frac{1-\sigma}{\sigma}\epsilon^{a(t-t_0)}\cos\omega t-\frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos(2\omega t-\omega t_0)\right.$$
$$\left.-\frac{1+\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\omega t_0\right]$$

$$y_s = \frac{-d}{b_i\omega}\sin\omega t - \frac{d}{b_i\omega}\cdot\frac{1-\sigma}{\sigma}\,\epsilon^{\alpha(t-t_0)}\sin\omega t + \frac{d}{b_i\omega}\cdot\frac{1-\sigma}{2\sigma}\,\epsilon^{\mu(t-t_0)}\sin(2\omega t-\omega t_0)$$

$$+\frac{d}{b_i\omega}\cdot\frac{1+\sigma}{2\sigma}\,\epsilon^{\mu(t-t_0)}\sin\omega t_0$$

$$=\frac{-d}{b_i\omega}\cdot\left[\sin\omega t + \frac{1-\sigma}{\sigma}\,\epsilon^{\alpha(t-t_0)}\sin\omega t - \frac{1-\sigma}{2\sigma}\,\epsilon^{\mu(t-t_0)}\sin(2\omega t-\omega t_0)\right.$$

$$\left.-\frac{1+\sigma}{2\sigma}\,\epsilon^{\mu(t-t_0)}\sin\omega t_0\right]$$

$$z_s = \frac{d}{c\omega}\cdot\frac{1-\sigma}{\sigma}\cdot\epsilon^{\alpha(t-t_0)} - \frac{d}{c\omega}\cdot\frac{1-\sigma}{\sigma}\cdot\epsilon^{\mu(t-t)}\cos\omega(t-t_0)$$

$$=\frac{1-\sigma}{\sigma}\cdot I_f\cdot[\epsilon^{\alpha(t-t_0)} - \epsilon^{\mu(t-t_0)}\cos\omega(t-t_0)]$$

It is interesting to see that the shape and size of the curve of z_s with $\omega(t-t_0)$ as abscissa is indifferent to ωt_0. It takes the shape as shown in figures $(1)_a$ and $(1)_b$, the shape depending upon the relative value of α and μ and the size depending upon the value of σ.

The shape and size of the curves of x_s and y_s change with the value of ωt_0. Curves representing approximately the maximum sudden short-circuit currents will be shown in the next article.

§ 4. Maximum sudden short-circuit currents.

Putting $\frac{\alpha}{\omega} \doteqdot \frac{\mu}{\omega} \doteqdot 0$ in the expressions of x_s, y_s and z_s just obtained, we have

$$x_s = k[\cos\alpha - M\cos(2\alpha-\beta) - N\cos\beta]$$

$$y_s = k[\sin\alpha - M\sin(2\alpha-\beta) - N\sin\beta]$$

$$z_s = k'[1-\cos(\alpha-\beta)]$$

where $k=\frac{-d}{b_i\omega\sigma}$, $k'=\frac{1-\sigma}{\sigma}I_f$, $M=\frac{1}{2}(1-\sigma)$, $N=\frac{1}{2}(1+\sigma)$, $\alpha=\omega t$ and $\beta=\omega t_0$.

Now to find the maximum values of x_s, we have, from

$$\frac{\partial x_s}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial x_s}{\partial \beta} = 0$$

$$\sin \alpha - 2M \sin (2\alpha - \beta) = 0 \quad \text{and} \quad M \sin (2\alpha - \beta) - N \sin \beta = 0$$

so that

$$\sin \alpha = 2N \sin \beta$$

accordingly

$$N \sin \beta - 4MN \sin \beta \,.\, \sqrt{1 - 4N^2 \sin^2 \beta} \,.\, \cos \beta + M (1 - 8N^2 \sin^2 \beta) \sin \beta = 0$$

so that

$$\sin \beta = 0 \quad \text{or} \quad N - 4MN \,.\, \sqrt{1 - 4N^2 \sin^2 \beta} \,.\, \cos \beta + M (1 - 8N^2 \sin^2 \beta) = 0$$

that is

$$\sin \beta = 0 \quad \text{or} \quad (M + N) - 8MN^2 \sin^2 \beta = 4MN \,.\, \sqrt{1 - 4N^2 \sin^2 \beta} \,.\, \cos \beta$$

that is

$$\sin \beta = 0 \quad \text{or} \quad \sin \beta = \sqrt{\frac{(M+N)^2 - 16M^2N^2}{16MN^3(1 - 4MN)}} = \sqrt{\frac{1 - (1 - \sigma^2)^2}{(1 - \sigma^2)(1 - \sigma)^2 \sigma^2}}$$

$$= \frac{1}{1 - \sigma} \sqrt{\frac{2 - \sigma^2}{1 - \sigma^2}}$$

which is > 1 for $1 > \sigma > 0$.

But $\sin \beta > 1$ is impossible.

Therefore $\sin \beta = 0$ only gives $(y_s)_{\text{max.}}$.

Now putting $\beta = 0$ we have

$$(x_s)_{\text{max.}} = k (\cos \alpha - M \cos 2\alpha - N)$$

that is

$$(x_s)_{\text{max.}} = k [\cos (\alpha - \beta) - M \cos 2 (\alpha - \beta) - N]$$

which is maximum when $\sin(\alpha-\beta)-2M\sin 2(\alpha-\beta)=0$ that is when $\sin(\alpha-\beta)=0$ or $\cos(\alpha-\beta)=\frac{1}{2(1-\sigma)}$ that is when $\alpha-\beta=0$ or π or $\cos^{-1}\frac{1}{2(1-\sigma)}$.

But when $\alpha-\beta=0$, we have

$$(x_s)_{\max.}=k\left[1-\frac{1}{2}(1-\sigma)-\frac{1}{2}(1+\sigma)\right]=0$$

and when $\alpha-\beta=\pi$

$$(x_s)_{\max.}=k\left[-1-\frac{1}{2}(1-\sigma)-\frac{1}{2}(1+\sigma)\right]=-2k=\frac{2d}{b_i\omega\sigma}$$

and when $\alpha-\beta=\cos^{-1}\frac{1}{2(1-\sigma)}$

$$(x_s)_{\max.}=k\left[\frac{1}{2(1-\sigma)}-\frac{1}{2}(1-\sigma)\left(\frac{1}{2(1-\sigma)^2}-1\right)-\frac{1}{2}(1+\sigma)\right]$$

$$=k\left(\frac{1}{4(1-\sigma)}-\sigma\right)$$

$$=\frac{-d}{b_i\omega\sigma}\cdot\frac{(1-2\sigma)^2}{4(1-\sigma)}$$

which is $<\frac{d}{b_i\omega\sigma}$ in magnitude when $\sigma<0{\cdot}5$.

Therefore $(x_s)_{\max.\,\max.}$ appears usually at $\alpha-\beta=\pi$ and $(x_s)_{\max.\,\max.}$ is $=\frac{2d}{b_i\omega\sigma}$.

Figures $(2)_a$ and $(2)_b$ represent the curves of

$$(x_s)_{\max.}=k\left[\cos(\alpha-\beta)-M\cos 2(\alpha-\beta)-N\right]$$

σ being taken $=0{\cdot}5$ in figure $(2)_a$ and $0{\cdot}1$ in figure $(2)_b$.

If we put $\beta=\pi$, then we have

$$(x_s)_{\text{max.}}=k(\cos\alpha+M\cos 2\alpha+N)$$

that is
$$(x_s)_{\text{max.}}=-k[\cos(\alpha-\beta)-M\cos 2(\alpha-\beta)-N]$$

which is equal in magnitude to that $(x_s)_{\text{max.}}$ when $\beta=0$ but has the opposite sign. This $(x_s)_{\text{max.}}$ traces the curves shown in figures $(2)_a$ and $(2)_b$ with the sign changed and we can immediately write down that $(x_s)_{\text{max.}}$ is $=0$ when $\alpha-\beta=0$, is $=\frac{-2d}{b_i\omega\sigma}$ when $\alpha-\beta=\pi$, and is $=\frac{d}{b_i\omega\sigma}\cdot\frac{(1-2\sigma)^2}{4(1-\sigma)}$ when $\alpha-\beta=\cos^{-1}\frac{1}{2(1-\sigma)}$ and $(x_s)_{\text{max. max.}}$ is $=\frac{-2d}{b_i\omega\sigma}$.

Next, from $\frac{\partial y_s}{\partial\alpha}=0$ and $\frac{\partial y_s}{\partial\beta}=0$ we have, similarly as before, $\cos\beta=0$ or $\frac{1}{1-\sigma}\sqrt{\frac{2-\sigma^2}{1-\sigma^2}}$, the latter of which is impossible.

Now putting $\beta=\frac{\pi}{2}$ we have

$$(y_s)_{\text{max.}}=k(\sin\alpha+M\cos 2\alpha-N)$$

that is

$$(y_s)_{\text{max.}}=k[\cos(\alpha-\beta)-M\cos 2(\alpha-\beta)-N]$$

which is the same as that $(x_s)_{\text{max.}}$ when $\beta=0$, as shown in figure (2).

If we put $\beta=3\frac{\pi}{2}$, then we have

$$(y_s)_{\text{max.}}=k(\sin\alpha-M\cos 2\alpha+N)$$

that is

$$(y_s)_{\text{max.}}=-k[\cos(\alpha-\beta)-M\cos 2(\alpha-\beta)-N]$$

which is the same as that $(x_s)_{\text{max.}}$ when $\beta=\pi$, as shown in figure (2) with the sign changed. Thus $(y_s)_{\text{max. max.}}$ is just the same as $(x_s)_{\text{max. max.}}$

To prove this more simply, writing y_s in the form

$$y_s = k\left\{\cos\left(\alpha - \frac{\pi}{2}\right) - M\cos\left[2\left(\alpha-\frac{\pi}{2}\right) - \left(\beta - \frac{\pi}{2}\right)\right] - N\cos\left(\beta-\frac{\pi}{2}\right)\right\}$$

we can, referring to what we saw in x_s, write down that y_s is maximum when $\beta - \frac{\pi}{2} = 0$ or π; and when $\beta - \frac{\pi}{2} = 0$ that is when $\beta = \frac{\pi}{2}$ we have

$$(y_s)_{\text{max.}} = k\left[\cos(\alpha-\beta) - M\cos 2(\alpha-\beta) - N\right]$$

which is the same as that $(x_s)_{\text{max.}}$ when $\beta = 0$; and when $\beta - \frac{\pi}{2} = \pi$ that is when $\beta = 3\frac{\pi}{2}$ we have

$$(y_s)_{\text{max.}} = -k\left[\cos(\alpha-\beta) - M\cos 2(\alpha-\beta) - N\right]$$

which is the same as that $(x_s)_{\text{max.}}$ when $\beta = \pi$.

Next $(z_s)_{\text{max.}}$ is obviously $2k'$ that is $2\frac{1-\sigma}{\sigma}I_f$ and so

$$(z_s + I_f)_{\text{max.}} = I_f\left(\frac{2}{\sigma} - 1\right)$$

It takes place at $\alpha - \beta = \pi$.

Note that, since we assume $\frac{\alpha}{\omega} \doteqdot \frac{\mu}{\omega} \doteqdot 0$, those maximum x_s, y_s and z_s as found above are not true maximum values but of course they are on the safe side.

Also note that putting $\omega t_0 = 0$ or π and $\omega t_0 = \frac{\pi}{2}$ or $3\frac{\pi}{2}$ respectively in the expressions of x_s and y_s given at the end of the previous article, we have

$$\begin{aligned} x_s &= y_s \\ &= \frac{\mp d}{b_i\omega}\left[\cos\omega(t-t_0) + \frac{1-\sigma}{\sigma}\epsilon^{\alpha(t-t_0)}\cos\omega(t-t_0)\right. \\ &\qquad \left. - \frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos 2\omega(t-t_0) - \frac{1+\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\right] \end{aligned}$$

which shows approximately the manner in which the instantaneous values of the maximum sudden short-circuit currents in the armature change with time, starting at $t - t_0 = 0$.

Curves showing these x_s or y_s with $\frac{a_1}{b_1 \omega} = \frac{1}{100}$, $\frac{a_i}{b_i \omega} = \frac{3}{100}$ and $\sigma = 0{\cdot}5$ or $0{\cdot}2$ so that $\frac{\alpha}{\omega} = \frac{-2}{100}$ or $\frac{-5}{100}$ and $\frac{\mu}{\omega} = \frac{-4{\cdot}5}{100}$ or $\frac{-9}{100}$ are given in figures (3) and (4).

Chapter II.

Three Phase Generator.

§ 5. Fundamental Equations.

Let

x, y and z = armature currents in the three phases,

w = alternating current in the field circuit,

a and b' = resistance and total inductance of each of the three phases,

a_1 and b_1 = those of the field circuit,

c, d and I_f = the same as in Chapter I,

c' = mutual inductance between any two of the three phases.

Then under the same assumptions of the field poles and flux distributions as in Chapter I, the fundamental equations will be

$$ax + b'\frac{dx}{dt} + c\frac{d}{dt}(w\cos\alpha) + c'\frac{d}{dt}(y+z) = d\sin\alpha \quad \text{.........(1)}$$

$$ay + b'\frac{dy}{dt} + c\frac{d}{dt}(w\cos\beta) + c'\frac{d}{dt}(z+x) = d\sin\beta \quad \text{.........(2)}$$

$$az + b'\frac{dz}{dt} + c\frac{d}{dt}(w\cos\gamma) + c'\frac{d}{dt}(x+y) = d\sin\gamma \quad \text{.........(3)}$$

$$a_1 w + b_1\frac{dw}{dt} + c\frac{d}{dt}(x\cos\alpha + y\cos\beta + z\cos\gamma) = 0 \quad \text{.........(4)}$$

where $\alpha = \omega t$, $\beta = \omega t - \frac{2\pi}{3}$ and $\gamma = \omega t + \frac{2\pi}{3}$.

Now, for simplifying deductions assume $x + y + z = 0$. (The case when

$x+y+z \mp 0$ will be explained in the next article.) Then putting $b'-c'=b$ the fundamental equations become

$$ax+b\frac{dx}{dt}+c\frac{d}{dt}(w\cos\alpha)=d\sin\alpha \quad \text{.........................(5)}$$

$$ay+b\frac{dy}{dt}+c\frac{d}{dt}(w\cos\beta)=d\sin\beta \quad \text{.........................(6)}$$

$$az+b\frac{dz}{dt}+c\frac{d}{dt}(w\cos\gamma)=d\sin\gamma \quad \text{.........................(7)}$$

$$a_1w+b_1\frac{dw}{dt}+c\frac{d}{dt}(x\cos\alpha+y\cos\beta+z\cos\gamma)=0$$

so that similarly as in Chapter I, we have

$$a\xi+b\frac{d\xi}{dt}+c\frac{d\chi}{dt}\cos\alpha=-\frac{d}{\omega}\cos\alpha \quad \text{.............................(8)}$$

$$a\eta+b\frac{d\eta}{dt}+c\frac{d\chi}{dt}\cos\beta=-\frac{d}{\omega}\cos\beta \quad \text{.............................(9)}$$

$$a\zeta+b\frac{d\zeta}{dt}+c\frac{d\chi}{dt}\cos\gamma=-\frac{d}{\omega}\cos\gamma \quad \text{..........................(10)}$$

$$a_1\chi+b_1\frac{d\chi}{dt}+c\left(\frac{d\xi}{dt}\cos\alpha+\frac{d\eta}{dt}\cos\beta+\frac{d\zeta}{dt}\cos\gamma\right)=0 \quad \text{........(11)}$$

Now from equations (8), (9) and (10), we have

$$\Sigma\left(\frac{d\xi}{dt}\cos\alpha\right)=\frac{-1}{b}\left[\left(\frac{d}{\omega}+c\frac{d\chi}{dt}\right)\Sigma\cos^2\alpha+a\Sigma(\xi\cos\alpha)\right]$$

But $$\Sigma\cos^2\alpha=\frac{1}{2}\Sigma(1+\cos 2\alpha)=\frac{3}{2}$$

Therefore from (11), we have

$$a_1\chi+b_1\frac{d\chi}{dt}-\frac{c}{b}\left[\frac{3}{2}\left(\frac{d}{\omega}+c\frac{d\chi}{dt}\right)+a\Sigma(\xi\cos\alpha)\right]=0$$

that is

$$\left(bb_1-\frac{3}{2}c^2\right)\frac{d\chi}{dt}+a_1b\chi-\frac{3}{2}\cdot\frac{cd}{\omega}-ca\Sigma(\xi\cos\alpha)=0 \quad \text{........(12)}$$

which by differentiation and insertion of equation (11) becomes

$$\left(bb_1 - \frac{3}{2}c^2\right)\frac{d^2\chi}{dt^2} + a_1 b\frac{d\chi}{dt} + a\left(a_1\chi + b_1\frac{d\chi}{dt}\right) + ca\omega\Sigma(\xi\sin\alpha) = 0$$

that is

$$\left(bb_1 - \frac{3}{2}c^2\right)\frac{d^2\chi}{dt^2} + (ab_1 + a_1 b)\frac{d\chi}{dt} + aa_1\chi + ca\omega\Sigma(\xi\sin\alpha) = 0 \quad \ldots(13)$$

so that

$$\left(bb_1 - \frac{3}{2}c^2\right)\frac{d^3\chi}{dt^3} + (ab_1 + a_1 b)\frac{d^2\chi}{dt^2} + aa_1\frac{d\chi}{dt} + ca\omega\left[\Sigma\left(\frac{d\xi}{dt}\sin\alpha\right) + \omega\Sigma(\xi\cos\alpha)\right] = 0$$

But

$$\Sigma\left(\frac{d\xi}{dt}\sin\alpha\right) = \frac{-1}{b}\left[\left(\frac{d}{\omega} - c\frac{d\chi}{dt}\right)\Sigma(\sin\alpha\cos\alpha) + a\Sigma(\xi\sin\alpha)\right] = -\frac{a}{b}\Sigma(\xi\sin\alpha)$$

Therefore

$$b\left(bb_1 - \frac{3}{2}c^2\right)\frac{d^3\chi}{dt^3} + b(ab_1 + a_1 b)\frac{d^2\chi}{dt^2} + aa_1 b\frac{d\chi}{dt} - ca\omega\left[a\Sigma(\xi\sin\alpha) - b\omega\Sigma(\xi\cos\alpha)\right] = 0$$

Therefore from equations (12) and (13)

$$b\left(bb_1 - \frac{3}{2}c^2\right)\frac{d^3\chi}{dt^3} + \left(2abb_1 + a_1 b^2 - \frac{3}{2}ac^2\right)\frac{d^2\chi}{dt^2} + \left[2aa_1 b + a^2 b_1 + b\left(bb_1 - \frac{3}{2}c^2\right)\omega^2\right]\frac{d\chi}{dt} + a_1(a^2 + b^2\omega^2)\chi - \frac{3}{2}bcd\omega = 0$$

accordingly

$$b\left(bb_1 - \frac{3}{2}c^2\right)\frac{d^3 w}{dt^3} + \left(2abb_1 + a_1 b^2 - \frac{3}{2}ac^2\right)\frac{d^2 w}{dt^2} + \left[2aa_1 b + a^2 b_1 + b\left(bb_1 - \frac{3}{2}c^2\right)\omega^2\right]\frac{dw}{dt} + a_1(a^2 + b^2\omega^2)w = 0 \quad \ldots\ldots\ldots\ldots(14)$$

Now to determine $bb_1 - \frac{3}{2}c^2$, if we assume the armature winding to be distributed uniformly and the flux distribution sinusoidal; and denote the number of turns per phase by n, the reluctance of the equivalent magnetic circuit for the total sine flux by $\mathcal{R}$ and the leakage coefficient of each phase by ν, then we have

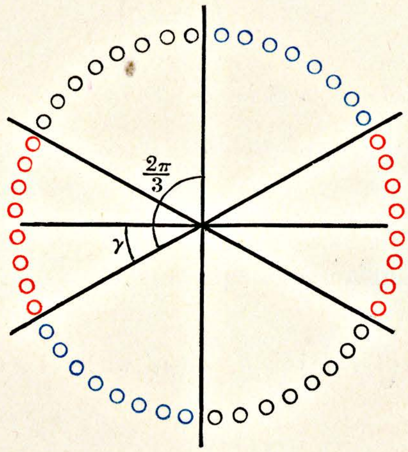


Fig. 5.

$$b' = \nu n \frac{1}{2\gamma}\int_{-\gamma}^{+\gamma}\left(\int_{\beta}^{\pi+\beta}\frac{4\pi n}{\mathcal{R}}\sin\alpha\, d\alpha\right)d\beta = \nu \,.\, \frac{4\pi n^2}{\mathcal{R}}\cdot\frac{1}{2\gamma}\int_{-\gamma}^{+\gamma} 2\cos\beta\, d\beta$$

$$= \nu \,.\, \frac{8\pi n^2}{\mathcal{R}}\cdot\frac{\sin\gamma}{\gamma}$$

$$c' = n\frac{1}{2\gamma}\int_{-\gamma+\frac{2\pi}{3}}^{+\gamma+\frac{2\pi}{3}}\left(\int_{\beta}^{\pi+\beta}\frac{4\pi n}{\mathcal{R}}\sin\alpha\, d\alpha\right)d\beta = \frac{4\pi n^2}{\mathcal{R}}\cdot\frac{1}{2\gamma}\int_{\frac{2\pi}{3}-\gamma}^{\frac{2\pi}{3}+\gamma} 2\cos\beta\, d\beta$$

$$= -\frac{4\pi n^2}{\mathcal{R}}\cdot\frac{\sin\gamma}{\gamma}$$

so that
$$b = b' - c' = b'\left(1 + \frac{1}{2\nu}\right)$$

accordingly

$$bb_1 - \frac{3}{2}c^2 = \left(1+\frac{1}{2\nu}\right)b'b_1\left(1-\frac{3}{2}\cdot\frac{1}{1+\frac{1}{2\nu}}\cdot\frac{c^2}{b'b_1}\right) = \left(1+\frac{1}{2\nu}\right)b'b_1\sigma'$$

$$= bb_1\sigma'$$

where $$\sigma' = 1 - \frac{1}{\left(\frac{2}{3}\nu+\frac{1}{3}\right)\nu_f}$$

where ν_f is the field leakage coefficient.

Note that σ' lies between 1 and 0.

§ 6. Solution of the fundamental equations.

Since $bb_1 - \frac{3}{2}c^2 = \left(1+\frac{1}{2\nu}\right)b'b_1\sigma' = bb_1\sigma'$ where $1 > \sigma' > 0$, equation (14) solves to

$$w = A\epsilon^{\alpha t} + D\epsilon^{\mu t}\sin(\nu t + \psi) \quad \text{.....................(15)}$$

where α and $\mu \pm j\nu$ are the roots of

$$b\left(bb_1 - \tfrac{3}{2}c^2\right)\lambda^3 + \left(2abb_1 + a_1b^2 - \tfrac{3}{2}ac^2\right)\lambda^2 + \left[2aa_1b + a^2b_1 + b\left(bb_1 - \tfrac{3}{2}c^2\right)\omega^2\right]\lambda$$
$$+ a_1(a^2 + b^2\omega^2) = 0$$

that is

$$b^2b_1\sigma'\lambda^3 + (abb_1 + a_1b^2 + abb_1\sigma')\lambda^2 + (2aa_1b + a^2b_1 + b^2b_1\sigma'\omega^2)\lambda + a_1(a^2 + b^2\omega^2) = 0$$

that is

$$\sigma'\left(\frac{\lambda}{\omega}\right)^3 + \left(\frac{a}{b\omega} + \frac{a_1}{b_1\omega} + \frac{a}{b\omega}\sigma'\right)\left(\frac{\lambda}{\omega}\right)^2 + \left(2\frac{a}{b\omega}\cdot\frac{a_1}{b_1\omega} + \frac{a^2}{b^2\omega^2} + \sigma'\right)\left(\frac{\lambda}{\omega}\right)$$
$$+ \frac{a_1}{b_1\omega}\left(\frac{a^2}{b^2\omega^2} + 1\right) = 0$$

that is

$$\sigma'\left(\frac{\lambda}{\omega}\right)^3 + (p+q+q\sigma')\left(\frac{\lambda}{\omega}\right)^2 + (2pq+q^2+\sigma')\left(\frac{\lambda}{\omega}\right) + p(q^2+1) = 0$$

where $$p = \frac{a_1}{b_1\omega} \text{ and } q = \frac{a}{b\omega}$$

so that, when p and q are small and σ not so small, the expressions of α, μ and ν are, referring to the theory of distortionless alternators,

$$\alpha = -\frac{p}{\sigma'}\omega, \quad q = -\frac{1}{2}(1+\sigma')\frac{q}{\sigma'}\omega \text{ and } \nu = \omega$$

Now since w takes the same form as z in Art. 2, the expressions of the armature current will be, similarly as in Art. 2,

$$x = X\sqrt{2}\sin(\omega t - \phi_x) + A_1\epsilon^{\alpha t}\sin(\omega t - \theta_\alpha) - D_1\epsilon^{\mu t}\sin(\overline{\nu+\omega}t + \psi + \psi_1)$$
$$- D_2\epsilon^{\mu t}\sin(\overline{\nu-\omega}t + \psi + \psi_2) + k_1\epsilon^{-\frac{a}{b}t} \quad \ldots\ldots\ldots\ldots\ldots(16)$$

$$y = X\sqrt{2}\sin\left(\omega t - \phi_x - \frac{2\pi}{3}\right) + A_1\epsilon^{\alpha t}\sin\left(\omega t - \theta_\alpha - \frac{2\pi}{3}\right)$$
$$- D_1\epsilon^{\mu t}\sin\left(\overline{\nu+\omega}t + \psi + \psi_1 - \frac{2\pi}{3}\right)$$
$$- D_2\epsilon^{\mu t}\sin\left(\overline{\nu-\omega}t + \psi + \psi_2 + \frac{2\pi}{3}\right) + k_2\epsilon^{-\frac{a}{b}t} \quad \ldots\ldots(17)$$

$$z = X\sqrt{2}\sin\left(\omega t - \phi_x + \frac{2\pi}{3}\right) + A_1\epsilon^{\alpha t}\sin\left(\omega t - \theta_\alpha + \frac{2\pi}{3}\right)$$
$$- D_1\epsilon^{\mu t}\sin\left(\overline{\nu+\omega}t + \psi + \psi_1 + \frac{2\pi}{3}\right)$$
$$- D_2\epsilon^{\mu t}\sin\left(\overline{\nu-\omega}t + \psi + \psi_2 - \frac{2\pi}{3}\right) + k_3\epsilon^{-\frac{a}{b}t} \quad \ldots\ldots(18)$$

where X, A_1, D_1, D_2, ϕ_x, θ_α, ψ_1 and ψ_2 denote the same things as in Art. 2.

Now put these x, y and z in the fundamental equation

$$a_1 w + b_1\frac{dw}{dt} + c\frac{d}{dt}(x\cos\alpha + y\cos\beta + z\cos\gamma) = 0$$

Then we have

$$a_1[A\epsilon^{at}+D\epsilon^{\mu t}\sin(\nu t+\psi)]+b_1\frac{d}{dt}[A\epsilon^{at}+D\epsilon^{\mu t}\sin(\nu t+\psi)]$$
$$-\frac{3}{2}c\frac{d}{dt}[A_1\epsilon^{at}\sin\theta_a+D_1\epsilon^{\mu t}\sin(\nu t+\psi+\psi_1)+D_2\epsilon^{\mu t}\sin(\nu t+\psi+\psi_2)]$$
$$+c\frac{d}{dt}[\epsilon^{-\frac{a}{b}t}(k_1\cos\alpha+k_2\cos\beta+k_3\cos\gamma)]=0$$

that is

$$\left[(a_1+b_1\alpha)A-\frac{3}{2}cA_1\alpha\sin\theta_a\right]\epsilon^{at}$$
$$+\left[(a_1+b_1\mu)D-\frac{3}{2}c\mu(D_1\cos\psi_1+D_2\cos\psi_2)+\frac{3}{2}c\nu(D_1\sin\psi_1+D_2\sin\psi_2)\right]\epsilon^{\mu t}\sin(\nu t+\psi)$$
$$+\left[b\nu D-\frac{3}{2}c\mu(D_1\sin\psi_1+D_2\sin\psi_2)-\frac{3}{2}c\nu(D_1\cos\psi_1+D_2\cos\psi_2)\right]\epsilon^{\mu t}\cos(\nu t+\psi)$$
$$+c\frac{d}{dt}\left[\left(k_1-\frac{1}{2}k_2-\frac{1}{2}k_3\right)\epsilon^{-\frac{a}{b}t}\cos\omega t+\frac{\sqrt{3}}{2}(k_2-k_3)\epsilon^{-\frac{a}{b}t}\sin\omega t\right]=0$$

But similarly as stated in the theory of distortionless alternators, we can prove that the coefficients of ϵ^{at} and $\epsilon^{\mu t}$ are $=0$.

Therefore we have

$$2k_1-k_2-k_3=0 \text{ and } k_2-k_3=0$$

But since $x+y+z=0$ as assumed in the previous article, we have, by summing up the equations (16), (17) and (18), $k_1+k_2+k_3=0$.

Therefore, we can conclude that $k_1=k_2=k_3=0$.

Note that heretofore we have assumed that $x+y+z=0$. We shall now explain the case when this sum is not equal to zero.

Summing up the equations (1), (2) and (3) and denoting $(b'+2c')$ by b'' we have

$$a(x+y+z)+b''\frac{d}{dt}(x+y+z)=0 \quad \text{..................(19)}$$

which gives

$$x + y + z = k\epsilon^{-\frac{a}{b''}t}$$

so that equations (1), (2) and (3) become

$$ax + b\frac{dx}{dt} + c\frac{d}{dt}(w\cos\alpha) + c'k\frac{d}{dt}\left(\epsilon^{-\frac{a}{b''}t}\right) = d\sin\alpha$$

$$ay + b\frac{dy}{dt} + c\frac{d}{dt}(w\cos\beta) + c'k\frac{d}{dt}\left(\epsilon^{-\frac{a}{b''}t}\right) = d\sin\beta$$

$$az + b\frac{dz}{dt} + c\frac{d}{dt}(w\cos\gamma) + c'k\frac{d}{dt}\left(\epsilon^{-\frac{a}{b''}t}\right) = d\sin\gamma$$

so that we have

$$a\xi + b\frac{d\xi}{dt} + c\frac{d\chi}{dt}\cos\alpha = -\frac{d}{\omega}\cos\alpha - c'k\epsilon^{-\frac{a}{b''}t}$$

$$a\eta + b\frac{d\eta}{dt} + c\frac{d\chi}{dt}\cos\beta = -\frac{d}{\omega}\cos\beta - c'k\epsilon^{-\frac{a}{b''}t}$$

$$a\zeta + b\frac{d\zeta}{dt} + c\frac{d\chi}{dt}\cos\gamma = -\frac{d}{\omega}\cos\gamma - c'k\epsilon^{-\frac{a}{b''}t}$$

from which we have the equation of w quite the same as equation (14), accordingly the expression of w the same as shown in equation (15), and those of x, y and z equal to those of equations (16), (17) and (18) respectively added all and each by

$$-\frac{1}{b}\cdot\frac{d}{dt}\left(c'k\epsilon^{-\frac{a}{b}t}\int\epsilon^{\left(\frac{a}{b}-\frac{a}{b''}\right)t}\,dt\right)$$

that is $$\frac{c'k}{b''-b}\cdot\epsilon^{-\frac{a}{b''}t}$$

that is $$\frac{1}{3}k\epsilon^{-\frac{a}{b''}t}$$

Now putting these x, y and z in equation (19), we have

$$\left(a - b''\frac{a}{b}\right)(k_1 + k_2 + k_3)\,\epsilon^{-\frac{a}{b}t} = 0 \text{ that is } k_1 + k_2 + k_3 = 0$$

and also putting the same x, y, and z in the fundamental equation

$$a_1 w + b_1 \frac{dw}{dt} + c\frac{d}{dt}(x\cos\alpha + y\cos\beta + z\cos\gamma) = 0$$

we have, similarly as before,

$$2k_1 - k_2 - k_3 = 0 \quad \text{and} \quad k_2 - k_3 = 0$$

so that we can as before conclude that

$$k_1 = k_2 = k_3 = 0$$

The solutions of x, y and z are therefore

$$x = X\sqrt{2}\sin(\omega t - \phi_x) + A_1\epsilon^{at}\sin(\omega t - \theta_a) - D_1\epsilon^{\mu t}\sin(\overline{\nu + \omega t} + \psi + \psi_1)$$
$$- D_2\epsilon^{\mu t}\sin(\overline{\nu - \omega t} + \psi + \psi_2) + \frac{1}{3}k\epsilon^{-\frac{a}{b''}t}$$

$$y = X\sqrt{2}\sin\left(\omega t - \phi_x - \frac{2\pi}{3}\right) + A_1\epsilon^{at}\sin\left(\omega t - \theta_a - \frac{2\pi}{3}\right)$$
$$- D_1\epsilon^{\mu t}\sin\left(\overline{\nu + \omega t} + \psi + \psi_1 - \frac{2\pi}{3}\right)$$
$$- D_2\epsilon^{\mu t}\sin\left(\overline{\nu - \omega t} + \psi + \psi_2 + \frac{2\pi}{3}\right) + \frac{1}{3}k\epsilon^{-\frac{a}{b''}t}$$

$$z = X\sqrt{2}\sin\left(\omega t - \phi_x + \frac{2\pi}{3}\right) + A_1\epsilon^{at}\sin\left(\omega t - \theta_a + \frac{2\pi}{3}\right)$$
$$- D_1\epsilon^{\mu t}\sin\left(\overline{\nu + \omega t} + \psi + \psi_1 + \frac{2\pi}{3}\right)$$
$$- D_2\epsilon^{\mu t}\sin\left(\overline{\nu - \omega t} + \psi + \psi_2 - \frac{2\pi}{3}\right) + \frac{1}{3}k\epsilon^{-\frac{a}{b''}t}$$

and, since the sum of these x, y and z is $k\epsilon^{-\frac{a}{b''}t}$, we can conclude that if $(x+y+z)_{t=t_0} = 0$, that is the load be balanced just before the sudden change of load, then k will be zero, that is, all the terms containing k in the solutions of x, y and z will vanish.

§ 7. Sudden short-circuit currents.

At short-circuit we have

$$\alpha = -\frac{p}{\sigma'}\omega, \quad q = -\frac{1}{2}(1+\sigma')\frac{q}{\sigma'}\omega \quad \text{and} \quad \nu = \omega$$

where

$$p = \frac{a_1}{b_1 \omega} \doteq 0 \quad \text{and} \quad q = \frac{a_i}{b_i \omega} \doteq 0$$

so that

$$\frac{\alpha}{\omega} \doteq \frac{\mu}{\omega} \doteq 0$$

and hence, similarly as explained in Chapter I, Art. 3,

$$X\sqrt{2} = Y\sqrt{2} = Z\sqrt{2} \doteq \frac{d}{b_i \omega}$$

$$\phi_x \doteq \frac{\pi}{2} \qquad \phi_y \doteq \frac{\pi}{2} + \frac{2\pi}{3} \qquad \phi_z \doteq \frac{\pi}{2} - \frac{2\pi}{3}$$

$$A_1 \doteq \frac{Ac}{b} \qquad D_1 \doteq \frac{1}{2}\cdot\frac{Dc}{b} \qquad D_2 \doteq \frac{1}{2}\cdot\frac{Dc}{b}\cdot\frac{1+\sigma}{1-\sigma}$$

$$\theta_\alpha \doteq \frac{\pi}{2} \qquad \psi_1 \doteq 0 \qquad \psi_2 \doteq 0$$

Therefore

$$x_s = \frac{-d}{b_i \omega}\cos\omega t - \frac{Ac}{b_i}\epsilon^{\alpha t}\cos\omega t - \frac{1}{2}\cdot\frac{Dc}{b_i}\epsilon^{\mu t}\sin(2\omega t + \psi)$$
$$- \frac{1}{2}\cdot\frac{Dc}{b_i}\cdot\frac{1+\sigma'}{1-\sigma'}\epsilon^{\mu t}\sin\psi + \frac{1}{3}k\epsilon^{-\frac{a_i}{b_i''}t}$$

$$y_s = \frac{-d}{b_i \omega}\cos\left(\omega t - \frac{2\pi}{3}\right) - \frac{Ac}{b_i}\epsilon^{\alpha t}\cos\left(\omega t - \frac{2\pi}{3}\right)$$
$$- \frac{1}{2}\cdot\frac{Dc}{b_i}\epsilon^{\mu t}\sin\left(2\omega t + \psi - \frac{2\pi}{3}\right)$$
$$- \frac{1}{2}\cdot\frac{Dc}{b_i}\cdot\frac{1+\sigma'}{1-\sigma'}\epsilon^{\mu t}\sin\left(\psi + \frac{2\pi}{3}\right) + \frac{1}{3}k\epsilon^{-\frac{a_i}{b_i''}t}$$

$$z_s = \frac{-d}{b_i\omega}\cos\left(\omega t + \frac{2\pi}{3}\right) - \frac{Ac}{b_i}\epsilon^{at}\cos\left(\omega t + \frac{2\pi}{3}\right)$$

$$-\frac{1}{2}\cdot\frac{Dc}{b_i}\epsilon^{\mu t}\sin\left(2\omega t + \psi + \frac{2\pi}{3}\right)$$

$$-\frac{1}{2}\cdot\frac{Dc}{b_i}\cdot\frac{1+\sigma'}{1-\sigma'}\epsilon^{\mu t}\sin\left(\psi - \frac{2\pi}{3}\right) + \frac{1}{3}k\epsilon^{-\frac{a_i}{b_i''}t}$$

$$w_s = A\epsilon^{at} + D\epsilon^{\mu t}\sin(\omega t + \psi)$$

Hence, denoting $D\cos\psi$ by E, $D\sin\psi$ by F and $\frac{1+\sigma'}{1-\sigma'}$ by k, we have

$$x_s = \frac{-d}{b_i\omega}\cos\omega t - \frac{Ac}{b_i}\epsilon^{at}\cos\omega t - \frac{1}{2}\cdot\frac{Ec}{b_i}\epsilon^{\mu t}\sin 2\omega t$$

$$-\frac{1}{2}\cdot\frac{Fc}{b_i}\epsilon^{\mu t}(k + \cos 2\omega t) + \frac{1}{3}k\epsilon^{-\frac{a_i}{b_i''}t}$$

$$y_s = \frac{-d}{b_i\omega}\cos\left(\omega t - \frac{2\pi}{3}\right) - \frac{Ac}{b_i}\epsilon^{at}\cos\left(\omega t - \frac{2\pi}{3}\right)$$

$$-\frac{1}{2}\cdot\frac{Ec}{b_i}\epsilon^{\mu t}\left[\sin\left(2\omega t - \frac{2\pi}{3}\right) + k\sin\frac{2\pi}{3}\right]$$

$$-\frac{1}{2}\cdot\frac{Fc}{b_i}\epsilon^{\mu t}\left[\cos\left(2\omega t - \frac{2\pi}{3}\right) + k\cos\frac{2\pi}{3}\right] + \frac{1}{3}k\epsilon^{-\frac{a_i}{b_i''}t}$$

$$z_s = \frac{-d}{b_i\omega}\cos\left(\omega t + \frac{2\pi}{3}\right) - \frac{Ac}{b_i}\epsilon^{at}\cos\left(\omega t + \frac{2\pi}{3}\right)$$

$$-\frac{1}{2}\cdot\frac{Ec}{b_i}\epsilon^{\mu t}\left[\sin\left(2\omega t + \frac{2\pi}{3}\right) - k\sin\frac{2\pi}{3}\right]$$

$$-\frac{1}{2}\cdot\frac{Fc}{b_i}\epsilon^{\mu t}\left[\cos\left(2\omega t + \frac{2\pi}{3}\right) + k\cos\frac{2\pi}{3}\right] + \frac{1}{3}k\epsilon^{-\frac{a_i}{b_i''}t}$$

$$w_s = A\epsilon^{at} + E\epsilon^{\mu t}\sin\omega t + F\epsilon^{\mu t}\cos\omega t$$

Now, to determine the arbitrary constants, suppose that $x=y=z=w=0$ at $t=t_0$ and denote ωt_0 by β. Then since $k=0$ in this case, as said at the end of the previous article, we have

$$A\epsilon^{at_0}\cos\beta+\frac{1}{2}E\epsilon^{\mu t_0}\sin 2\beta+\frac{1}{2}F\epsilon^{\mu t_0}(k+\cos 2\beta)=\frac{-d}{c\omega}\cos\beta \quad \ldots\ldots\ldots\ldots(20)$$

$$A\epsilon^{at_0}\cos\left(\beta-\frac{2\pi}{3}\right)+\frac{1}{2}E\epsilon^{\mu t_0}\left[\sin\left(2\beta-\frac{2\pi}{3}\right)+k\sin\frac{2\pi}{3}\right]$$
$$+\frac{1}{2}F\epsilon^{\mu t_0}\left[\cos\left(2\beta-\frac{2\pi}{3}\right)+k\cos\frac{2\pi}{3}\right]=\frac{-d}{c\omega}\cos\left(\beta-\frac{2\pi}{3}\right)$$
$$\ldots\ldots\ldots\ldots(21)$$

$$A\epsilon^{at_0}\cos\left(\beta+\frac{2\pi}{3}\right)+\frac{1}{2}E\epsilon^{\mu t_0}\left[\sin\left(2\beta+\frac{2\pi}{3}\right)-k\sin\frac{2\pi}{3}\right]$$
$$+\frac{1}{2}F\epsilon^{\mu t_0}\left[\cos\left(2\beta+\frac{2\pi}{3}\right)+k\cos\frac{2\pi}{3}\right]=\frac{-d}{c\omega}\cos\left(\beta+\frac{2\pi}{3}\right)$$
$$\ldots\ldots\ldots\ldots(22)$$

$$A\epsilon^{at_0}+E\epsilon^{\mu t_0}\sin\beta+F\epsilon^{\mu t_0}\cos\beta=0 \quad \ldots\ldots\ldots\ldots(23)$$

the two middle ones of which give (by subtraction)

$$A\epsilon^{at_0}\sin\beta+\frac{1}{2}E\epsilon^{\mu t_0}(k-\cos 2\beta)+\frac{1}{2}F\epsilon^{\mu t_0}\sin 2\beta=\frac{-d}{c\omega}\sin\beta \quad \ldots\ldots\ldots\ldots(24)$$

Thus we have three equations (20), (24) and (23) for determining A, E and F, just the same as those we had in the case of the two phase generator (see Art. 3). Therefore

$$A=\frac{d}{c\omega}\cdot\frac{1-\sigma'}{\sigma'}\epsilon^{-at_0},\quad E=\frac{-d}{c\omega}\cdot\frac{1-\sigma'}{\sigma'}\epsilon^{\mu t_0}\sin\beta,\quad F=\frac{-d}{c\omega}\cdot\frac{1-\sigma'}{\sigma'}\epsilon^{\mu t_0}\cos\beta$$

accordingly the expressions of the sudden short-circuit currents are

$$x_s = \frac{-d}{b_i\omega}\left[\cos\omega t + \frac{1-\sigma'}{\sigma'}\epsilon^{\alpha(t-t_0)}\cos\omega t - \frac{1-\sigma'}{2\sigma'}\epsilon^{\mu(t-t_0)}\cos(2\omega t - \omega t_0) - \frac{1+\sigma'}{2\sigma'}\epsilon^{\mu(t-t_0)}\cos\omega t_0\right]$$

$$y_s = \frac{-d}{b_i\omega}\left[\cos\left(\omega t - \frac{2\pi}{3}\right) + \frac{1-\sigma'}{\sigma'}\epsilon^{\alpha(t-t_0)}\cos\left(\omega t - \frac{2\pi}{3}\right) - \frac{1-\sigma'}{2\sigma'}\epsilon^{\mu(t-t_0)}\cos\left(2\omega t - \frac{2\pi}{3} - \omega t_0\right) - \frac{1+\sigma'}{2\sigma'}\epsilon^{\mu(t-t_0)}\cos\left(\omega t_0 - \frac{2\pi}{3}\right)\right]$$

$$z_s = \frac{-d}{b_i\omega}\left[\cos\left(\omega t + \frac{2\pi}{3}\right) + \frac{1-\sigma'}{\sigma'}\epsilon^{\alpha(t-t_0)}\cos\left(\omega t + \frac{2\pi}{3}\right) - \frac{1-\sigma'}{2\sigma'}\epsilon^{\mu(t-t_0)}\cos\left(2\omega t + \frac{2\pi}{3} - \omega t_0\right) - \frac{1+\sigma'}{2\sigma'}\epsilon^{\mu(t-t_0)}\cos\left(\omega t_0 + \frac{2\pi}{3}\right)\right]$$

$$w_s = \frac{1-\sigma'}{\sigma'} I_f\left[\epsilon^{\alpha(t-t_0)} - \epsilon^{\mu(t-t_0)}\cos(\omega t - \omega t_0)\right]$$

Now to find the maximum values of these sudden short-circuit currents, put $\frac{\alpha}{\omega} \doteqdot \frac{\mu}{\omega} \doteqdot 0$. Then, denoting

$\frac{-d}{b_i\omega\sigma'}$ by k, $\frac{1-\sigma'}{\sigma'}I_f$ by k', $\frac{1}{2}(1-\sigma')$ by M, $\frac{1}{2}(1+\sigma')$ by N,

ωt by α and ωt_0 by β, we have

$$x_s = k[\cos\alpha - M\cos(2\alpha-\beta) - N\cos\beta]$$

$$y_s = k\left[\cos\left(\alpha - \frac{2\pi}{3}\right) - M\cos\left(2\alpha - \frac{2\pi}{3} - \beta\right) - N\cos\left(\beta - \frac{2\pi}{3}\right)\right]$$

$$= k\left\{\cos\left(\alpha - \frac{2\pi}{3}\right) - M\cos\left[2\left(\alpha - \frac{2\pi}{3}\right) - \left(\beta - \frac{2\pi}{3}\right)\right] - N\cos\left(\beta - \frac{2\pi}{3}\right)\right\}$$

$$z_s = k\left[\cos\left(\alpha+\frac{2\pi}{3}\right) - M\cos\left(2\alpha+\frac{2\pi}{3}-\beta\right) - N\cos\left(\beta+\frac{2\pi}{3}\right)\right]$$

$$= k\left\{\cos\left(\alpha+\frac{2\pi}{3}\right) - M\cos\left[2\left(\alpha+\frac{2\pi}{3}\right)-\left(\beta+\frac{2\pi}{3}\right)\right] - N\cos\left(\beta+\frac{2\pi}{3}\right)\right\}$$

$$w_s = k'\left[1-\cos(\alpha-\beta)\right]$$

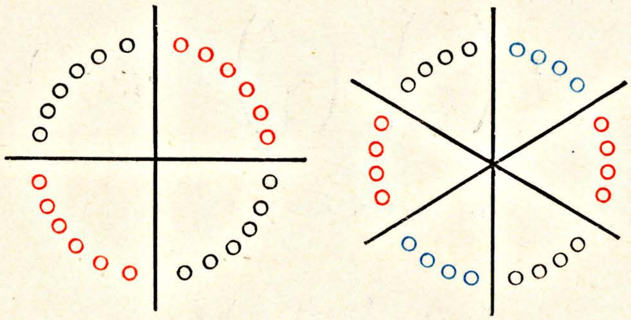


Fig. 6.

Therefore, as explained in Art. 4, $(x_s)_{\text{max.}}$ takes place at $\beta=0$ or π and $\alpha-\beta=\pi$ and $(x_s)_{\text{max.}}$ is $=\dfrac{\pm 2d}{b_i\omega\sigma'}$.

Similarly $(y_s)_{\text{max.}}$ takes place at $\beta-\dfrac{2\pi}{3}=0$ or π and $\alpha-\beta=\pi$ and $(y_s)_{\text{max.}}$ is $=(x_s)_{\text{max.}}$

$(z_s)_{\text{max.}}$ takes place at $\beta+\dfrac{2\pi}{3}=0$ or π and $\alpha-\beta=\pi$ and $(z_s)_{\text{max.}}$ is $=(x_s)_{\text{max.}}$

$(w_s)_{\text{max.}}$ is, as said in Art. 4, equal to $2k'$ that is $2\dfrac{1-\sigma'}{\sigma'}I_f$ and $(w_s+I_f)_{\text{max.}}=I_f\left(\dfrac{2}{\sigma}-1\right)$. It takes place at $\alpha-\beta=\pi$.

Note that the shapes of the field current and the approximate maximum sudden short-circuit currents in the armature are the same as given in Chapter I.

Now, for comparing the maximum values of the permanent and sudden short-circuit currents in the two and three phase generators as shown in fig. 6, let L_{i2} and L_{i3} be the total internal inductance of each phase of the two and three phase windings respectively. Then since the numbers of turns of the two and three phase armatures per phase are in the ratio 2:3, the ratio L_{i3} to L_{i2} is, referring to Art. 5,

$$=\left(\frac{2}{3}\right)^2 \nu_3\left(\frac{\sin\frac{\pi}{6}}{\frac{\pi}{6}}\right)\Bigg/\nu_2\left(\frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}}\right)=\left(\frac{2}{3}\right)^2\cdot\frac{\nu_3}{\nu_2}\cdot\left(\frac{3}{2\sqrt{2}}\right)$$

But
$$b_{i2}=L_{i2} \quad\text{and}\quad b_{i3}=\left(1+\frac{1}{2\nu_3}\right)L_{i3}$$

Therefore
$$\frac{b_{i3}}{b_{i2}}=\left(1+\frac{1}{2\nu_3}\right)\left(\frac{2}{3}\right)^2\cdot\left(\frac{\nu_3}{\nu_2}\right)\cdot\left(\frac{3}{2\sqrt{2}}\right)$$

But the ratio of the induced E.M.F. in each phase of the two and three phase armatures is

$$\frac{E_3}{E_2}\doteqdot\left(\frac{2}{3}\right)\left(\frac{3}{2\sqrt{2}}\right)$$

which is exactly so when the winding is in fine distribution. Therefore the ratio of the magnitudes of the permanent short-circuit currents in the two and three phase armatures is

$$\frac{X_{s3}}{X_{s2}}=\frac{\left(\frac{2}{3}\right)\left(\frac{3}{2\sqrt{2}}\right)}{\left(1+\frac{1}{2\nu_3}\right)\left(\frac{2}{3}\right)^2\left(\frac{\nu_3}{\nu_2}\right)\left(\frac{3}{2\sqrt{2}}\right)}=\frac{3}{2}\cdot\frac{1}{1+\frac{1}{2\nu_3}}\cdot\frac{\nu_2}{\nu_3}$$

which is roughly $=1$ for $1+\frac{1}{2\nu_3}\doteqdot\frac{3}{2}$ and $\frac{\nu_3}{\nu_2}\doteqdot 1$.

Also since $\frac{2}{3}+\frac{1}{3\nu_3}\doteqdot 1$ we have, referring to Art. 5, $\sigma'\doteqdot\sigma$ and hence the maximum sudden short-circuit currents of the two and three phase armatures are roughly equal in magnitude.

§ 8. Ordinary treatment of the two and three phase generators.

In the ordinary treatment of the three phase generator we are used to consider the armature M.M.F., of which the amount is $\frac{3}{2}nI\sqrt{2}$, where n is the number of turns and I the effective current of each phase, acting in the direction as shown in figure 7 (ϕ being the phase angle between the current I and the induced E.M.F. due to the original field N—S). Therefore the E.M.F. induced in each phase of the armature due to the reactive M.M.F. will be

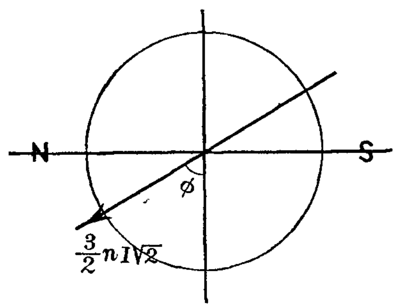


Fig. 7.

$$= -knk\frac{\frac{3}{2}nI\sqrt{2}}{\mathcal{R}}\cdot\frac{d}{dt}[\sin(\omega t-\phi)]$$

$$= -\frac{3}{2}k^2\frac{n^2}{\mathcal{R}}\cdot\frac{d}{dt}[I\sqrt{2}\sin(\omega t-\phi)]$$

$$= -\frac{3}{2}\cdot\frac{b_i'}{\nu}\cdot\frac{di}{dt}$$

where k is a factor due to the distribution of the winding, $\mathcal{R}$ the reluctance of the equivalent magnetic circuit for the sine flux produced by the armature current, and b_i' and ν are as before.

Thus the fundamental equation in the ordinary treatment of the three phase generator will be

$$(a_e+a_i)\,i+(b_e+b_l')\frac{di}{dt}=d\sin\omega t-\frac{3}{2}\cdot\frac{1}{\nu}b_i'\frac{di}{dt}$$

where a_e and b_e are the load resistance and inductance and b_l' the leakage inductance of the armature winding of each phase.

That is

$$(a_e+a_i)\,i+\left(b_e+b_l'+\frac{3}{2}\cdot\frac{1}{\nu}b_i'\right)\frac{di}{dt}=d\sin\omega t$$

which solves to

$$i = \frac{d}{\sqrt{(a_e + a_i)^2 + \left(b_e + b_l' + \frac{3}{2} \cdot \frac{1}{\nu} b_i'\right)^2 \omega^2}} \sin \left(\omega t - \tan^{-1} \frac{\left(b_e + b_l' + \frac{3}{2} \cdot \frac{1}{\nu} b_i'\right) \omega}{a_e + a_i} \right)$$

But $\frac{b_i' - b_l'}{b_i'} = \frac{1}{\nu}$ that is $b_l' = \left(1 - \frac{1}{\nu}\right) b_i'$

so that

$$b_l' + \frac{3}{2} \cdot \frac{1}{\nu} b_i' = b_i' \left(1 - \frac{1}{\nu} + \frac{3}{2} \cdot \frac{1}{\nu}\right) = \left(1 + \frac{1}{2\nu}\right) b_i'$$

and hence

$$i = \frac{d}{\sqrt{(a_e + a_i)^2 + \left[b_e + \left(1 + \frac{1}{2\nu}\right) b_i'\right]^2 \omega^2}} \sin \left(\omega t - \tan^{-1} \frac{\left[b_e + \left(1 + \frac{1}{2\nu}\right) b_i'\right] \omega}{a_e + a_i} \right)$$

which coincides with the result obtained before by the complete analytical solution.

At short circuit, that is when $a_e = b_e = 0$ and a_i is negligible compared with $\left(1 + \frac{1}{2\nu}\right) b_i'$, i.e. b_i, we have

$$i_s = \frac{-d}{\left(1 + \frac{1}{2\nu}\right) b_i' \omega} \cos \omega t = \frac{-d}{b_i \omega} \cos \omega t$$

As to the maximum sudden short-circuit current we have, according to Mr Berg,

$$(i)_{\text{sudden max.}} \doteqdot \frac{\pm 2d}{b_l' \omega}$$

while the complete analytical solution gives

$$(i)_{\text{sudden max.}} = \frac{2d}{b_i \omega} \cdot \frac{1}{\sigma'} \doteqdot \frac{2d}{b_i \omega \sigma} \doteqdot \frac{2d}{b_i \omega (\tau + \tau_f)} \; *$$

where τ and τ_f are the ratios of the leakage flux to the difference of the total and the leakage flux produced by the armature and the field current respectively.

* See the theory of distortionless alternator, Art. 22.

In the case of the two phase generator we have, from ordinary treatment, the amplitude of the permanent short-circuit current

$$= \frac{d}{\sqrt{(a_e + a_i)^2 + \left(b_e + b_l + \frac{1}{\nu} b_i\right)^2 \omega^2}} = \frac{d}{\sqrt{(a_e + a_i)^2 + (b_e + b_i)^2 \omega^2}}$$

which coincides with the result from the complete analytical solution. Similarly as in the three phase generator, maximum value of the sudden short-circuit current in the two phase armature, according to Mr Berg, deviates very much from that obtained by the complete analytical solution.

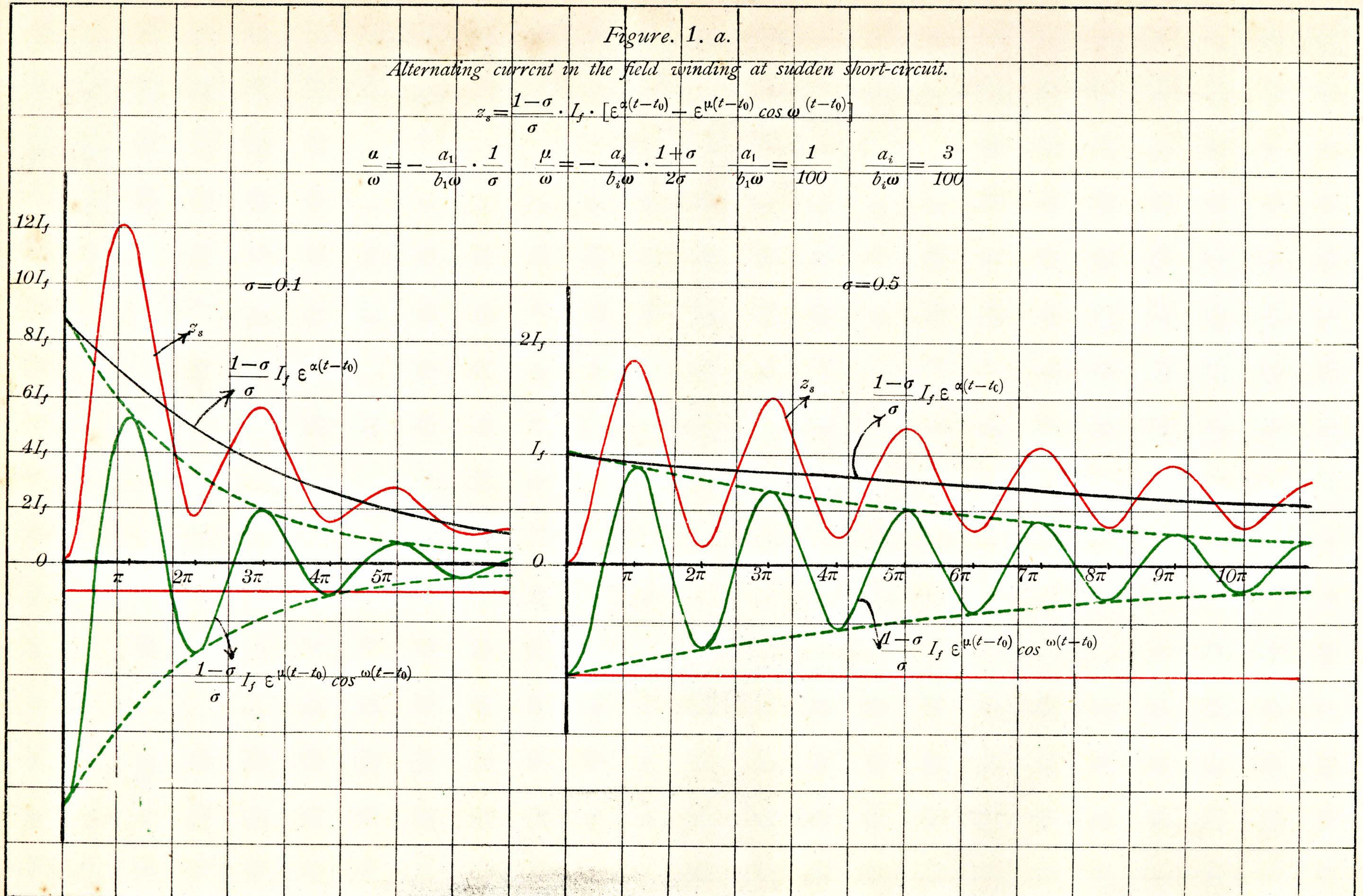


Figure. 1. a.

Alternating current in the field winding at sudden short-circuit.

$$z_s = \frac{1-\sigma}{\sigma} \cdot I_f \cdot \left[\varepsilon^{\alpha(t-t_0)} - \varepsilon^{\mu(t-t_0)} \cos\omega^{(t-t_0)}\right]$$

$$\frac{\alpha}{\omega} = -\frac{a_1}{b_1\omega} \cdot \frac{1}{\sigma} \qquad \frac{\mu}{\omega} = -\frac{a_i}{b_i\omega} \cdot \frac{1+\sigma}{2\sigma} \qquad \frac{a_1}{b_1\omega} = \frac{1}{100} \qquad \frac{a_i}{b_i\omega} = \frac{3}{100}$$

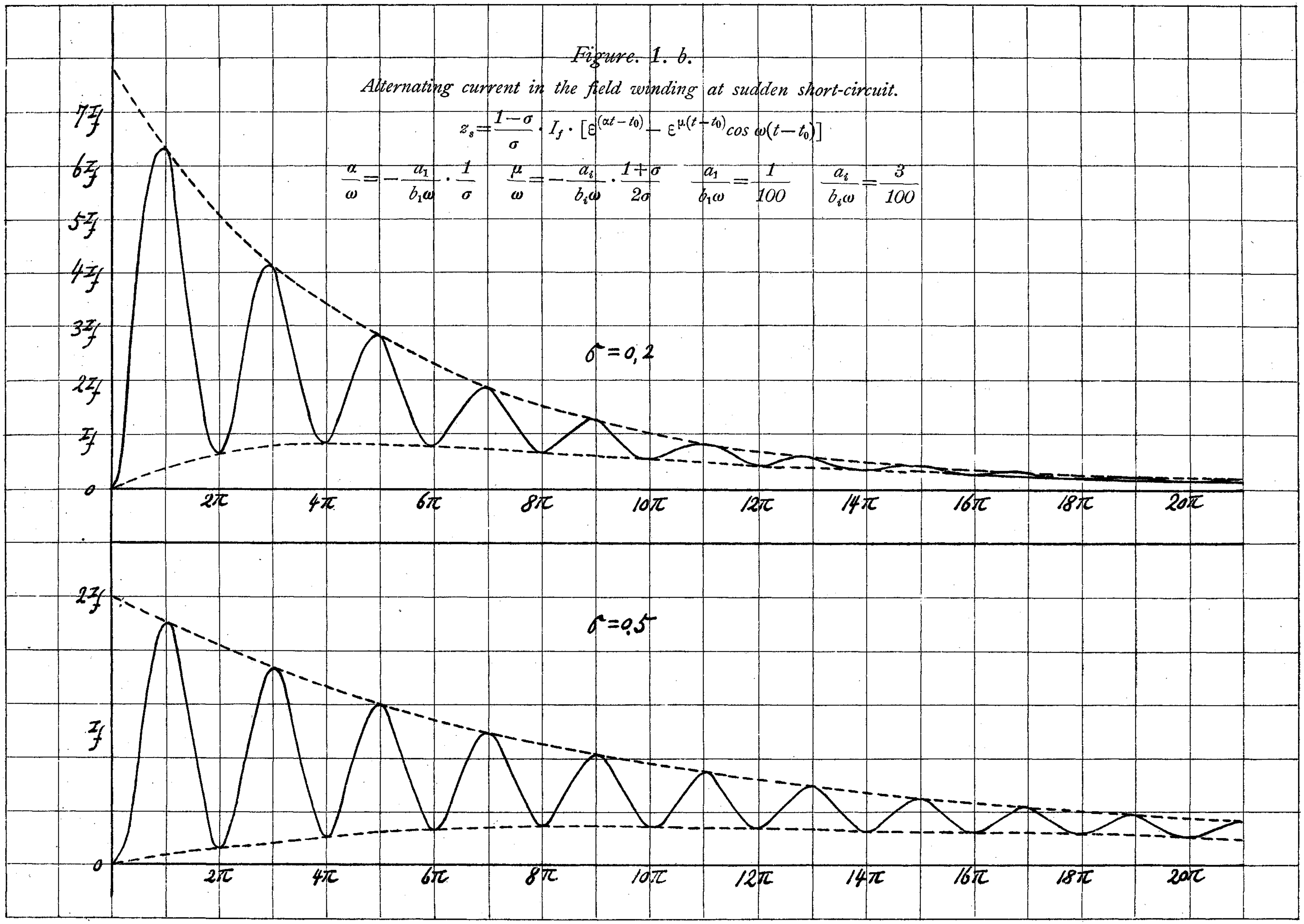


Figure. 1. b.

Alternating current in the field winding at sudden short-circuit.

$$z_s = \frac{1-\sigma}{\sigma} \cdot I_f \cdot \left[\varepsilon^{(\alpha t - t_0)} - \varepsilon^{\mu(t-t_0)} \cos \omega (t-t_0)\right]$$

$$\frac{\alpha}{\omega} = -\frac{a_1}{b_1\omega} \cdot \frac{1}{\sigma} \qquad \frac{\mu}{\omega} = -\frac{a_i}{b_i\omega} \cdot \frac{1+\sigma}{2\sigma} \qquad \frac{a_1}{b_1\omega} = \frac{1}{100} \qquad \frac{a_i}{b_i\omega} = \frac{3}{100}$$

Figure. 2. a.

$$x_{max}=\frac{-d}{b_i\omega\sigma}\cdot\left(\cos\omega t-\frac{1-\sigma}{2}\cos 2\,\omega t-\frac{1+\sigma}{2}\right)$$

$\sigma=0.5$

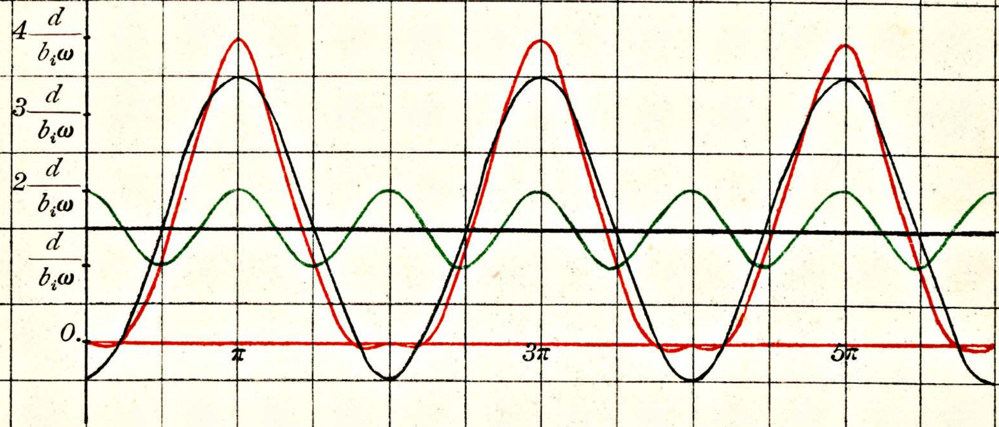
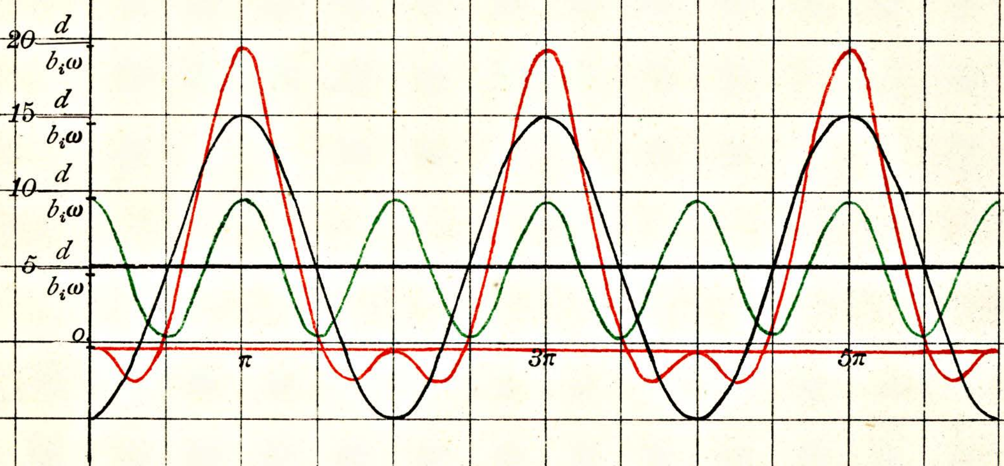


Figure. 2. b.

The same as above,

but with $\sigma=0.1$



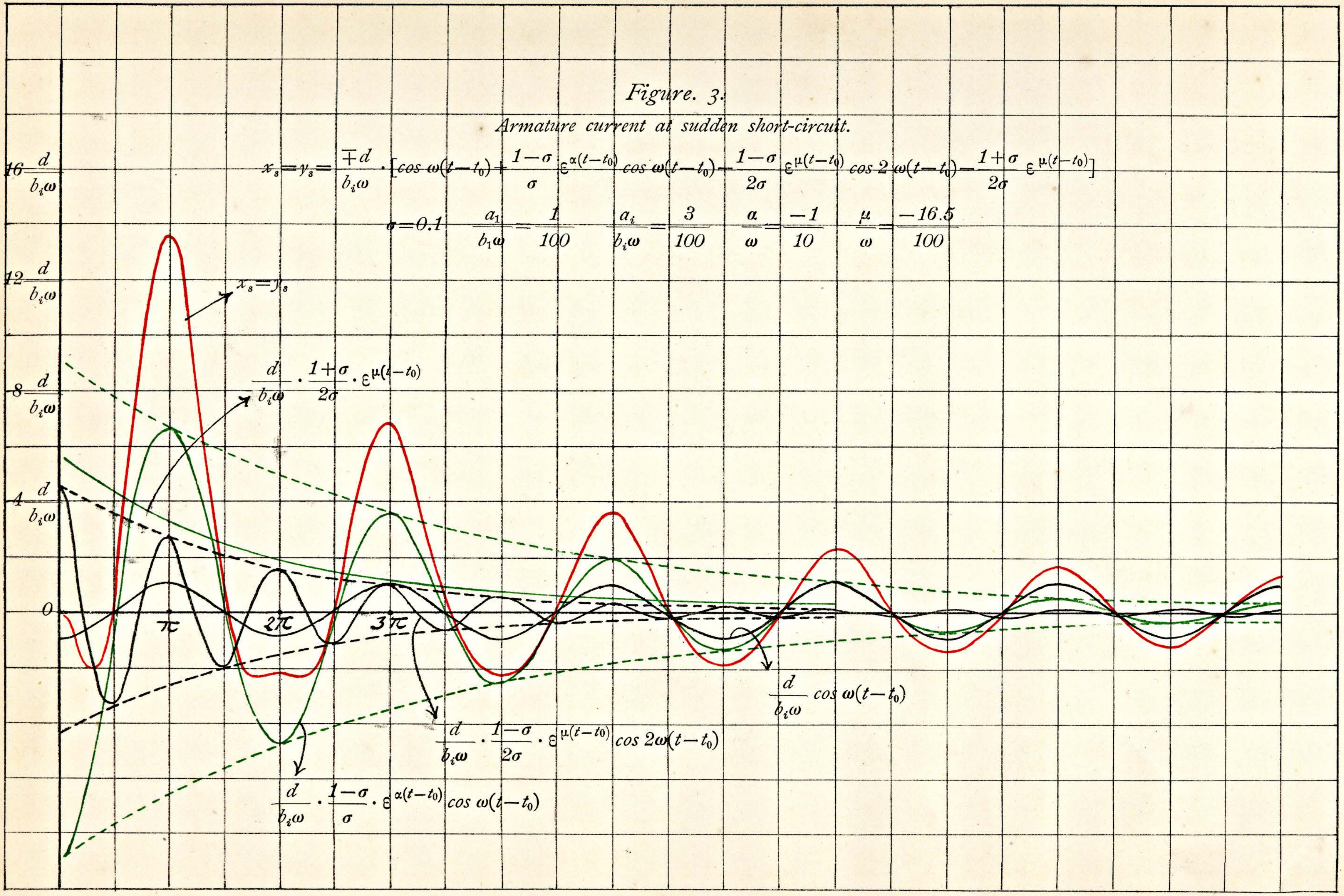


Figure. 3.

Armature current at sudden short-circuit.

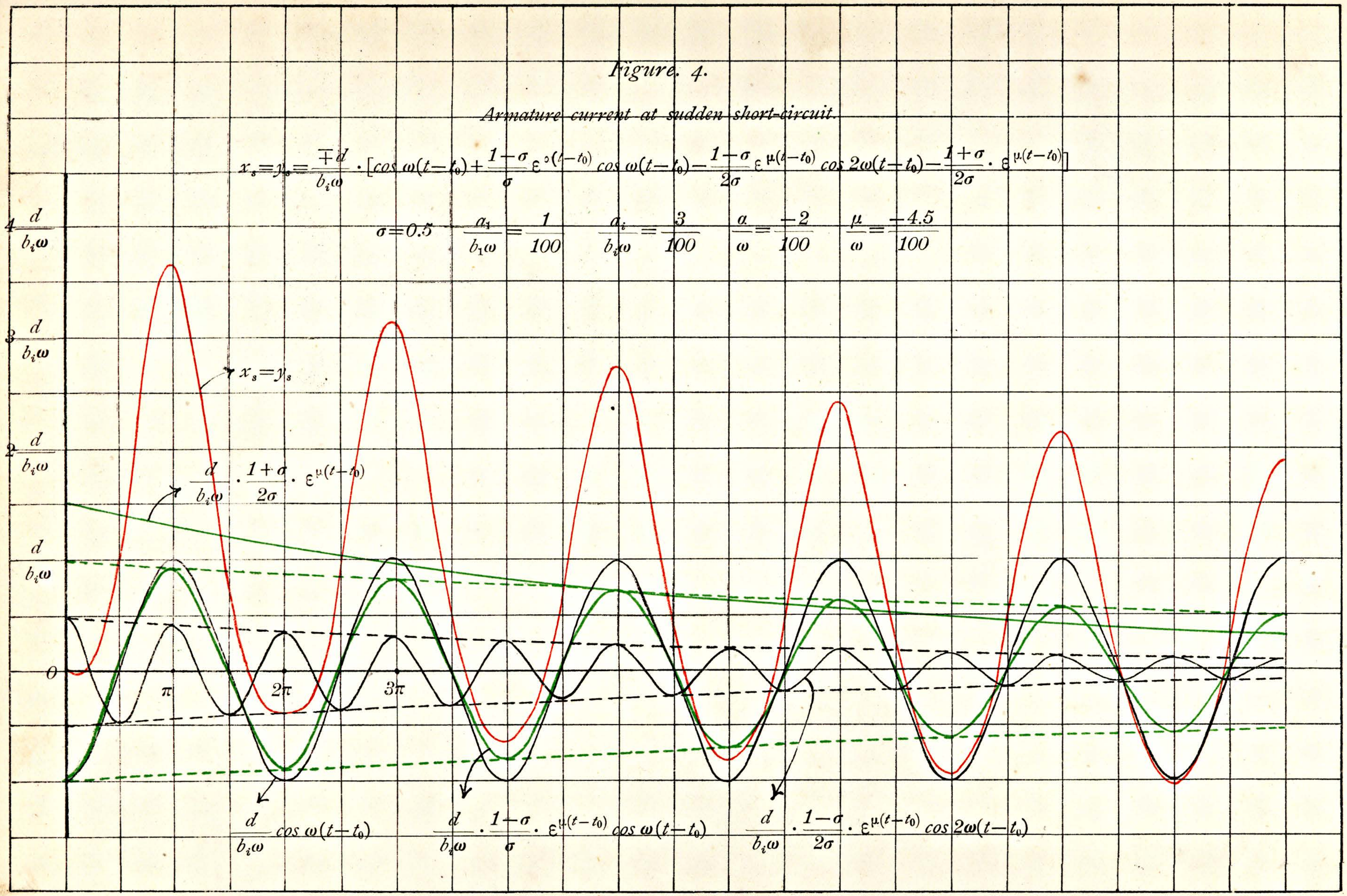


Figure. 4.

Armature current at sudden short-circuit.