Theory of Distortionless Alternators

AUTHOR(S):

Shimidzu, Giichi

CITATION:

Shimidzu, Giichi. Theory of Distortionless Alternators. Memoirs of the College of Engineering, Kyoto Imperial University 1925, 2(7): 181-278

ISSUE DATE:

1925-01-30

URL:

http://hdl.handle.net/2433/280041

Theory of Distortionless Alternators.

By

Giichi Shimidzu.

(received, December 20, 1921)

Contents.

Chapter I. Permanent Phenomena.

Chapter II. Transient Phenomena.

Chapter I.

Permanent Phenomena.

§ 1. No distortion of wave form in the alternator with two field windings.

The following investigation is started on the idea that distortion of wave form in the ordinary alternator due to armature reaction arises from the current induced in the field winding.

It seems possible to damp out the distortion by providing another field winding* displaced magnetically by 90 degrees from the ordinary one. Here for simplicity therefore will be considered an alternator with two field windings, identical in construction and placed magnetically crosswise to each other, and it will be proved that in this alternator, the poles being non-salient and laminated and both the field and armature windings being so arranged that the distribution of the magnetic flux circumferentially along the air gap is sinusoidal, there will be no distortion of wave form under load.

The ordinary field winding will be called the direct field winding and the other the cross field winding; and for simplicity the field iron will be considered as finely laminated.

Let $\mathcal{R}$ be the reluctance of the equivalent passage of the total flux produced by unit armature current, n the number of turns of the armature winding and

$$i = I_{2m+1}\sqrt{2}\sin(\overline{2m+1}\omega t - \phi_{2m+1})$$

the current flowing in the armature, the origin of time being that instant when the armature coil is situated at the neutral point with regard to the direct field winding.

---

* Boucherot.—Atti del Congresso Internazionale delle Applicazioni Elettriche. Volume 2. (1911.)

Wechselstromtechnik by E. Arnold. Bd. IV.

Transactions A. I. E. E. 1911. p. 170 and G. E. R. 1920. p. 177.

Then calling the flux in the direction of the axis of the direct field winding the direct flux and that magnetically crosswise to it the cross flux, the total direct flux produced by the armature current is, referring to figure 1,

$$= \frac{1}{\mathcal{R}} . kn I_{2m+1} \sqrt{2} \times \sin(\overline{2m+1}\omega t - \phi_{2m+1}) \cos \omega t$$

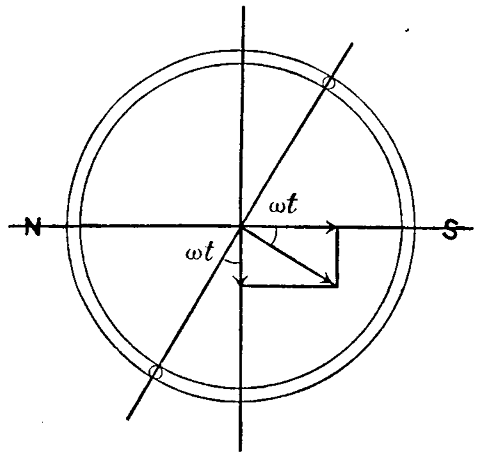


Fig. 1.

where k is a factor due to the distribution of the armature conductors; and the total cross flux is

$$= \frac{1}{\mathcal{R}} . kn I_{2m+1} \sqrt{2} \sin(\overline{2m+1}\omega t - \phi_{2m+1}) \sin \omega t$$

Therefore the direct flux produced by the armature current and linking with the armature winding itself is

$$= \frac{k^2 n^2}{\mathcal{R}} . I_{2m+1} \sqrt{2} \sin(\overline{2m+1}\omega t - \phi_{2m+1}) \cos^2 \omega t$$

$$= L . I_{2m+1} \sqrt{2} \sin(\overline{2m+1}\omega t - \phi_{2m+1}) \cos^2 \omega t$$

where L is the total flux linkage of the armature winding itself when unit current flows in it, that is, the total inductance of the armature winding.

Similarly the cross flux produced by the armature current and linking with the armature itself is

$$= L . I_{2m+1} \sqrt{2} \sin(\overline{2m+1}\omega t - \phi_{2m+1}) \sin^2 \omega t$$

Therefore if the field winding be open, then the E.M.F. induced in the armature due to the current in itself is

$$= - L . I_{2m+1} \sqrt{2} \frac{d}{dt} . [\sin(\overline{2m+1}\omega t - \phi_{2m+1})]$$

$$= - L . \frac{di}{dt}$$

which is a result that can be expected because, except that there is an air gap, there is no difference between our armature and a coil rotating together with all iron in connection with it.

Next, considering the flux produced by the armature current and linking with the direct field winding, it is

$$= \frac{k_f n_f}{\nu} \cdot \frac{kn}{\mathcal{R}} I_{2m+1} \sqrt{2} \sin (\overline{2m+1}\omega t - \phi_{2m+1}) \cos \omega t$$

$$= \frac{1}{2} M I_{2m+1} \sqrt{2} \,.\, [\sin (\overline{2m+2}\omega t - \phi_{2m+1}) + \sin (2m\omega t - \phi_{2m+1})]$$

where ν is the armature leakage coefficient, n_f the number of turns of the field winding, k_f a factor due to the distribution of the field winding and M is the maximum mutual inductance between the field and the armature winding.

Therefore the E.M.F. induced in the direct field winding is

$$= -\frac{1}{2} M I_{2m+1} \sqrt{2} \,.\, [(2m+2)\,\omega \cos (\overline{2m+2}\omega t - \phi_{2m+1}) + 2m\omega \cos (2m\omega t - \phi_{2m+1})]$$

and hence the current induced in the direct field winding is

$$= -\frac{1}{2} M I_{2m+1} \sqrt{2} \,.\, \left[\frac{(2m+2)\,\omega}{(z_f)_{2m+2}} \cos (\overline{2m+2}\omega t - \phi_{2m+1} - \theta_{2m+2}) + \frac{2m\omega}{(z_f)_{2m}} \cos (2m\omega t - \phi_{2m+1} - \theta_{2m}) \right]$$

where $(z_f)_{2m+2}$ and $(z_f)_{2m}$ are the impedances of the field winding at frequencies of $2m+2$ and $2m$ times the fundamental, and θ_{2m+2} and θ_{2m} are the corresponding phase angles.

Similarly the flux produced by the armature current and linking with the cross field winding is

$$= \frac{1}{2} M I_{2m+1} \sqrt{2} \,.\, [\cos (2m\omega t - \phi_{2m+1}) - \cos (\overline{2m+2}\omega t - \phi_{2m+1})]$$

so that the current induced in the cross field winding by the armature current i is

$$= \frac{1}{2} M I_{2m+1} \sqrt{2} \,.\, \left[\frac{2m\omega}{(z_f)_{2m}} \sin (2m\omega t - \phi_{2m+1} - \theta_{2m}) - \frac{(2m+2)\,\omega}{(z_f)_{2m+2}} \sin (\overline{2m+2}\omega t - \phi_{2m+1} - \theta_{2m+2}) \right]$$

Now considering the flux produced by these currents induced in the direct and cross field windings and linking with the armature coil, it is

$$= -\frac{kn}{\nu_f}\cdot\frac{k_f n_f}{\mathcal{R}_f}\cdot\frac{1}{2}MI_{2m+1}\sqrt{2}\,.\left[\frac{(2m+2)\,\omega}{(z_f)_{2m+2}}\cos\left(\overline{2m+2}\omega t-\phi_{2m+1}-\theta_{2m+2}\right)\right.$$
$$\left.+\frac{2m\omega}{(z_f)_{2m}}\cos\left(2m\omega t-\phi_{2m+1}-\theta_{2m}\right)\right]\cos\omega t$$
$$+\frac{kn}{\nu_f}\cdot\frac{k_f n_f}{\mathcal{R}_f}\cdot\frac{1}{2}MI_{2m+1}\sqrt{2}\,.\left[\frac{2m\omega}{(z_f)_{2m}}\sin\left(2m\omega t-\phi_{2m+1}-\theta_{2m}\right)\right.$$
$$\left.-\frac{(2m+2)\,\omega}{(z_f)_{2m+2}}\sin\left(\overline{2m+2}\omega t-\phi_{2m+1}-\theta_{2m+2}\right)\right]\sin\omega t$$
$$= -\frac{1}{2}M^2 I_{2m+1}\sqrt{2}\,.\left[\frac{(2m+2)\,\omega}{(z_f)_{2m+2}}\cos\left(\overline{2m+1}\omega t-\phi_{2m+1}-\theta_{2m+2}\right)\right.$$
$$\left.+\frac{2m\omega}{(z_f)_{2m}}\cos\left(\overline{2m+1}\omega t-\phi_{2m+1}-\theta_{2m}\right)\right]$$

where ν_f is the field leakage coefficient and $\mathcal{R}_f$ the reluctance of the equivalent passage of the total flux produced by the field current, direct or cross, so that

$$\frac{kn}{\nu_f}\cdot\frac{k_f n_f}{\mathcal{R}_f}$$

is also equal to M.

Hence the E.M.F. induced in the armature due to the currents induced in the direct and cross field windings is

$$= -\frac{1}{2}M^2 I_{2m+1}\sqrt{2}\,(2m+1)\,\omega\,.\left[\frac{(2m+2)\,\omega}{(z_f)_{2m+2}}\sin\left(\overline{2m+1}\omega t-\phi_{2m+1}-\theta_{2m+2}\right)\right.$$
$$\left.+\frac{2m\omega}{(z_f)_{2m}}\sin\left(\overline{2m+1}\omega t-\phi_{2m+1}-\theta_{2m}\right)\right]$$

which contains no other harmonics than that of the armature current itself.

We can therefore conclude that in our alternator with direct and cross field windings there is no distortion of wave form due to armature reaction; and hence we shall hereafter call our alternator with the direct and cross field windings the distortionless alternator with two field windings.

Here note that in this distortionless alternator it is not necessary that the two field windings be short-circuited.

§ 2. Field and armature currents of the distortionless alternator with two field windings.

Considering for simplicity the case when only the direct field winding is excited (case when both field windings are excited will be treated later), the

E.M.F. induced in the armature due to the exciting current I_f is

$$= -\frac{d}{dt}(MI_f \cos \omega t) = MI_f\omega \sin \omega t$$

so that if $z = \sqrt{r^2 + L^2\omega^2}$ be the total impedance of the armature circuit with fundamental frequency, then in the armature circuit there will flow a current i_a' which can be expressed by

$$i_a' = I_a' \sqrt{2} \sin (\omega t - \phi_a)$$

where

$$I_a' \sqrt{2} = \frac{MI_f\omega}{z} \text{ and } \phi_a = \tan^{-1} \frac{L\omega}{r}$$

Now on account of this current i_a' in the armature there will be a current i_{fd}' induced in the direct field winding which, as shown in the previous article, can be expressed by

$$i_{fd}' = -\frac{1}{2} MI_a' \sqrt{2} \frac{2\omega}{z_f} \cos (2\omega t - \phi_a - \phi_f)$$

where $z_f = \sqrt{r_f^2 + 4L_f^2\omega^2}$ and $\phi_f = \tan^{-1} \frac{L_f 2\omega}{r_f}$, in which r_f and L_f are the resistance and inductance of the field circuit.

Similarly the current i_{fc}' induced in the cross field winding will be

$$i_{fc}' = -\frac{1}{2} MI_a' \sqrt{2} \frac{2\omega}{z_f} \sin (2\omega t - \phi_a - \phi_f)$$

Now in turn these two field currents will induce a current i_a'' in the armature circuit, which, referring to the previous article, can be expressed by

$$i_a'' = -\frac{1}{2} M^2 I_a' \sqrt{2} \cdot \frac{\omega}{z} \cdot \frac{2\omega}{z_f} \sin (\omega t - 2\phi_a - \phi_f)$$

This i_a'' will now induce a current i_{fd}'' and i_{fc}'' in the direct and cross field windings respectively, where

$$i_{fd}'' = \left(\frac{1}{2}\right)^2 M^3 I_a' \sqrt{2} \cdot \frac{\omega}{z} \cdot \left(\frac{2\omega}{z_f}\right)^2 \cos (2\omega t - 2\phi_a - 2\phi_f)$$

and

$$i_{fc}'' = \left(\frac{1}{2}\right)^2 M^3 I_a' \sqrt{2} \cdot \frac{\omega}{z} \cdot \left(\frac{2\omega}{z_f}\right)^2 \sin (2\omega t - 2\phi_a - 2\phi_f)$$

and now in turn, these field currents will induce in the armature circuit a current i_a''' of which the expression is

$$i_a''' = \left(\frac{1}{2}\right)^2 M^4 I_a' \sqrt{2} \, . \left(\frac{\omega}{z}\right)^2 \left(\frac{2\omega}{z_f}\right)^2 \sin(\omega t - 3\phi_a - 2\phi_f)$$

and so on.

Thus the total current in the armature circuit is

$$\begin{aligned} i_a = I_a' \sqrt{2} \, . \Bigg[& \sin(\omega t - \phi_a) - \frac{1}{2} M^2 \frac{\omega}{z} \, . \frac{2\omega}{z_f} \sin(\omega t - 2\phi_a - \phi_f) \\ & + \left(\frac{1}{2}\right)^2 M^4 \left(\frac{\omega}{z}\right)^2 \left(\frac{2\omega}{z_f}\right)^2 \sin(\omega t - 3\phi_a - 2\phi_f) \\ & - \left(\frac{1}{2}\right)^3 M^6 \left(\frac{\omega}{z}\right)^3 \left(\frac{2\omega}{z_f}\right)^3 \sin(\omega t - 4\phi_a - 3\phi_f) + \text{etc.} \Bigg] \end{aligned}$$

$$\begin{aligned} &= I_a' \sqrt{2} \, . [\cos\phi_a + c\cos(\phi_a + \theta) + c^2 \cos(\phi_a + 2\theta) + \ldots] \sin\omega t \\ &\quad - I_a' \sqrt{2} \, . [\sin\phi_a + c\sin(\phi_a + \theta) + c^2 \sin(\phi_a + 2\theta) + \ldots] \cos\omega t \end{aligned}$$

where $$c = -\frac{1}{2} M^2 \frac{\omega}{z} \, . \frac{2\omega}{z_f} < \frac{1}{2} \frac{M^2}{LL_f} = \frac{1}{2} \, . \frac{1}{\nu\nu_f} < 1$$

and $$\theta = \phi_a + \phi_f$$

The total alternating current i_{fd} in the direct field winding is

$$\begin{aligned} i_{fd} = - I_a' \sqrt{2} \, . \frac{1}{2} M \frac{2\omega}{z_f} \, . [& \cos(2\omega t - \phi_a - \phi_f) + c\cos(2\omega t - 2\phi_a - 2\phi_f) \\ & + c^2 \cos(2\omega t - 3\phi_a - 3\phi_f) \\ & + \text{etc.}] \end{aligned}$$

$$\begin{aligned} &= - I_a' \sqrt{2} \frac{M\omega}{z_f} (\cos\theta + c\cos 2\theta + c^2 \cos 3\theta + \ldots) \cos 2\omega t \\ &\quad - I_a' \sqrt{2} \frac{M\omega}{z_f} (\sin\theta + c\sin 2\theta + c^2 \sin 3\theta + \ldots) \sin 2\omega t \end{aligned}$$

and similarly the total current in the cross field winding is

$$\begin{aligned} i_{fc} &= - I_a' \sqrt{2} \frac{M\omega}{z_f} (\cos\theta + c\cos 2\theta + c^2 \cos 3\theta + \ldots) \sin 2\omega t \\ &\quad + I_a' \sqrt{2} \frac{M\omega}{z_f} (\sin\theta + c\sin 2\theta + c^2 \sin 3\theta + \ldots) \cos 2\omega t \end{aligned}$$

that is

$$\begin{aligned} i_a &= I_a' \sqrt{2} \, . \left(\frac{\cos\phi_a - c\cos\phi_f}{1 - 2c\cos\theta + c^2} \sin\omega t - \frac{\sin\phi_a + c\sin\phi_f}{1 - 2c\cos\theta + c^2} \cos\omega t \right) \\ &= \frac{I_a' \sqrt{2}}{\sqrt{1 - 2c\cos\theta + c^2}} \sin\left(\omega t - \tan^{-1} \frac{\sin\phi_a + c\sin\phi_f}{\cos\phi_a - c\cos\phi_f}\right) \end{aligned}$$

$$i_{fd} = -I_a'\sqrt{2}\,.\,\frac{M\omega}{z_f}\left(\frac{\cos\theta - c}{1 - 2c\cos\theta + c^2}\cos 2\omega t + \frac{\sin\theta}{1 - 2c\cos\theta + c^2}\sin 2\omega t\right)$$

$$= \frac{-I_a'\sqrt{2}}{\sqrt{1 - 2c\cos\theta + c^2}}\,.\,\frac{M\omega}{z_f}\cos\left(2\omega t - \tan^{-1}\frac{\sin\theta}{\cos\theta - c}\right)$$

and similarly

$$i_{fc} = \frac{-I_a'\sqrt{2}}{\sqrt{1 - 2c\cos\theta + c^2}}\,.\,\frac{M\omega}{z_f}\sin\left(2\omega t - \tan^{-1}\frac{\sin\theta}{\cos\theta - c}\right)$$

But

$$1 - 2c\cos\theta + c^2 = \left(\frac{1}{zz_f}\right)^2.\left[z^2 z_f^2 + 2M^2\omega^2(rr_f - 2LL_f\omega^2) + M^4\omega^4\right]$$

$$= \left(\frac{1}{zz_f}\right)^2.\left\{[rr_f - (2LL_f - M^2)\,\omega^2]^2 + (Lr_f + 2L_f r)^2\omega^2\right\}$$

$$= \left(\frac{1}{z}\right)^2.\left[(r + \rho^2 r_f)^2 + (L - 2\rho^2 L_f)^2\omega^2\right]$$

where

$$\rho = \frac{M\omega}{z_f}$$

$$\tan^{-1}\frac{\sin\phi_a + c\sin\phi_f}{\cos\phi_a - c\cos\phi_f} = \tan^{-1}\frac{L\omega z_f^2 - 2L_f M^2\omega^3}{rz_f^2 + r_f M^2\omega^2}$$

$$= \tan^{-1}\frac{(L - 2\rho^2 L_f)\,\omega}{r + \rho^2 r_f}$$

$$= \tan^{-1}\frac{(Lr_f + 2L_f r)\,\omega}{rr_f - (2LL_f - M^2)\,\omega^2} - \tan^{-1}\frac{2L_f\omega}{r_f}$$

and

$$\tan^{-1}\frac{\sin\theta}{\cos\theta - c} = \frac{(Lr_f + 2L_f r)\,\omega}{rr_f - (2LL_f - M^2)\,\omega^2}$$

$$= \tan^{-1}\frac{(L - 2\rho^2 L_f)\,\omega}{r + \rho^2 r_f} + \tan^{-1}\frac{2L_f\omega}{r_f}$$

Therefore

$$i_a = \frac{MI_f\omega z_f}{\sqrt{[rr_f - (2LL_f - M^2)\,\omega^2]^2 + (Lr_f + 2L_f r)^2\omega^2}}$$

$$\times\sin\left(\omega t - \tan^{-1}\frac{(Lr_f + 2L_f r)\,\omega}{rr_f - (2LL_f - M^2)\,\omega^2} + \tan^{-1}\frac{2L_f\omega}{r_f}\right)$$

$$= \frac{MI_f\omega}{\sqrt{(r + \rho^2 r_f)^2 + (L - 2\rho^2 L_f)^2\omega^2}}\sin\left(\omega t - \tan^{-1}\frac{(L - 2\rho^2 L_f)\,\omega}{r + \rho^2 r_f}\right)$$

$$i_{fd}=\frac{M^2I_f\omega^2}{\sqrt{[rr_f-(2LL_f-M^2)\,\omega^2]^2+(Lr_f+2L_fr)^2\,\omega^2}}$$

$$\times\sin\left(2\omega t-\frac{\pi}{2}-\tan^{-1}\frac{(Lr_f+2L_fr)\,\omega}{rr_f-(2LL_f-M^2)\,\omega^2}\right)$$

$$=\frac{MI_f\omega\rho}{\sqrt{(r+\rho^2r_f)^2+(L-2\rho^2L_f)^2\,\omega^2}}$$

$$\times\sin\left(2\omega t-\frac{\pi}{2}-\tan^{-1}\frac{(L-2\rho^2L_f)\,\omega}{r+\rho^2r_f}-\tan^{-1}\frac{2L_f\omega}{r_f}\right)$$

and

$$i_{fc}=\frac{M^2I_f\omega^2}{\sqrt{[rr_f-(2LL_f-M^2)\,\omega^2]^2+(Lr_f+2L_fr)^2\,\omega^2}}$$

$$\times\sin\left(2\omega t-\pi-\tan^{-1}\frac{(Lr_f+2L_fr)\,\omega}{rr_f-(2LL_f-M^2)\,\omega^2}\right)$$

$$=\frac{MI_f\omega\rho}{\sqrt{(r+\rho^2r_f)^2+(L-2\rho^2L_f)^2\,\omega^2}}$$

$$\times\sin\left(2\omega t-\pi-\tan^{-1}\frac{(L-2\rho^2L_f)\,\omega}{r+\rho^2r_f}-\tan^{-1}\frac{2L_f\omega}{r_f}\right)$$

Bearing in mind that currents of double the fundamental frequency flow in the field windings, note that the above expressions of i_a, i_{fd} and i_{fc} are quite the same as those of the primary and secondary currents of the air core transformer.

§ 3. Fundamental equations of the distortionless alternator with two field windings.

The direct field winding only being excited, the fundamental equations are

$$ri_a+L\frac{di_a}{dt}+M\frac{d}{dt}(i_{fd}\cos\omega t+i_{fc}\sin\omega t)=MI_f\omega\sin\omega t$$

$$r_fi_{fd}+L_f\frac{di_{fd}}{dt}+M\frac{d}{dt}(i_a\cos\omega t)=0$$

$$r_fi_{fc}+L_f\frac{di_{fc}}{dt}+M\frac{d}{dt}(i_a\sin\omega t)=0$$

so that denoting i_a by x, i_{fd} by y, i_{fc} by z, r by a, L by b, r_f by a_1, L_f by b_1, M by c and $MI_f\omega$ by d, they become

$$ax + b\frac{dx}{dt} + c\frac{d}{dt}(y\cos\omega t + z\sin\omega t) = d\sin\omega t \quad\text{.........(1)}$$

$$a_1 y + b_1\frac{dy}{dt} + c\frac{d}{dt}(x\cos\omega t) = 0 \quad\text{.........(2)}$$

$$a_1 z + b_1\frac{dz}{dt} + c\frac{d}{dt}(x\sin\omega t) = 0 \quad\text{.........(3)}$$

that is

$$a\xi + b\frac{d\xi}{dt} + c\left(\frac{d\eta}{dt}\cos\omega t + \frac{d\zeta}{dt}\sin\omega t\right) = -\frac{d}{\omega}\cos\omega t \quad\text{.........(4)}$$

$$a_1\eta + b_1\frac{d\eta}{dt} + c\frac{d\xi}{dt}\cos\omega t = 0 \quad\text{.........(5)}$$

$$a_1\zeta + b_1\frac{d\zeta}{dt} + c\frac{d\xi}{dt}\sin\omega t = 0 \quad\text{.........(6)}$$

where $\xi = \int x\,dt - \frac{1}{a}k_1 \qquad \eta = \int y\,dt - \frac{1}{a_1}k_2 \qquad \zeta = \int z\,dt - \frac{1}{a_1}k_3$

where k_1, k_2 and k_3 are integration constants, accordingly

$$x = \frac{d\xi}{dt} \quad y = \frac{d\eta}{dt} \text{ and } z = \frac{d\zeta}{dt}$$

Now equations (5) and (6) give the relation

$$b_1\left(\frac{d\eta}{dt}\cos\omega t + \frac{d\zeta}{dt}\sin\omega t\right) + a_1(\eta\cos\omega t + \zeta\sin\omega t) + c\frac{d\xi}{dt} = 0$$

so that from equation (4) we have

$$(bb_1 - c^2)\frac{d\xi}{dt} + ab_1\xi = ca_1(\eta\cos\omega t + \zeta\sin\omega t) - \frac{b_1 d}{\omega}\cos\omega t \quad\text{......(7)}$$

Differentiating this and inserting equation (4) we have

$$(bb_1 - c^2)\frac{d^2\xi}{dt^2} + (ab_1 + a_1 b)\frac{d\xi}{dt} + aa_1\xi$$

$$= b_1 d\sin\omega t - \frac{a_1 d}{\omega}\cos\omega t - ca_1\omega(\eta\sin\omega t - \zeta\cos\omega t) \quad\text{......(8)}$$

Differentiating this and inserting the relation

$$b_1\left(\frac{d\eta}{dt}\sin\omega t - \frac{d\zeta}{dt}\cos\omega t\right) + a_1(\eta\sin\omega t - \zeta\cos\omega t) = 0$$

which is obtained from equations (5) and (6), we have

$$(bb_1 - c^2)\frac{d^3\xi}{dt^3} + (ab_1 + a_1 b)\frac{d^2\xi}{dt^2} + aa_1\frac{d\xi}{dt}$$

$$= b_1 d\omega \cos\omega t + a_1 d \sin\omega t + \frac{ca_1^2\omega}{b_1}(\eta \sin\omega t - \zeta\cos\omega t)$$

$$- ca_1\omega^2(\eta\cos\omega t + \zeta\sin\omega t)$$

which, by insertion of the relations (7) and (8), becomes

$$b_1(bb_1 - c^2)\frac{d^3\xi}{dt^3} + (ab_1^2 + 2a_1bb_1 - a_1c^2)\frac{d^2\xi}{dt^2}$$

$$+ [2aa_1b_1 + a_1^2 b + b_1(bb_1 - c^2)\omega^2]\frac{d\xi}{dt} + a(a_1^2 + b_1^2\omega^2)\xi$$

$$= \frac{a_1 d}{\omega}\cdot\sqrt{a_1^2 + 4b_1^2\omega^2}\sin\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

Thus we have obtained a linear differential equation with constant coefficients, which can easily be solved as will be explained in the next article.

Note that this equation of ξ can also be deduced symbolically as follows: Equation (2) $\times \cos\omega t +$ equation (3) $\times \sin\omega t$ gives

$$a_1 f + b_1\left(\frac{df}{dt} - \omega g\right) + c\frac{dx}{dt} = 0$$

where $f = y\cos\omega t + z\sin\omega t$ and $g = z\cos\omega t - y\sin\omega t$, because

$$\frac{dx}{dt} = \left[\frac{d}{dt}(x\cos\omega t)\right]\cos\omega t + \left[\frac{d}{dt}(x\sin\omega t)\right]\sin\omega t$$

Therefore putting $\frac{d}{dt} = D$ and $a_1 + b_1 D = D_1$ we have

$$D_1 f - b_1\omega g = -cDx \quad \text{..........................(9)}$$

Next, equation (3) $\times \cos\omega t -$ equation (2) $\times \sin\omega t$ gives

$$a_1 g + b_1\left(\frac{dg}{dt} + \omega f\right) + c\omega x = 0$$

because $$\omega x = \left[\frac{d}{dt}(x\sin\omega t)\right]\cos\omega t - \left[\frac{d}{dt}(x\cos\omega t)\right]\sin\omega t$$

so that symbolically

$$D_1 g + b_1\omega f = -c\omega x \quad \text{..........................(10)}$$

Next, equation (1) gives

$$D_2 x + cDf = d\sin\omega t \quad \text{.......................(11)}$$

where $$D_2 = a + bD$$

Now eliminating g from equations (9) and (10) we have

$$(D_1^2 + b_1^2\omega^2) f = -c(DD_1 + b_1\omega^2)x \quad \text{.................(12)}$$

Accordingly from (11) and (12) we have

$$\left(D_2 - \frac{DD_1 + b_1\omega^2}{D_1^2 + b_1^2\omega^2}c^2 D\right)x = d\sin\omega t$$

that is

$$(a + bD)[(a_1 + b_1 D)^2 + b_1^2\omega^2]x - c^2[(a_1 + b_1 D)D + b_1\omega^2]Dx$$
$$= d[(a_1 + b_1 D)^2 + b_1^2\omega^2]\sin\omega t$$

that is

$$b_1(bb_1 - c^2)D^3 x + (ab_1^2 + 2a_1 bb_1 - a_1 c^2)D^2 x$$
$$+ [2aa_1 b_1 + a_1^2 b + b_1(bb_1 - c^2)\omega^2]Dx + a(a_1^2 + b_1^2\omega^2)x$$
$$= a_1 d(a_1 + 2b_1 D)\sin\omega t$$
$$= a_1 d\sqrt{a_1^2 + 4b_1^2\omega^2}\cos\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

which is nothing other than that equation of ξ differentiated.

§ 4. Solution of the fundamental equations.

First in order to obtain the expression of the armature current, let us take up the equation

$$\frac{d^3x}{dt^3} + \frac{ab_1^2 + 2a_1 bb_1 - a_1 c^2}{b_1(bb_1 - c^2)}\cdot\frac{d^2x}{dt^2} + \frac{2aa_1 b_1 + a_1^2 b + b_1(bb_1 - c^2)\omega^2}{b_1(bb_1 - c^2)}\cdot\frac{dx}{dt}$$
$$+ \frac{a(a_1^2 + b_1^2\omega^2)}{b_1(bb_1 - c^2)}x = a_1 d\frac{\sqrt{a_1^2 + 4b_1^2\omega^2}}{b_1(bb_1 - c^2)}\cos\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

The solution of this equation is

$$x = \epsilon^{\alpha t}\int\chi\frac{D_1}{D}dt + \epsilon^{\beta t}\int\chi\frac{D_2}{D}dt + \epsilon^{\gamma t}\int\chi\frac{D_3}{D}dt + A\epsilon^{\alpha t} + B\epsilon^{\beta t} + C\epsilon^{\gamma t}$$

where $$\chi = N\cos(\omega t - \theta)$$

in which $$N = a_1 d\frac{\sqrt{a_1^2 + 4b_1^2\omega^2}}{b_1(bb_1 - c^2)} \quad \text{and} \quad \theta = \tan^{-1}\frac{a_1}{2b_1\omega}$$

$$D = K\epsilon^{(\alpha+\beta+\gamma)t}$$

in which

$$K = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$$

$$D_1 = \begin{vmatrix} 1 & 1 \\ \beta & \gamma \end{vmatrix} \epsilon^{(\beta+\gamma)t} = -(\beta-\gamma)\,\epsilon^{(\beta+\gamma)t}$$

$$D_2 = -\begin{vmatrix} 1 & 1 \\ \alpha & \gamma \end{vmatrix} \epsilon^{(\alpha+\gamma)t} = -(\gamma-\alpha)\,\epsilon^{(\gamma+\alpha)t}$$

$$D_3 = \begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} \epsilon^{(\alpha+\beta)t} = -(\alpha-\beta)\,\epsilon^{(\alpha+\beta)t}$$

A, B and C = integration constants, and α, β and γ = roots of the equation

$$b_1(bb_1-c^2)\lambda^3 + (ab_1^2 + 2a_1bb_1 - a_1c^2)\lambda^2 + [2aa_1b_1 + a_1^2b + b_1(bb_1-c^2)\omega^2]\lambda + a(a_1^2 + b_1^2\omega^2) = 0$$

Hence neglecting for the moment the transient terms

$$A\epsilon^{\alpha t} + B\epsilon^{\beta t} + C\epsilon^{\gamma t}$$

the solution becomes

$$x = \frac{-N}{K}\Big[(\beta-\gamma)\,\epsilon^{\alpha t}\int\epsilon^{-\alpha t}\cos(\omega t-\theta)\,dt + (\gamma-\alpha)\,\epsilon^{\beta t}\int\epsilon^{-\beta t}\cos(\omega t-\theta)\,dt + (\alpha-\beta)\,\epsilon^{\gamma t}\int\epsilon^{-\gamma t}\cos(\omega t-\theta)\,dt\Big]$$

$$= \frac{N}{K}\cdot\left[\frac{\alpha(\beta-\gamma)}{\alpha^2+\omega^2} + \frac{\beta(\gamma-\alpha)}{\beta^2+\omega^2} + \frac{\gamma(\alpha-\beta)}{\gamma^2+\omega^2}\right]\cos(\omega t-\theta) - \frac{N}{K}\cdot\omega\cdot\left(\frac{\beta-\gamma}{\alpha^2+\omega^2} + \frac{\gamma-\alpha}{\beta^2+\omega^2} + \frac{\alpha-\beta}{\gamma^2+\omega^2}\right)\sin(\omega t-\theta)$$

But

$$\alpha(\beta-\gamma)(\beta^2+\omega^2)(\gamma^2+\omega^2) + \beta(\gamma-\alpha)(\gamma^2+\omega^2)(\alpha^2+\omega^2) + \gamma(\alpha-\beta)(\alpha^2+\omega^2)(\beta^2+\omega^2) = K[\omega^2(\alpha+\beta+\gamma) - \alpha\beta\gamma]$$

$$(\beta-\gamma)(\beta^2+\omega^2)(\gamma^2+\omega^2) + (\gamma-\alpha)(\gamma^2+\omega^2)(\alpha^2+\omega^2) + (\alpha-\beta)(\alpha^2+\omega^2)(\beta^2+\omega^2) = K[\omega^2 - (\beta\gamma+\gamma\alpha+\alpha\beta)]$$

and

$$(\alpha^2+\omega^2)(\beta^2+\omega^2)(\gamma^2+\omega^2) = [\omega^2(\alpha+\beta+\gamma) - \alpha\beta\gamma]^2 + \omega^2[\omega^2 - (\beta\gamma+\gamma\alpha+\alpha\beta)]^2$$

Therefore the solution is

$$x = \frac{N}{\sqrt{[\omega^2(\alpha+\beta+\gamma)-\alpha\beta\gamma]^2+\omega^2[\omega^2-(\beta\gamma+\gamma\alpha+\alpha\beta)]^2}}$$

$$\times \cos\left(\omega t - \theta + \tan^{-1}\frac{\omega[\omega^2-(\beta\gamma+\gamma\alpha+\alpha\beta)]}{\omega^2(\alpha+\beta+\gamma)-\alpha\beta\gamma}\right)$$

But

$$\omega^2(\alpha+\beta+\gamma)-\alpha\beta\gamma = -\frac{\omega^2(ab_1^2+2a_1bb_1-a_1c^2)-a(a_1^2+b_1^2\omega^2)}{b_1(bb_1-c^2)}$$

$$= \frac{a_1[aa_1-(2bb_1-c^2)\omega^2]}{b_1(bb_1-c^2)}$$

and

$$\omega^2-(\beta\gamma+\gamma\alpha+\alpha\beta) = \omega^2 - \frac{2aa_1b_1+a_1^2b+b_1(bb_1-c^2)\omega^2}{b_1(bb_1-c^2)} = \frac{-a_1(2ab_1+a_1b)}{b_1(bb_1-c^2)}$$

Therefore

$$x = \frac{d\sqrt{a_1^2+4b_1^2\omega^2}}{\sqrt{[aa_1-(2bb_1-c^2)\omega^2]^2+(a_1b+2ab_1)^2\omega^2}}$$

$$\times \cos\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega} - \tan^{-1}\frac{(a_1b+2ab_1)\omega}{aa_1-(2bb_1-c^2)\omega^2}\right)$$

$$= \frac{d\sqrt{a_1^2+4b_1^2\omega^2}}{\sqrt{[aa_1-(2bb_1-c^2)\omega^2]^2+(a_1b+2ab_1)^2\omega^2}}$$

$$\times \sin\left(\omega t + \tan^{-1}\frac{2b_1\omega}{a_1} - \tan^{-1}\frac{(a_1b+2ab_1)\omega}{aa_1-(2bb_1-c^2)\omega^2}\right)$$

which coincides with that result obtained in Art. 2.

Now putting this expression of x in the fundamental equation No. 2 we have

$$\frac{dy}{dt} + \frac{a_1}{b_1}y = -\frac{c}{b_1}\cdot\frac{d}{dt}\cdot[X\sqrt{2}\sin(\omega t-\phi_x)\cos\omega t]$$

$$= -\frac{c}{b_1}X\sqrt{2}\,\omega\cos(2\omega t-\phi_x)$$

where $$X\sqrt{2} = \frac{d}{\sqrt{(a+\rho^2a_1)^2+(b-2\rho^2b_1)^2\omega^2}}$$

and $$\phi_x = \tan^{-1}\frac{(b-2\rho^2b_1)\omega}{a+\rho^2a_1}$$

which solution is

$$y = -\frac{c}{b_1} \cdot X\sqrt{2}\,\omega\epsilon^{-\frac{a_1}{b_1}t}\int \epsilon^{\frac{a_1}{b_1}t}\cos(2\omega t - \phi_x)\,dt + K\epsilon^{-\frac{a_1}{b_1}t}$$

$$= -X\sqrt{2}\,\frac{c\omega}{\sqrt{a_1^2 + 4b_1^2\omega^2}}\cos\left(2\omega t - \phi_x - \tan^{-1}\frac{2b_1\omega}{a_1}\right) + K\epsilon^{-\frac{a_1}{b_1}t}$$

where K is an integration constant.

Thus if we neglect the transient term $K\epsilon^{-\frac{a_1}{b_1}t}$ then the expression of the alternating current y in the direct field winding is

$$y = \rho X\sqrt{2}\sin\left(2\omega t - \phi_x - \frac{\pi}{2} - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

Similarly from the fundamental equation No. 3 we have

$$z = -\frac{c}{b_1} X\sqrt{2}\,\epsilon^{-\frac{a_1}{b_1}t}\int \epsilon^{\frac{a_1}{b_1}t}\sin(2\omega t - \phi_x)\,dt + K'\epsilon^{-\frac{a_1}{b_1}t}$$

$$= -X\sqrt{2}\,\frac{c\omega}{\sqrt{a_1^2 + 4b_1^2\omega^2}}\sin\left(2\omega t - \phi_x - \tan^{-1}\frac{2b_1\omega}{a_1}\right) + K'\epsilon^{-\frac{a_1}{b_1}t}$$

which, when the transient term is neglected, becomes

$$z = \rho X\sqrt{2}\sin\left(2\omega t - \phi_x - \pi - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

Thus we have arrived at the same results as obtained in Art. 2.

§ 5. Other methods of obtaining the expressions of the field and armature currents.

(*a*) The fundamental equation

$$P\frac{d^3x}{dt^3} + Q\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Sx = V\sqrt{2}\cos\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

$$= V\sqrt{2}\sin\left(\omega t + \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

where

$$P = b_1(bb_1 - c^2)$$

$$Q = ab_1^2 + 2a_1bb_1 - a_1c^2$$

$$R = 2aa_1b_1 + a_1^2b + b_1(bb_1 - c^2)\omega^2$$

$$S = a(a_1^2 + b_1^2\omega^2)$$

$$V\sqrt{2} = a_1d\sqrt{a_1^2 + 4b_1^2\omega^2}$$

shows immediately that x is a simple sine function of t with the frequency $\omega/2\pi$.

Therefore we have symbolically

$$(Pj^3\omega^3 + Qj^2\omega^2 + Rj\omega + S)\,\dot{X} = \dot{V}$$

that is

$$\dot{X} = \frac{\dot{V}}{(S - Q\omega^2) + j\omega(R - P\omega^2)}$$

so that

$$x = \frac{V\sqrt{2}}{\sqrt{(S-Q\omega^2)^2 + (R-P\omega^2)^2\,\omega^2}}\sin\left(\omega t + \tan^{-1}\frac{2b_1\omega}{a_1} - \tan^{-1}\frac{(R-P\omega^2)\,\omega}{S - Q\omega^2}\right)$$

But $S - Q\omega^2 = a_1[aa_1 - (2bb_1 - c^2)\,\omega^2]$ and $R - P\omega^2 = a_1(a_1 b + 2ab_1)$

Therefore

$$x = \frac{d\sqrt{a_1^2 + 4b_1^2\omega^2}}{\sqrt{[aa_1 - (2bb_1 - c^2)\,\omega^2]^2 + (a_1 b + 2ab_1)^2\,\omega^2}}$$

$$\times \sin\left(\omega t - \tan^{-1}\frac{(ab_1 + 2a_1 b)\,\omega}{aa_1 - (2bb_1 - c^2)\,\omega^2} + \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

(*b*) Putting $x \equiv \sum_{n=1}^{\infty} X_n\sqrt{2}\sin(n\omega t - \phi_n)$ in the equation

$$P\frac{d^3x}{dt^3} + Q\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Sx = a_1 d\sqrt{a_1^2 + 4b_1^2\omega^2}\sin\left(\omega t + \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

we have

$$\sum_{n=1}^{\infty}[2aa_1b_1 + a_1^2 b - b_1(bb_1 - c^2)(n^2 - 1)\,\omega^2]\,n\omega\,X_n\sqrt{2}\cos(n\omega t - \phi_n)$$

$$+ \sum_{n=1}^{\infty}[a(a_1^2 + b_1^2\omega^2) - (ab_1^2 + 2a_1bb_1 - a_1c^2)\,n^2\omega^2]\,X_n\sqrt{2}\sin(n\omega t - \phi_n)$$

$$\equiv a_1 d\sqrt{a_1^2 + 4b_1^2\omega^2}\sin\left(\omega t + \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

so that

$$(a_1 b + 2ab_1)\,\omega X_1\sqrt{2}\cos(\omega t - \phi_1) + [aa_1 - (2bb_1 - c^2)\,\omega^2]\,X_1\sqrt{2}\sin(\omega t - \phi_1)$$

$$\equiv d\sqrt{a_1^2 + 4b_1^2\omega^2}\sin\left(\omega t + \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

and $$A_n X_n\sqrt{2}\cos(n\omega t - \phi_n) + B_n X_n\sqrt{2}\sin(n\omega t - \phi_n) \equiv 0$$

where $$A_n = [2aa_1b_1 + a_1^2 b - b_1(bb_1 - c^2)(n^2 - 1)\,\omega^2]\,n\omega$$

and $$B_n = a(a_1^2 + b_1^2\omega^2) - (ab_1^2 + 2a_1bb_1 - a_1c^2)\,n^2\omega^2$$

in which n is any positive integer greater than 1, the latter of which shows that

$$X_n\sqrt{2}\sqrt{A_n^2 + B_n^2}\sin\left(n\omega t - \phi_n + \tan^{-1}\frac{A_n}{B_n}\right) \equiv 0$$

that is $$X_n = 0$$

and the former shows that

$$X_1\sqrt{2}\sqrt{[aa_1-(2bb_1-c^2)\,\omega^2]^2+(a_1b+2ab_1)^2\,\omega^2}$$
$$\times\sin\left(\omega t-\phi_1+\tan^{-1}\frac{(a_1b+2ab_1)\,\omega}{aa_1-(2bb_1-c^2)\,\omega^2}\right)$$
$$\equiv d\sqrt{a_1^2+4b_1^2\omega^2}\sin\left(\omega t+\tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

so that
$$X_1\sqrt{2}=\frac{d\sqrt{a_1^2+4b_1^2\omega^2}}{\sqrt{[aa_1-(2bb_1-c^2)\,\omega^2]^2+(a_1b+2ab_1)^2\,\omega^2}}$$

and
$$\phi_1=\tan^{-1}\frac{(a_1b+2ab_1)\,\omega}{aa_1-(2bb_1-c^2)\,\omega^2}-\tan^{-1}\frac{2b_1\omega}{a_1}$$

(*c*) We saw in Art. 1 that the armature current contains the fundamental harmonic only and the field currents the second order harmonic only; so that we can write

$$x\equiv X\sqrt{2}\sin(\omega t-\phi_x)\qquad y\equiv Y\sqrt{2}\sin(2\omega t-\phi_y)$$
$$z\equiv Z\sqrt{2}\sin(2\omega t-\phi_z)$$

Now put these x and y in fundamental equation No. 2. Then

$$a_1Y\sqrt{2}\sin(2\omega t-\phi_y)+2b_1\omega\,Y\sqrt{2}\cos(2\omega t-\phi_y)+cX\sqrt{2}\,\omega\cos(2\omega t-\phi_x)=0$$

that is

$$(a_1Y\cos\phi_y+2b_1\omega Y\sin\phi_y+cX\omega\sin\phi_x)\sin 2\omega t$$
$$-(a_1Y\sin\phi_y-2b_1\omega Y\cos\phi_y-cX\omega\cos\phi_x)\cos 2\omega t=0$$

so that

$$a_1Y\cos\phi_y+2b_1\omega Y\sin\phi_y+cX\omega\sin\phi_x=0\quad\ldots\ldots\ldots\ldots(1)$$

and

$$a_1Y\sin\phi_y-2b_1\omega Y\cos\phi_y-cX\omega\cos\phi_x=0\quad\ldots\ldots\ldots\ldots(2)$$

Now subtract (2) from (1) multiplied by j. Then

$$a_1Y(j\cos\phi_y-\sin\phi_y)+2b_1\omega Y(\cos\phi_y+j\sin\phi_y)+cX\omega(\cos\phi_x+j\sin\phi_x)=0$$

that is

$$(2b_1\omega+ja_1)\,Y.\epsilon^{j\phi_y}=-c\omega X.\epsilon^{j\phi_x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Similarly from the fundamental equation No. 3, we have

$$(2b_1\omega+ja_1)\,Z.\epsilon^{j\phi_z}=-jc\omega X.\epsilon^{j\phi_x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Therefore

$$Z \,.\, \epsilon^{j\phi_z} = jY \,.\, \epsilon^{j\phi_y} = Y \,.\, \epsilon^{j\left(\phi_y + \frac{\pi}{2}\right)}$$

so that

$$Z = Y \quad \text{and} \quad \phi_z = \phi_y + \frac{\pi}{2} \quad \text{......................(5)}$$

and hence

$$y \cos \omega t + z \sin \omega t = Y\sqrt{2} \sin(\omega t - \phi_y)$$

Therefore from fundamental equation No. 1, we have

$$(aX \cos \phi_x + b\omega X \sin \phi_x + c\omega Y \sin \phi_y)\sqrt{2} \sin \omega t$$
$$- (aX \sin \phi_x - b\omega X \cos \phi_x - c\omega Y \cos \phi_y)\sqrt{2} \cos \omega t = d \sin \omega t$$

so that

$$aX \cos \phi_x + b\omega X \sin \phi_x + c\omega Y \sin \phi_y = \frac{1}{\sqrt{2}} d \quad \text{...........(6)}$$

and

$$aX \sin \phi_x - b\omega X \cos \phi_x - c\omega Y \cos \phi_y = 0 \quad \text{.................(7)}$$

Now subtract (7) from (6) multiplied by j. Then

$$aX(j \cos \phi_x - \sin \phi_x) + b\omega X(\cos \phi_x + j \sin \phi_x) + c\omega Y(\cos \phi_y + j \sin \phi_y) = j\frac{1}{\sqrt{2}} d$$

that is

$$(b\omega + ja) X \,.\, \epsilon^{j\phi_x} + c\omega Y \,.\, \epsilon^{j\phi_y} = j\frac{1}{\sqrt{2}} d \quad \text{....................(8)}$$

Accordingly from (3) we have

$$X\sqrt{2}\epsilon^{j\phi_x} = \frac{jd(2b_1\omega + ja_1)}{(b\omega + ja)(2b_1\omega + ja_1) - c^2\omega^2}$$
$$= \frac{d(a_1 - j2b_1\omega)}{aa_1 - (2bb_1 - c^2)\omega^2 - j(2ab_1 + a_1 b)\omega}$$

Therefore

$$X\sqrt{2} = \frac{d\sqrt{a_1^2 + 4b_1^2\omega^2}}{\sqrt{[aa_1 - (2bb_1 - c^2)\omega^2]^2 + (2ab_1 + a_1 b)^2\omega^2}}$$

and $$\phi_x = \tan^{-1}\frac{(2ab_1 + a_1 b)\omega}{aa_1 - (2bb_1 - c^2)\omega^2} - \tan^{-1}\frac{2b_1\omega}{a_1}$$

and from (3) and (5)

$$Y\sqrt{2} = Z\sqrt{2} = \frac{dc\omega}{\sqrt{[aa_1 - (2bb_1 - c^2)\omega^2]^2 + (2ab_1 + a_1 b)^2\omega^2}} = X\sqrt{2}\rho$$

and $$\phi_y = \phi_x + \pi - \tan^{-1}\frac{a_1}{2b_1\omega} = \phi_x + \frac{\pi}{2} + \tan^{-1}\frac{2b_1\omega}{a_1}$$

$$= \frac{\pi}{2} + \tan^{-1}\frac{(2ab_1 + a_1 b)\,\omega}{aa_1 - (2bb_1 - c^2)\,\omega^2}$$

and $$\phi_z = \phi_x + \pi + \tan^{-1}\frac{2b_1\omega}{a_1} = \pi + \tan^{-1}\frac{(2ab_1 + a_1 b)\,\omega}{aa_1 - (2bb_1 - c^2)\,\omega^2}$$

§ 6. Distortionless alternator with two field windings containing r, L and C in series in the armature circuit.

In this case the fundamental equations are

$$ax + b\frac{dx}{dt} + e\int x\,dt + c\frac{d}{dt}(y\cos\omega t + z\sin\omega t) = d\sin\omega t$$

$$a_1 y + b_1\frac{dy}{dt} + c\frac{d}{dt}(x\cos\omega t) = 0$$

$$a_1 z + b_1\frac{dz}{dt} + c\frac{d}{dt}(x\sin\omega t) = 0$$

where $e = \frac{1}{C}$ and all other notations are the same as in Art. 3.

Hence similarly as in Art. 3 we have

$$a\xi + b\frac{d\xi}{dt} + e\int\left(\xi + \frac{K_1}{a}\right)dt + c\left(\frac{d\eta}{dt}\cos\omega t + \frac{d\zeta}{dt}\sin\omega t\right) = -\frac{d}{\omega}\cos\omega t \ldots(1)$$

$$a_1\eta + b_1\frac{d\eta}{dt} + c\frac{d\xi}{dt}\cos\omega t = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$a_1\zeta + b_1\frac{d\zeta}{dt} + c\frac{d\xi}{dt}\sin\omega t = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

so that similarly as in Art. 3

$$(bb_1 - c^2)\frac{d\xi}{dt} + ab_1\xi + b_1 e\int\left(\xi + \frac{K_1}{a}\right)dt - ca_1(\eta\cos\omega t + \zeta\sin\omega t) = -\frac{b_1 d}{\omega}\cos\omega t \ \ldots\ldots(4)$$

Now differentiate this and insert the relation (1). Then we have

$$(bb_1 - c^2)\frac{d^2\xi}{dt^2} + (ab_1 + a_1 b)\frac{d\xi}{dt} + (aa_1 + b_1 e)\,\xi + \frac{b_1 e}{a}K_1 + a_1 e\int\left(\xi + \frac{K_1}{a}\right)dt$$

$$= b_1 d\sin\omega t - \frac{a_1 d}{\omega}\cos\omega t - ca_1\omega(\eta\sin\omega t - \zeta\cos\omega t) \ldots(5)$$

which by differentiation becomes

$$(bb_1 - c^2)\frac{d^3\xi}{dt^3} + (ab_1 + a_1 b)\frac{d^2\xi}{dt^2} + (aa_1 + b_1 e)\frac{d\xi}{dt} + a_1 e\left(\xi + \frac{K_1}{a}\right)$$

$$= b_1 d\omega \cos \omega t + a_1 d \sin \omega t + \frac{ca_1^2\omega}{b_1}(\eta \sin \omega t - \zeta \cos \omega t) - ca_1\omega^2(\eta \cos \omega t + \zeta \sin \omega t)$$

Hence inserting here the relations (4) and (5) we have

$$b_1(bb_1 - c^2)\frac{d^3\xi}{dt^3} + (ab_1^2 + 2a_1 bb_1 - a_1 c^2)\frac{d^2\xi}{dt^2}$$

$$+ [2aa_1 b_1 + a_1^2 b + b_1(bb_1 - c^2)\omega^2 + b_1^2 e]\frac{d\xi}{dt} + [a(a_1^2 + b_1^2\omega^2) + 2a_1 b_1 e]\,\xi$$

$$+ 2\frac{a_1 b_1 e}{a}K_1 + (a_1^2 + b_1^2\omega^2)e\int\left(\xi + \frac{K_1}{a}\right)dt$$

$$= \frac{a_1 d}{\omega}\sqrt{a_1^2 + 4b_1^2\omega^2}\sin\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

so that

$$b_1(bb_1 - c^2)\frac{d^4\xi}{dt^4} + (ab_1^2 + 2a_1 bb_1 - a_1 c^2)\frac{d^3\xi}{dt^3} + [2aa_1 b_1 + a_1^2 b + b_1(bb_1 - c^2)\omega^2 + b_1^2 e]$$

$$\times\frac{d^2\xi}{dt^2} + [a(a_1^2 + b_1^2\omega^2) + 2a_1 b_1 e]\frac{d\xi}{dt} + (a_1^2 + b_1^2\omega^2)e\left(\xi + \frac{K_1}{a}\right)$$

$$= a_1 d\sqrt{a_1^2 + 4b_1^2\omega^2}\cos\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

so that

$$b_1(bb_1 - c^2)\frac{d^4x}{dt^4} + (ab_1^2 + 2a_1 bb_1 - a_1 c^2)\frac{d^3x}{dt^3}$$

$$+ [2aa_1 b_1 + a_1^2 b + b_1(bb_1 - c^2)\omega^2 + b_1^2 e]\frac{d^2x}{dt^2} + [a(a_1^2 + b_1^2\omega^2) + 2a_1 b_1 e]\frac{dx}{dt}$$

$$+ (a_1^2 + b_1^2\omega^2)ex = -a_1 d\omega\sqrt{a_1^2 + 4b_1^2\omega^2}\sin\left(\omega t - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

This is a linear differential equation with constant coefficients and its right-hand side is a simple sine function; so that x will also be a simple sine function and therefore symbolically we have

$$(Pj^4\omega^4 + Qj^3\omega^3 + Rj^2\omega^2 + Sj\omega + T)\,\underset{\cdot}{X} = \underset{\cdot}{V}$$

that is

$$\underset{\cdot}{X} = \frac{\underset{\cdot}{V}}{(P\omega^4 - R\omega^2 + T) - j(Q\omega^2 - S)\,\omega}$$

where $P = b_1(bb_1 - c^2)$ $\qquad Q = ab_1^2 + 2a_1bb_1 - a_1c^2$

$R = 2aa_1b_1 + a_1^2b + b_1(bb_1 - c^2)\,\omega^2 + b_1^2e \qquad S = a(a_1^2 + b_1^2\omega^2) + 2a_1b_1e$

$T = (a_1^2 + b_1^2\omega^2)\,e$ and $V\sqrt{2} = -a_1 d\omega\sqrt{a_1^2 + 4b_1^2\omega^2}$

But
$$P\omega^4 - R\omega^2 + T = a_1[a_1e - (a_1b + 2ab_1)\,\omega]$$
and
$$Q\omega^2 - S = a_1[(2bb_1 - c^2)\,\omega^2 - (aa_1 + 2b_1e)]$$

so that
$$\dot{X} = \frac{\dfrac{\dot{V}}{a_1\omega}}{\dfrac{a_1e}{\omega} - (a_1b + 2ab_1)\,\omega - j[(2bb_1 - c^2)\,\omega^2 - (aa_1 + 2b_1e)]}$$

and therefore
$$x = \frac{-MI_f\omega\sqrt{r_f^2 + 4L_f^2\omega^2}}{\sqrt{\left[\dfrac{r_f}{C\omega} - (Lr_f + 2L_fr)\,\omega\right]^2 + \left[(2LL_f - M^2)\,\omega^2 - \left(rr_f + \dfrac{2L_f}{C}\right)\right]^2}}$$
$$\times \sin\left(\omega t - \tan^{-1}\frac{r_f}{2L_f\omega} + \tan^{-1}\frac{(2LL_f - M^2)\,\omega^2 - \left(rr_f + \dfrac{2L_f}{C}\right)}{\dfrac{r_f}{C\omega} - (Lr_f + 2L_fr)\,\omega}\right)$$

But
$$\left[\frac{r_f}{C\omega} - (Lr_f + 2L_fr)\,\omega\right]^2 + \left[(2LL_f - M^2)\,\omega^2 - \left(rr_f + \frac{2L_f}{C}\right)\right]^2$$
$$= (r_f^2 + 4L_f^2\omega^2) + \left[(r + \rho^2r_f)^2\left(L\omega - \frac{1}{C\omega} - 2\rho^2L_f\omega\right)^2\right]$$

and
$$\tan^{-1}\frac{(2LL_f - M^2)\,\omega^2 - \left(rr_f + \dfrac{2L_f}{C}\right)}{\dfrac{r_f}{C\omega} - (Lr_f + 2L_fr)\,\omega} - \tan^{-1}\frac{r_f}{2L_f\omega}$$
$$= \pi - \tan^{-1}\frac{\left(L\omega - \dfrac{1}{C\omega}\right) - 2\rho^2L_f\omega}{r + \rho^2r_f}$$

where $\rho = \dfrac{M\omega}{z_f}$ in which $z_f = \sqrt{r_f^2 + 2L_f^2\omega^2}$

Therefore
$$x = \frac{MI_f\omega}{\sqrt{(r + \rho^2r_f)^2 + \left(L\omega - \dfrac{1}{C\omega} - 2\rho^2L_f\omega\right)^2}}$$
$$\times \sin\left(\omega t - \tan^{-1}\frac{L\omega - \dfrac{1}{C\omega} - 2\rho^2L_f\omega}{r + \rho^2r_f}\right)$$

and similarly as in Art. 4

$$y = \rho X \sqrt{2} \sin\left(2\omega t - \tan^{-1}\frac{L\omega - \frac{1}{C\omega} - 2\rho^2 L_f\omega}{r + \rho^2 r_f} - \frac{\pi}{2} - \tan^{-1}\frac{2L_f\omega}{r_f}\right)$$

$$= \rho X \sqrt{2} \sin\left(2\omega t - \frac{\pi}{2} - \tan^{-1}\frac{(Lr_f + 2L_f r)\,\omega - \frac{r_f}{C\omega}}{\left(rr_f + \frac{2L_f}{C}\right) - (2LL_f - M^2)\,\omega^2}\right)$$

and

$$z = \rho X \sqrt{2} \sin\left(2\omega t - \tan^{-1}\frac{L\omega - \frac{1}{C\omega} - 2\rho^2 L_f\omega}{r + \rho^2 r_f} - \pi - \tan^{-1}\frac{2L_f\omega}{r_f}\right)$$

$$= \rho X \sqrt{2} \sin\left(2\omega t - \pi - \tan^{-1}\frac{(Lr_f + 2L_f r)\,\omega - \frac{r_f}{C\omega}}{\left(rr_f + \frac{2L_f}{C}\right) - (2LL_f - M^2)\,\omega^2}\right)$$

§ 7. Distortionless alternator with two field windings when both fields are excited.

In this case two simple sine fields displaced in space by magnetical 90 degrees superpose on each other; so that we have the fundamental equations

$$ax + b\frac{dx}{dt} + c\frac{d}{dt}(y\cos\omega t + z\sin\omega t) = d\sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right)$$

$$a_1 y + b_1\frac{dy}{dt} + c\frac{d}{dt}(x\cos\omega t) = 0$$

$$a_1 z + b_1\frac{dz}{dt} + c\frac{d}{dt}(x\sin\omega t) = 0$$

where a, b, c etc. all denote the same things as in Arts. 3 and 4. So that we have

$$b_1(bb_1 - c^2)\frac{d^3x}{dt^3} + (ab_1^2 + 2a_1bb_1 - a_1c^2)\frac{d^2x}{dt^2} + [2aa_1b_1 + a_1^2b - b_1(bb_1 - c^2)\,\omega^2]\frac{dx}{dt}$$

$$+ a(a_1^2 + b_1^2\omega^2)\,x = \sqrt{2}a_1 d\sqrt{a_1^2 + 4b_1^2\omega^2}\cos\left(\omega t - \frac{\pi}{4} - \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

so that similarly as in Art. 4 we arrive at the results

$$x = \frac{\sqrt{2}d\sqrt{a_1^2 + 4b_1^2\omega^2}}{\sqrt{[aa_1 - (2bb_1 - c^2)\omega^2]^2 + (a_1 b + 2ab_1)^2\omega^2}} \times \sin\left(\omega t - \frac{\pi}{4} - \tan^{-1}\frac{(a_1 b + 2ab_1)\omega}{aa_1 - (2bb_1 - c^2)\omega^2} + \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

$$= \frac{\sqrt{2}d}{\sqrt{(a + \rho^2 a_1)^2 + (b - 2\rho^2 b_1)^2\omega^2}} \sin\left(\omega t - \frac{\pi}{4} - \tan^{-1}\frac{(b - 2\rho^2 b_1)\omega}{a + \rho^2 a_1}\right)$$

$$y = \frac{\sqrt{2}dc\omega}{\sqrt{[aa_1 - (2bb_1 - c^2)\omega^2]^2 + (a_1 b + 2ab_1)^2\omega^2}} \times \sin\left(2\omega t - 3\frac{\pi}{4} - \tan^{-1}\frac{(a_1 b + 2ab_1)\omega}{aa_1 - (2bb_1 - c^2)\omega^2}\right)$$

$$= \frac{\sqrt{2}d\rho}{\sqrt{(a + \rho^2 a_1)^2 + (b - 2\rho^2 b_1)^2\omega^2}} \sin\left(\omega t - 3\frac{\pi}{4} - \tan^{-1}\frac{(b - 2\rho^2 b_1)\omega}{a + \rho^2 a_1} - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

$$z = \frac{\sqrt{2}dc\omega}{\sqrt{[aa_1 - (2bb_1 - c^2)\omega^2]^2 + (a_1 b + 2ab_1)^2\omega^2}} \times \sin\left(2\omega t - 5\frac{\pi}{4} - \tan^{-1}\frac{(a_1 b + 2ab_1)\omega}{aa_1 - (2bb_1 - c^2)\omega^2}\right)$$

$$= \frac{\sqrt{2}d\rho}{\sqrt{(a + \rho^2 a_1)^2 + (b - 2\rho^2 b_1)^2\omega^2}} \times \sin\left(2\omega t - 5\frac{\pi}{4} - \tan^{-1}\frac{(b - 2\rho^2 b_1)\omega}{a + \rho^2 a_1} - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

§ 8. Distortionless alternator with single field winding.

The distortionless alternator with two field windings is theoretically perfect. It requires however an extra space and copper for the cross field winding and, in order to make use of the cross field winding by exciting it as described in the preceding article, a little complication of the apparatus is unavoidable.

A better practical solution of the distortionless alternator is obtained by the use of a field winding and a large reactance in series with it.

As shown by Prof. Lyle*, and also as will be proved by another method later in the theory of the single phase alternator, the armature current i

* Philosophical Magazine 1909.

of the ordinary single phase alternator with non-salient and laminated poles and with simple sine flux distribution along the air gap is

$$i = \sum_{n=0}^{\infty} I_{2n+1} \sqrt{2} \sin (\overline{2n+1}\, \omega t - \phi_{2n+1})$$

which component harmonics have the amplitudes and the phase angles as follows:

$$I_1 (\sin \phi_1 - j \cos \phi_1) = \frac{I_f \sqrt{2}}{s_1}$$

$$I_3 (\sin \phi_3 - j \cos \phi_3) = \frac{I_f \sqrt{2}}{s_1 \,.\, s_2 \,.\, s_3}$$

$$I_5 (\sin \phi_5 - j \cos \phi_5) = \frac{I_f \sqrt{2}}{s_1 \,.\, s_2 \,.\, s_3 \,.\, s_4 \,.\, s_5}$$

etc.

where

$$s_1 = \left(\frac{2b}{c} + j \frac{2a}{c\omega}\right) - \cfrac{1}{\cfrac{2b_1}{c} + j \cfrac{2a_1}{2c\omega} - \cfrac{1}{\cfrac{2b}{c} + j \cfrac{2a}{3c\omega} - \text{etc. to } \infty}}$$

$$s_2 = \left(\frac{2b_1}{c} + j \frac{2a_1}{2c\omega}\right) - \cfrac{1}{\cfrac{2b}{c} + j \cfrac{2a}{3c\omega} - \cfrac{1}{\cfrac{2b_1}{c} + j \cfrac{2a_1}{4c\omega} - \text{etc. to } \infty}}$$

$$s_3 = \left(\frac{2b}{c} + j \frac{2a}{3c\omega}\right) - \cfrac{1}{\cfrac{2b_1}{c} + j \cfrac{2a_1}{4c\omega} - \cfrac{1}{\cfrac{2b}{c} + j \cfrac{2a}{5c\omega} - \text{etc. to } \infty}}$$

etc.

where $a = r$, $b = L$, $a_1 = r_f$, $b_1 = L_f$, $c = M$ and $I_f =$ the exciting current as in Art. 3.

Hence putting $\frac{L_f}{M} = \infty$ that is $\frac{b_1}{c} = \infty$ so that

$$s_2 = s_4 = s_6 = \text{etc.} = \infty$$

we have
$$I_1 \sqrt{2} (\sin \phi_1 - j \cos \phi_1) = \frac{2 I_f}{\frac{2L}{M} + j \frac{2r}{M\omega}} = \frac{M I_f \omega}{L\omega + jr}$$

so that

$$i = \frac{M I_f \omega}{\sqrt{r^2 + L^2 \omega^2}} \sin \left(\omega t - \tan^{-1} \frac{L\omega}{r}\right)$$

and all higher harmonics vanish.

The alternating current in the field circuit being

$$i_f = \sum_{n=1}^{\infty} I_{2n}\sqrt{2}\sin(2n\omega t - \phi_{2n})$$

where

$$I_2(\sin\phi_2 - j\cos\phi_2) = \frac{-I_f\sqrt{2}}{s_1 \,.\, s_2}$$

$$I_4(\sin\phi_4 - j\cos\phi_4) = \frac{-I_f\sqrt{2}}{s_1 \,.\, s_2 \,.\, s_3 \,.\, s_4}$$

etc.

it is obviously equal to zero when $\frac{L_f}{M} = \infty$, accordingly

$$s_2 = s_4 = s_6 = \text{etc.} = \infty$$

Thus by the use of a large reactance in the field circuit we can have a distortionless alternator with single field winding.

The field and armature current of this distortionless alternator with single field winding can also be found as a special case of the distortionless alternator with two field windings. Putting $\frac{c}{b_1} = 0$ so that $\rho = 0$ in the expressions of i_a, i_{fd} and i_{fc} in Art. 4, we have

$$i_a = \frac{MI_f\omega}{\sqrt{r^2 + L^2\omega^2}} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{r}\right)$$

and

$$i_{fd} = i_{fc} = 0$$

Thus we see that if a large reactance be inserted in series in the field circuit, then the armature circuit can be considered as independent of the field circuit, giving the fundamental equation

$$ri_a + L\frac{di_a}{dt} = MI_f\omega\sin\omega t.$$

§ 9. Distortion of wave form due to pole projection.

If the field poles be projecting so that the reluctance of the passage of the magnetic flux produced by the armature current be varying with time, then, notwithstanding the insertion of a large reactance in the field circuit, the armature current will be distorted. In this case, as in the case treated in the preceding article, the large reactance put in series in the field circuit suppresses any alternating current induced there and makes the armature circuit quite independent of the field circuit; so that if we assume the armature resistance and the load impedance constant and the armature inductance varying with double the fundamental frequency due to pole projection following a simple rule shown by $L(1+p\cos 2\omega t)$, where p is a proper fraction, then the fundamental equation for the armature current i will be

$$ri + L\frac{d}{dt}[(1+p\cos 2\omega t)\,i] = E\sqrt{2}\sin\omega t$$

where $$E\sqrt{2} = MI_f\omega$$

that is $$L(1+p\cos 2\omega t)\frac{di}{dt} + (r - 2pL\omega\sin 2\omega t)\,i = E\sqrt{2}\sin\omega t$$

which solves to

$$i\epsilon^{\frac{r}{L\omega}\cdot\frac{1}{\sqrt{1-p^2}}\tan^{-1}\left(\sqrt{\frac{1-p}{1+p}}\tan\omega t\right)+\log(1+p\cos 2\omega t)}$$

$$= \frac{E\sqrt{2}}{L}\int \epsilon^{\frac{r}{L\omega}\cdot\frac{1}{\sqrt{1-p^2}}\tan^{-1}\left(\sqrt{\frac{1-p}{1+p}}\tan\omega t\right)+\log(1+p\cos 2\omega t)}\cdot\frac{\sin\omega t}{1+p\cos 2\omega t}\,dt + C$$

This however seems hard to integrate and therefore we shall at present be contented with the following solution of the permanent current only.

Now put

$$i \equiv \sum_{n=1}^{\infty} I_n\sqrt{2}\sin(n\omega t - \phi_n)$$

then the fundamental equation becomes

$$L\omega(1+p\cos 2\omega t)\sum_{n=1}^{\infty} nI_n\cos(n\omega t-\phi_n)$$
$$+ (r - 2pL\omega\sin 2\omega t)\sum_{n=1}^{\infty} I_n\sin(n\omega t-\phi_n) \equiv E\sin\omega t$$

that is

$$L\omega\sum_{n=1}^{\infty} nI_n\cos(n\omega t-\phi_n) + r\sum_{n=1}^{\infty} I_n\sin(n\omega t-\phi_n)$$
$$+\frac{1}{2}pL\omega\sum_{n=1}^{\infty} nI_n[\cos(\overline{n-2}\,\omega t-\phi_n)+\cos(\overline{n+2}\,\omega t-\phi_n)]$$
$$-pL\omega\sum_{n=1}^{\infty} I_n[\cos(\overline{n-2}\,\omega t-\phi_n)-\cos(\overline{n+2}\,\omega t-\phi_n)] \equiv E\sin\omega t$$

that is

$$I_1[L\omega\cos(\omega t-\phi_1)+r\sin(\omega t-\phi_1)] - \frac{1}{2}pL\omega I_1\cos(\omega t+\phi_1) + \frac{1}{2}pL\omega I_3\cos(\omega t-\phi_3)$$
$$+ I_2[2L\omega\cos(2\omega t-\phi_2)+r\sin(2\omega t-\phi_2)] + pL\omega I_4\cos(2\omega t-\phi_4)$$
$$+ L\omega\sum_{n=3}^{\infty} nI_n\cos(n\omega t-\phi_n) + r\sum_{n=3}^{\infty} I_n\sin(n\omega t-\phi_n)$$
$$+\frac{1}{2}pL\omega\sum_{n=3}^{\infty}[I_{n+2}n\cos(n\omega t-\phi_{n+2}) + I_{n-2}n\cos(n\omega t-\phi_{n-2})]$$
$$\equiv E\sin\omega t$$

so that

$$L\omega I_1 \cos\phi_1 - rI_1 \sin\phi_1 - \frac{1}{2} pL\omega I_1 \cos\phi_1 + \frac{1}{2} pL\omega I_3 \cos\phi_3 = 0 \quad \ldots(1)$$

$$L\omega I_1 \sin\phi_1 + rI_1 \cos\phi_1 + \frac{1}{2} pL\omega I_1 \sin\phi_1 + \frac{1}{2} pL\omega I_3 \sin\phi_3 = E \ldots(1')$$

$$2L\omega I_2 \cos\phi_2 - rI_2 \sin\phi_2 + \frac{1}{2}. 2pL\omega I_4 \cos\phi_4 = 0 \quad \ldots(2)$$

$$2L\omega I_2 \sin\phi_2 + rI_2 \cos\phi_2 + \frac{1}{2}. 2pL\omega I_4 \sin\phi_4 = 0 \quad \ldots(2')$$

$$3L\omega I_3 \cos\phi_3 - rI_3 \sin\phi_3 + \frac{1}{2}. 3pL\omega (I_5 \cos\phi_5 + I_1 \cos\phi_1) = 0 \quad \ldots(3)$$

$$3L\omega I_3 \sin\phi_3 + rI_3 \cos\phi_3 + \frac{1}{2}. 3pL\omega (I_5 \sin\phi_5 + I_1 \sin\phi_1) = 0 \quad \ldots(3')$$

$$4L\omega I_4 \cos\phi_4 - rI_4 \sin\phi_4 + \frac{1}{2}. 4pL\omega (I_6 \cos\phi_6 + I_2 \cos\phi_2) = 0 \quad \ldots(4)$$

$$4L\omega I_4 \sin\phi_4 + rI_4 \cos\phi_4 + \frac{1}{2}. 4pL\omega (I_6 \sin\phi_6 + I_2 \sin\phi_2) = 0 \quad \ldots(4')$$

etc.

$$nL\omega I_n \cos\phi_n - rI_n \sin\phi_n + \frac{1}{2} npL\omega (I_{n+2} \cos\phi_{n+2} + I_{n-2} \cos\phi_{n-2}) = 0$$

$$nL\omega I_n \sin\phi_n + rI_n \cos\phi_n + \frac{1}{2} npL\omega (I_{n+2} \sin\phi_{n+2} + I_{n-2} \sin\phi_{n-2}) = 0$$

etc.

so that adding (1′), (2′), (3′) etc. multiplied by j to (1), (2), (3) etc. and dividing by $\frac{1}{2} pL\omega I_1 \epsilon^{j\phi_1}$, $\frac{1}{2}. 2pL\omega I_2 \epsilon^{j\phi_2}$, $\frac{1}{2}. 3pL\omega I_3 \epsilon^{j\phi_3}$ etc. respectively, we have

$$\frac{2}{p}\left(1 + j\frac{r}{L\omega}\right) - \epsilon^{-j2\phi_1} + \frac{I_3 \epsilon^{j\phi_3}}{I_1 \epsilon^{j\phi_1}} = \frac{2}{pL\omega} . \frac{jE}{I_1 \epsilon^{j\phi_1}} \quad \ldots\ldots\ldots\ldots(\mathrm{I})$$

$$\frac{2}{p}\left(1 + j\frac{r}{2L\omega}\right) + \frac{I_4 \epsilon^{j\phi_4}}{I_2 \epsilon^{j\phi_2}} = 0 \quad \ldots\ldots\ldots\ldots(\mathrm{II})$$

$$\frac{2}{p}\left(1 + j\frac{r}{3L\omega}\right) + \frac{I_5 \epsilon^{j\phi_5}}{I_3 \epsilon^{j\phi_3}} + \frac{I_1 \epsilon^{j\phi_1}}{I_3 \epsilon^{j\phi_3}} = 0 \quad \ldots\ldots\ldots\ldots(\mathrm{III})$$

$$\frac{2}{p}\left(1 + j\frac{r}{4L\omega}\right) + \frac{I_6 \epsilon^{j\phi_6}}{I_4 \epsilon^{j\phi_4}} + \frac{I_2 \epsilon^{j\phi_2}}{I_4 \epsilon^{j\phi_4}} = 0 \quad \ldots\ldots\ldots\ldots(\mathrm{IV})$$

etc.

$$\frac{2}{p}\left(1 + j\frac{r}{nL\omega}\right) + \frac{I_{n+2} \epsilon^{j\phi_{n+2}}}{I_n \epsilon^{j\phi_n}} + \frac{I_{n-2} \epsilon^{j\phi_{n-2}}}{I_n \epsilon^{j\phi_n}} = 0$$

etc.

Therefore taking up the relations (III) (V) (VII) etc. we have

$$\frac{-I_1\epsilon^{j\phi_1}}{I_3\epsilon^{j\phi_3}}=s_3$$

where $$s_3=\frac{2}{p}\left(1+j\frac{r}{3L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{5L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{7L\omega}\right)-\text{etc.}}}$$

$$\frac{-I_3\epsilon^{j\phi_3}}{I_5\epsilon^{j\phi_5}}=s_5$$

where $$s_5=\frac{2}{p}\left(1+j\frac{r}{5L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{7L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{9L\omega}\right)-\text{etc.}}}$$

etc.

and from the relation (I)

$$s_1-\epsilon^{-j2\phi_1}=\frac{2}{pL\omega}\cdot\frac{jE}{I_1\epsilon^{j\phi_1}}$$

that is

$$s_1I_1\epsilon^{j\phi_1}-I_1\epsilon^{-j\phi_1}=\frac{2}{pL\omega}\cdot jE$$

that is

$$(s_1-1)\,I_1\cos\phi_1+j\,(s_1+1)\,I_1\sin\phi_1=\frac{2}{pL\omega}\cdot jE$$

where $$s_1=\frac{2}{p}\left(1+j\frac{r}{L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{3L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{5L\omega}\right)-\text{etc.}}}$$

so that if $s_1=a+jb$, then

$$I_1\left[(a-1)\cos\phi_1-b\sin\phi_1\right]+jI_1\left[b\cos\phi_1+(a+1)\sin\phi_1\right]=\frac{2}{pL\omega}\cdot jE$$

accordingly

$$I_1\sin\phi_1=\frac{a-1}{a^2+b^2-1}\cdot\frac{2}{pL\omega}\cdot E$$

and

$$I_1\cos\phi_1=\frac{b}{a^2+b^2-1}\cdot\frac{2}{pL\omega}\cdot E$$

Next taking up the relations (II) (IV) (VI) etc. we have

$$\frac{0}{I_2\epsilon^{j\phi_2}}=s_2$$

where $$s_2=\frac{2}{p}\left(1+j\frac{r}{2L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{4L\omega}\right)-\frac{1}{\frac{2}{p}\left(1+j\frac{r}{6L\omega}\right)-\text{etc.}}}$$

$$\frac{-I_2\epsilon^{j\phi_2}}{I_4\epsilon^{j\phi_4}}=s_4$$

where $$s_4=\frac{2}{p}\left(1+j\frac{r}{4L\omega}\right)-\cfrac{1}{\frac{2}{p}\left(1+j\frac{r}{6L\omega}\right)-\cfrac{1}{\frac{2}{p}\left(1+j\frac{r}{8L\omega}\right)-\text{etc.}}}$$

etc.

so that $$I_2\epsilon^{j\phi_2}=I_4\epsilon^{j\phi_4}=I_6\epsilon^{j\phi_6}=\text{etc.}=0$$

that is $$I_2=I_4=I_6=\text{etc.}=0$$

Here note that the component harmonics of the armature current i found above diminish in amplitude with the orders of the harmonics, which can be proved as follows.—

Since s_{2m+1} may be written in the form

$$(a+jb)-\frac{1}{c+jd}\quad\text{that is}\quad\left(a-\frac{c}{c^2+d^2}\right)+j\left(b+\frac{d}{c^2+d^2}\right)$$

where $$a=\frac{2}{p}$$

we can say that $[s_{2m+1}]>1$ when $a-\dfrac{c}{c^2+d^2}>1$

that is when $$\frac{2}{p}-1>\frac{1}{c}$$

But $$\frac{2}{p}-1>1\quad\text{for}\quad p<1$$

Therefore $[s_{2m+1}]>1$ when $c>1$

But now

$$c+jd=(e+jf)-\frac{1}{g+jh}\quad\text{where}\quad e=\frac{2}{p}$$

so that similarly as above we can say that if $g>1$ then $c>1$.

Thus proceeding the condition sufficient for $[s_{2m+1}]>1$ is that the real term of s_{2m+1} is >1 when m is sufficiently large.

But, when m is sufficiently large, s_{2m+1} becomes

$$s_{2m+1}=\frac{2}{p}-\cfrac{1}{\frac{2}{p}-\cfrac{1}{\frac{2}{p}-\text{etc. to }\infty}}$$

$$=\frac{1}{p}+\sqrt{\frac{1}{p^2}-1}$$

which is >1 for $p<1$.

Therefore $[s_{2m+1}] > 1$ in general. And since $[I_{2m+1}] = \frac{[I_{2m-1}]}{[s_{2m+1}]}$, we can conclude that the component harmonics diminish in amplitude with the orders of the harmonics.

§ 10. Permanent short-circuit currents of the distortionless alternators.

(*a*) Distortionless alternator with two field windings.

Usually a_1 is negligible compared with $b_1\omega$, so that $\rho \doteqdot c/2b_1$.

Therefore if we denote the internal resistance and inductance of the armature by a_i and b_i respectively, then under the assumption that the direct field only is excited, the expression of the short-circuit current x_s in the armature is

$$x_s = \frac{d}{\sqrt{\left(a_i + \frac{c^2}{4b_1^2}a_1\right)^2 + b_i^2\left(1 - \frac{c^2}{2b_ib_1}\right)^2\omega^2}} \cdot \sin\left(\omega t - \tan^{-1}\frac{b_i\left(1 - \frac{c^2}{2b_ib_1}\right)\omega}{a_i + \frac{c^2}{4b_1^2}a_1}\right)$$

But

$$1 - \frac{c^2}{2b_ib_1} = 1 - \frac{1}{2\nu\nu_f} = 1 - \frac{1}{2}(1-\sigma) = \frac{1}{2}(1+\sigma) \text{ where } \sigma = 1 - \frac{1}{\nu\nu_f}$$

and

$$\frac{c^2}{4b_1^2}a_1 = \frac{1}{4}\cdot\frac{c^2}{b_ib_1}\cdot\frac{b_i}{b_1}a_1 = \frac{1}{4}(1-\sigma)\,b_i\cdot\frac{a_1}{b_1}\cdot$$

so that

$$\frac{a_i + \frac{c^2}{4b_1^2}a_1}{b_i\left(1 - \frac{c^2}{2b_ib_1}\right)\omega} = \frac{\frac{a_i}{b_i\omega} + \frac{1}{4}(1-\sigma)\frac{a_1}{b_1\omega}}{\frac{1}{2}(1+\sigma)}$$

accordingly $a_i + \frac{c^2}{4b_1^2}a_1$ is negligible compared with $b_i\left(1 - \frac{c^2}{2b_ib_1}\right)\omega$ and the expression of the short-circuit current in the armature is

$$x_s = \frac{2d}{b_i\omega(1+\sigma)}\cdot\sin\left(\omega t - \frac{\pi}{2}\right) = \frac{-2d}{b_i\omega(1+\sigma)}\cdot\cos\omega t$$

and those of the alternating currents in the direct and cross field windings are

$$y_s = \frac{c}{2b_1} \cdot \frac{2d}{b_i \omega (1+\sigma)} \cdot \sin\left(2\omega t - \frac{\pi}{2} - \frac{\pi}{2} - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

$$= I_f \cdot \frac{1-\sigma}{1+\sigma} \cdot \sin\left(2\omega t - 3\frac{\pi}{2}\right)$$

$$= I_f \cdot \frac{1-\sigma}{1+\sigma} \cdot \cos 2\omega t$$

and

$$z_s = I_f \cdot \frac{1-\sigma}{1+\sigma} \cdot \sin(2\omega t - 2\pi) = I_f \cdot \frac{1-\sigma}{1+\sigma} \cdot \sin 2\omega t$$

These expressions coincide with those obtained by Mr Boucherot*.

In the case when both direct and cross field windings are excited, we have

$$X_s = \frac{2\sqrt{2}d}{b_i \omega (1+\sigma)} \quad \text{and} \quad Y_s = Z_s = \sqrt{2} I_f \cdot \frac{1-\sigma}{1+\sigma}$$

But here note that for the same amplitude of the induced E.M.F. the exciting current in this case is $\frac{1}{\sqrt{2}}$ times that when the direct field only is excited, and therefore the amplitude of the field and armature current at short-circuit is just the same whether one or both field windings are excited.

(*b*) **Distortionless alternator with single field winding.**

In this case if the pole be non-salient then the short-circuit current x_s in the armature is

$$x_s = \frac{d}{b_i \omega} \sin\left(\omega t - \frac{\pi}{2}\right) = \frac{-d}{b_i \omega} \cos \omega t$$

If the pole be salient and $p = \frac{1}{2}$ (see Art. 9), then since

$$s_1 = \cfrac{2}{p} - \cfrac{1}{\cfrac{2}{p} - \cfrac{1}{\cfrac{2}{p} - \text{etc. to } \infty}} = \frac{1}{p} + \sqrt{\frac{1}{p^2} - 1} = 3{\cdot}73$$

we have

$$I_1 \sqrt{2} \sin \phi_1 = \frac{3{\cdot}73 - 1}{3{\cdot}73^2 - 1} \cdot \frac{4}{b_i \omega} \cdot d = 0{\cdot}846 \frac{d}{b_i \omega}$$

$$I_1 \sqrt{2} \cos \phi_1 = 0$$

* See footnote p. 183.

$$I_3\sqrt{2}\sin\phi_3 = \frac{-I_1\sqrt{2}\sin\phi_1}{3{\cdot}73} = -0{\cdot}247\frac{d}{b_i\omega}$$

$$I_3\sqrt{2}\cos\phi_3 = 0$$

$$I_5\sqrt{2}\sin\phi_5 = \frac{-I_3\sqrt{2}\sin\phi_3}{3{\cdot}73} = 0{\cdot}0663\frac{d}{b_i\omega}$$

$$I_5\sqrt{2}\cos\phi_5 = 0$$

etc.

so that

$$x_s = -0{\cdot}846\frac{d}{b_i\omega}\left(\cos\omega t - \frac{1}{3{\cdot}73}\cos 3\omega t + \frac{1}{3{\cdot}73^2}\cos 5\omega t - \text{etc.}\right)$$

(*c*) **Comparison of the magnitudes of the permanent short-circuit currents of the ordinary and distortionless alternators.**

As will be explained later in the theory of the single phase generator, the maximum value of the short-circuit current in the armature of the ordinary single phase generator with non-salient and laminated poles and with simple sine flux distribution along the air gap is

$$(x_s)_{\text{max.}} = \frac{d}{b_i\omega}\cdot\frac{1}{\sqrt{\sigma}}$$

and the maximum values of the alternating current in the field circuit are

$$(y_s)_{\text{positive max.}} = I_f\frac{1-\sqrt{\sigma}}{\sqrt{\sigma}}$$

$$(y_s)_{\text{negative max.}} = -I_f(1-\sqrt{\sigma})$$

so that the maximum value of the field current including the exciting current is

$$\frac{1}{\sqrt{\sigma}}I_f$$

and the effective values are

$$(x_s)_{\text{eff.}} = \frac{1}{\sqrt{2}}\cdot\sigma^{-\frac{1}{2}}\frac{d}{b_i\omega}$$

$$(y_s)_{\text{eff.}} = \frac{1}{\sqrt{2}}\cdot\sigma^{-\frac{1}{2}}(1-\sqrt{\sigma})I_f$$

Thus the ratio of the maximum values of the short-circuit currents in the armature of the ordinary alternator with non-salient poles and the distortionless alternators with two and single field winding is

$$\frac{1}{\sqrt{\sigma}} : \frac{2}{1+\sigma} : 1$$

which is $\quad 1{\cdot}414 : 1{\cdot}333 : 1$ when $\sigma = 0{\cdot}5$

and $\quad 1{\cdot}58 \ : 1{\cdot}43 \ : 1$,, $\sigma = 0{\cdot}4$

and $\quad 3{\cdot}16 \ : 1{\cdot}82 \ : 1$,, $\sigma = 0{\cdot}1$

The ratio of the maximum values of the field currents including the exciting current I_f is

$$\frac{1-\sqrt{\sigma}}{\sqrt{\sigma}} + 1 : \frac{1-\sigma}{1+\sigma} + 1 : 1 \quad \text{that is} \quad \frac{1}{\sqrt{\sigma}} : \frac{2}{1+\sigma} : 1$$

which is the same as the ratio of the armature currents.

The ratio of the effective values of the armature currents is

$$\sigma^{-\frac{1}{4}} : \frac{2}{1+\sigma} : 1$$

which is $\quad 1{\cdot}19 : 1{\cdot}333 : 1$ when $\sigma = 0{\cdot}5$

and $\quad 1{\cdot}26 : 1{\cdot}43 \ : 1$,, $\sigma = 0{\cdot}4$

and $\quad 1{\cdot}78 : 1{\cdot}82 \ : 1$,, $\sigma = 0{\cdot}1$

The ratio of the effective values of the field currents including the exciting current is

$$\sqrt{1 + \frac{1}{2}\sigma^{-\frac{1}{2}}(1-\sqrt{\sigma})^2} : \sqrt{1 + \frac{1}{2}\left(\frac{1-\sigma}{1+\sigma}\right)^2} : 1$$

that is

$$\frac{1}{\sqrt{2}}\sigma^{-\frac{1}{4}}\sqrt{1+\sigma} : \sqrt{1 + \frac{1}{2}\left(\frac{1-\sigma}{1+\sigma}\right)^2} : 1$$

which is $\quad 1{\cdot}031 : 1{\cdot}055 : 1$ when $\sigma = 0{\cdot}5$

and $\quad 1{\cdot}053 : 1{\cdot}092 : 1$,, $\sigma = 0{\cdot}4$

and $\quad 1{\cdot}321 : 1{\cdot}335 : 1$,, $\sigma = 0{\cdot}1$

§ 11. Terminal voltage and voltage regulation of the distortionless alternator with two field windings.

Let a_i and b_i be the internal resistance and inductance of the armature, a_e and b_e the external ones at full load, and a_1 and b_1 those of the field circuit.

Then as shown in Arts. 2 and 4 the expression of the full load current I (effective) is

$$I = \frac{E}{\sqrt{(a_e + a_i + \rho^2 a_1)^2 + (b_e + b_i - 2\rho^2 b_1)^2 \omega^2}}$$

$$= \frac{E}{\sqrt{(a_e + a_i + \rho^2 a_1)^2 + \left[b_e + \frac{1}{2} b_i (1 + \sigma)\right]^2 \omega^2}}$$

where E is the effective value of the induced E.M.F.

But the expression of the terminal voltage V (effective) is

$$V = I\sqrt{a_e^2 + b_e^2\omega^2} = I a_e \sec\phi = I b_e \omega \operatorname{cosec}\phi$$

where ϕ is the phase angle between the full load current and the terminal voltage, that is $\tan^{-1}(b_e\omega/a_e)$

Accordingly $\quad Ia_e = V\cos\phi \quad$ and $\quad Ib_e\omega = V\sin\phi$

Therefore if we denote $a_i + \rho^2 a_1$ by a_{i1}, $\frac{1}{2} b_i (1 + \sigma)$ by b_{i1}, $\sqrt{a_{i1}^2 + b_{i1}^2\omega^2}$ by z_{i1} and $\tan^{-1}(b_{i1}\omega/a_{i1})$ by α, then

$$(V\cos\phi + Iz_{i1}\cos\alpha)^2 + (V\sin\phi + Iz_{i1}\sin\alpha)^2 = E^2$$

that is

$$V^2 + 2VIz_{i1}\cos(\alpha - \phi) + I^2 z_{i1}^2 - E^2 = 0$$

so that the expression of the terminal voltage V (effective) is

$$V = -Iz_{i1}\cos(\alpha - \phi) + \sqrt{E^2 - I^2 z_{i1}^2 \sin^2(\alpha - \phi)}$$

In the case when the load circuit contains in series a condenser of which the capacity is C, then as shown in Art. 6 we have

$$I = \frac{E}{\sqrt{(a_e + a_i + \rho^2 a_1)^2 + \left[b_e\omega - \frac{1}{C\omega} + \frac{1}{2} b_i (1 + \sigma)\omega^2\right]^2}}$$

which, when $b_e\omega > \frac{1}{C\omega}$, gives the same expression of the terminal voltage V as above, the difference being only that $\left(b_e\omega - \frac{1}{C\omega}\right)$ is to be put in place of $b_e\omega$, but when $b_e\omega < \frac{1}{C\omega}$ gives the expression

$$V = -Iz_{i1}\cos(\alpha + \phi) + \sqrt{E^2 - I^2 z_{i1}^2 \sin^2(\alpha + \phi)}$$

where

$$\phi = \tan^{-1}\frac{\frac{1}{C\omega} - b_e\omega}{a_e}$$

Here note that, since the expressions of the load current I from Arts. 2, 4 and 6 show that with regard to the load current we can consider, in place of the field and armature coils interlinked magnetically, a simple circuit having resistance $a_i + \rho^2 a_1$ and reactance $\frac{1}{2} b_i \omega (1 + \sigma)$ put in series with the load circuit, we can immediately write down the above expressions of the terminal voltage V from the diagrams shown in figures 2 and 3.

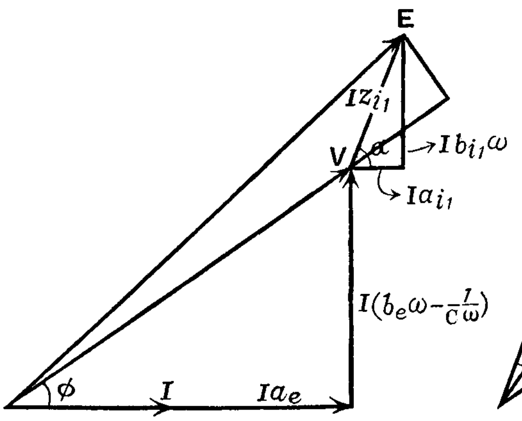


Fig. 2.

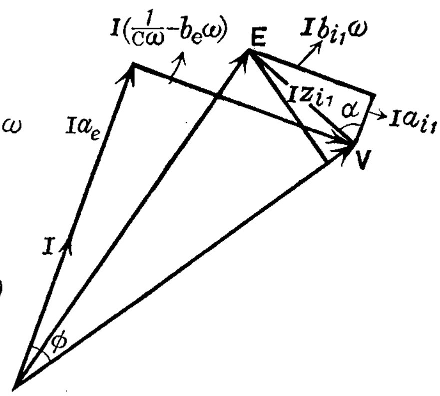


Fig. 3.

Next in order to find the expression of voltage regulation, let the ratio of the short-circuit current to the full load current be m. Then first considering the case when the load circuit contains no condenser, we have

$$m = \sqrt{\frac{(a_e + a_{i1})^2 + (b_e \omega + b_{i1} \omega)^2}{a_{i1}^2 + b_{i1}^2 \omega^2}}$$

that is

$$(a_e + a_{i1})^2 + (a_e \tan \phi + b_{i1} \omega)^2 = m^2 (a_{i1}^2 + b_{i1}^2 \omega^2)$$

that is

$$a_e^2 \sec^2 \phi + 2a_e . (a_{i1} + b_{i1} \omega \tan \phi) - (m^2 - 1)(a_{i1}^2 + b_{i1}^2 \omega^2) = 0$$

so that

$$a_e \sec \phi = -(a_{i1} \cos \phi + b_{i1} \omega \sin \phi) + \sqrt{m^2 (a_{i1}^2 + b_{i1}^2 \omega^2) - (a_{i1} \sin \phi - b_{i1} \omega \cos \phi)^2}$$

But now

$$V = I a_e \sec \phi$$

and

$$E = I \sqrt{(a_e + a_{i1})^2 + (b_e \omega + b_{i1} \omega)^2} = I . m . \sqrt{a_{i1}^2 + b_{i1}^2 \omega^2}$$

Therefore

$$\frac{V}{E}=\frac{-(a_{i1}\cos\phi+b_{i1}\omega\sin\phi)+\sqrt{m^2(a_{i1}^2+b_{i1}^2\omega^2)-(a_{i1}\sin\phi-b_{i1}\omega\cos\phi)^2}}{m\sqrt{a_{i1}^2+b_{i1}^2\omega^2}}$$

$$=\frac{-\left(\frac{a_{i1}}{b_{i1}\omega}\cos\phi+\sin\phi\right)+\sqrt{m^2\left(1+\frac{a_{i1}^2}{b_{i1}^2\omega^2}\right)-\left(\frac{a_{i1}}{b_{i1}\omega}\sin\phi-\cos\phi\right)^2}}{m\sqrt{1+\frac{a_{i1}^2}{b_{i1}^2\omega^2}}}$$

But $\frac{a_{i1}^2}{b_{i1}^2\omega^2}$ is usually negligible compared with unity.

Therefore

$$\frac{V}{E}\doteqdot\frac{1}{m}\cdot\left(\frac{a_{i1}}{b_{i1}\omega}\cos\phi+\sin\phi\right)+\sqrt{1-\frac{1}{m^2}\left(\frac{a_{i1}}{b_{i1}\omega}\sin\phi-\cos\phi\right)^2}$$

Thus V/E having been found, the voltage regulation $(E-V)/V$ can now be easily calculated.

If the load circuit contains capacity, the same expression of V/E as above holds when $b_e\omega>1/C\omega$; but when $1/C\omega>b_e\omega$ the expression of V/E should read

$$\frac{V}{E}=\frac{-\left(\frac{a_{i1}}{b_{i1}\omega}\cos\phi-\sin\phi\right)+\sqrt{m^2\left(1+\frac{a_{i1}^2}{b_{i1}^2\omega^2}\right)-\left(\frac{a_{i1}}{b_{i1}\omega}\sin\phi+\cos\phi\right)^2}}{m\sqrt{1+\frac{a_{i1}^2}{b_{i1}^2\omega^2}}}$$

$$\doteqdot-\frac{1}{m}\cdot\left(\frac{a_{i1}}{b_{i1}\omega}\cos\phi-\sin\phi\right)+\sqrt{1-\frac{1}{m^2}\left(\frac{a_{i1}}{b_{i1}\omega}\sin\phi+\cos\phi\right)^2}$$

Here note that if the current, which flows in the armature when field windings are open and E volt (effective) is applied to the armature from some external source, be m_1 times the full load current, that is

$$m_1I=\frac{E}{\sqrt{a_i^2+b_i^2\omega^2}}\doteqdot\frac{E}{b_i\omega}$$

then since

$$mI=\frac{E}{\sqrt{a_{i1}^2+b_{i1}^2\omega^2}}\doteqdot\frac{E}{\frac{1}{2}b_i\omega(1+\sigma)}$$

we have

$$m=m_1\frac{2}{1+\sigma}$$

Thus m and accordingly m_1 is an important factor affecting the voltage regulation; the greater the value of m and m_1, the smaller being the voltage regulation, and the smaller the value of m and m_1, the greater the voltage regulation. But now the magnitude of m_1 and accordingly m depends mainly upon the value of b_i and therefore upon the radial length of the air gap of the alternator. Therefore the radial length of the air gap is an important factor affecting the voltage regulation; the larger the gap length, the smaller being b_i so that the greater the value of m and m_1 and hence the smaller the voltage regulation; and the shorter the gap length, the larger being b_i so that the smaller the value of m_1 and m and hence the greater the voltage regulation.

§ 12. Evaluation of the ratio of resistance to reactance of the field and armature windings.

Let E be the induced E.M.F. (effective) and I the full load armature current (effective). Then since

$$\frac{a_i}{b_i\omega} = \frac{Ia_i}{E}\frac{\frac{E}{b_i\omega}}{I} = m_1\frac{Ia_i}{E}$$

we can say that $a_i/b_i\omega$ is the product of the ratio of the resistance drop to the induced E.M.F. and the ratio of $E/b_i\omega$ to the full load current. Note that this $E/b_i\omega$ is the current which flows in the armature when the field circuit is open and E volt is applied from some external source to the armature.

Next

$$m_1 I = \frac{1}{\sqrt{2}}\frac{cI_f\omega}{\sqrt{a_i^2+b_i^2\omega^2}} \doteqdot \frac{1}{\sqrt{2}}\cdot\frac{cI_f\omega}{b_i\omega} = \frac{1}{\sqrt{2}}\cdot\frac{c}{b_i}\cdot I_f = \frac{1}{\sqrt{2}}\cdot\frac{n_f}{n}\cdot\frac{1}{\nu}\cdot I_f$$

where ν is the armature leakage coefficient, and n_f and n are the field and armature number of turns.

Therefore $n_f I_f = \sqrt{2}\, m_1 \nu n I$ that is ampereconductors of the direct field winding due to the exciting current is $\sqrt{2}\, m_1\nu$ times that of the armature winding.

For obtaining the total ampereconductors in the direct field winding, however, we have to consider also the induced current ρI which is $\doteqdot \frac{c}{2b_1}I = \frac{1}{2}\cdot\frac{n}{n_f}\cdot\frac{1}{\nu_f}\cdot I$ where ν_f is the field leakage coefficient.

Therefore the total ampereconductors in the direct field winding is

$$nI\sqrt{2m_1^2\nu^2 + \frac{1}{4\nu_f^2}} \quad \text{that is} \quad \sqrt{2}m\nu nI\sqrt{1 + \frac{1}{8m_1^2\nu^2\nu_f^2}}$$

that is

$$\sqrt{2}m_1\nu nI\sqrt{1 + \frac{1}{8}\cdot\left(\frac{1-\sigma}{m_1}\right)^2}$$

But $(1-\sigma)^2$ is usually small compared with $8m_1^2$. Therefore we can for approximation neglect the second term and take up $\sqrt{2}m_1\nu nI$ as the total ampereconductors in the direct field winding.

Thus if the same amount of the ampereconductors per unit length along the circumference is adopted for the field and armature winding, then the space for the direct field winding must be $\sqrt{2}m_1\nu$ times that for the armature winding. But owing to the smaller iron loss in the field core and better cooling of the field winding we can take the ampereconductors per unit length of the field winding much higher than that of the armature winding. If the ratio of the ampereconductors per unit length of the field winding to that of the armature winding be equal to K then the space for the direct field winding will be $\sqrt{2}m_1\nu/K$ times that for the armature winding.

Now both resistance and reactance are proportional to the square of the number of turns for fixed space, so that if the space for winding be constant, then the ratio of the resistance to reactance of the winding remains unchanged with the number of turns in the winding; while change of the winding space causes inversely proportional change of the resistance of the winding. Therefore neglecting a slight change of reluctance accompanying the change of the winding space, we can say that the ratio of $a_1/b_1\omega$ to $a_i/b_i\omega$ is equal to the reciprocal of the ratio of the winding space for the field winding to that of the armature winding, that is equal to $K/\sqrt{2}m_1\nu$; and hence finally we have

$$\frac{a_1}{b_1\omega} = \frac{K}{\sqrt{2}\nu}\cdot\frac{Ia_i}{E}$$

Note that in the above we assume that the circumference occupied by one field winding, direct or cross, is equal to that occupied by the armature winding and also assume that the direct field winding only is excited. If both field windings be excited then, since $\frac{1}{\sqrt{2}}$ times that exciting current when the direct field winding only is excited will do for the same induced E.M.F., we have the relation $n_f I_f = m\nu n I$ in place of $n_f I_f = \sqrt{2} m\nu n I$. But, since we have then heat of an equal amount produced in the two field windings, the ampere-turns per unit length in the field winding should in this case be taken smaller than that when the direct field winding only is excited. Thus taking K' in place of K in the previous case we have here

$$\frac{a_1}{b_1\omega} = \frac{K'}{\nu} \cdot \frac{Ia_i}{E}$$

Next proceeding to calculate $\frac{a_{i1}}{b_{i1}\omega}$ that is $\dfrac{a_i + \rho^2 a_1}{\frac{1}{2} b_i \omega (1+\sigma)}$

it is

$$= \frac{\frac{a_i}{b_i\omega} + \frac{c^2}{4b_1^2} \cdot \frac{a_1}{b_i\omega}}{\frac{1}{2}(1+\sigma)} = \frac{\frac{a_i}{b_i\omega} + \frac{1}{4}\frac{c^2}{b_i b_1} \cdot \frac{a_1}{b_1\omega}}{\frac{1}{2}(1+\sigma)}$$

$$= \frac{\frac{a_i}{b_i\omega}\left[1 + \frac{1}{4}(1-\sigma)\frac{K}{2m\nu}\right]}{\frac{1}{2}(1+\sigma)}.$$

As an example if we take the case when $\frac{Ia_i}{E} = \frac{1}{100}$, $\sigma = 0{\cdot}4$, $\frac{K}{2\nu} = 1$ then for $m_1 = 4$, 3 and 2 we have the results as shown in the following table.

	$\frac{a_1}{b_1\omega}$	$\frac{a_i}{b_i\omega}$	$\frac{a_{i1}}{b_{i1}\omega}$	m	Voltage regulation in % for		
					$\cos\phi = 1$	$\cos\phi = 0{\cdot}9$ lagging	$\cos\phi = 0{\cdot}8$ lagging
$m_1 = 4$	$\frac{1}{100}$	$\frac{1}{25}$	$\frac{1}{17}$	5·7	2·7	10·7	14·0
$m_1 = 3$	$\frac{1}{100}$	$\frac{1}{33}$	$\frac{1}{22}$	4·28	4·0	15·3	19·8
$m_1 = 2$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{1}{33}$	2·9	7·8	26·0	33·8

§ 13. Terminal voltage and voltage regulation of the distortionless alternator with single field winding.

Similarly as in Art. 11, the expression of the terminal voltage V is, when the load current is lagging,

$$V = -Iz_i \cos(\alpha - \phi) + \sqrt{E^2 - I^2 z_i \sin(\alpha - \phi)}$$

where $\quad z_i = \sqrt{a_i^2 + b_i^2 \omega^2}$ and $\alpha = \tan^{-1} \frac{b_i \omega}{a_i}$

and the ratio of the terminal voltage to the induced E.M.F. is

$$\frac{V}{E} \doteqdot -\frac{1}{m} \cdot \left(\frac{a_i}{b_i \omega} \cos\phi + \sin\phi \right) + \sqrt{1 - \frac{1}{m^2} \cdot \left(\frac{a_i}{b_i \omega} \sin\phi - \cos\phi \right)^2}$$

where m is as said before the ratio of the short-circuit current to the full load current, and is here specially equal to m_1, for the short-circuit current is in this case equal to $E/b_i \omega$.

In the case when the load current is leading we have to put $-\phi$ in place of ϕ in the above expressions of V and V/E.

As an example if we take the case when $\frac{Ia_i}{E} = \frac{1}{100}$, as in the previous article, then for $m = 6$, 4 and 3 we have the results as shown in the following table.

	$\frac{a_1}{b_1 \omega}$	$\frac{a_i}{b_i \omega}$	Voltage regulation in % for		
			$\cos\phi = 1$	$\cos\phi = 0{\cdot}9$ lagging	$\cos\phi = 0{\cdot}8$ lagging
$m = m_1 = 6$	$\frac{1}{100}$	$\frac{3}{50}$	2·5	10·2	13·4
$m = m_1 = 4$	$\frac{1}{100}$	$\frac{1}{25}$	4·2	15·2	22·0
$m = m_1 = 3$	$\frac{1}{100}$	$\frac{1}{33}$	7·2	24·0	32·0

Comparing this with that in the previous article, we see that for the same value of m_1, the distortionless alternator with two field windings has a smaller, that is a better, voltage regulation compared with that with single field winding.

Chapter II.

Transient Phenomena.

§ 14. Complete solution of the distortionless alternator with two field windings.

As explained in Art. 4 the complete solution of the armature current x is

$$x = X\sqrt{2}\sin(\omega t - \phi_x) + A\epsilon^{\alpha t} + B\epsilon^{\beta t} + C\epsilon^{\gamma t}$$

where A, B and C are the arbitrary constants and α, β and γ are the roots of

$$b_1(bb_1 - c^2)\lambda^3 + (ab_1^2 + 2a_1bb_1 - a_1c^2)\lambda^2 + [2aa_1b_1 + a_1^2b + b_1(bb_1 - c^2)\omega^2]\lambda$$
$$+ a(a_1^2 + b_1^2\omega^2) = 0$$

Now putting this complete solution of x in the fundamental equation of y we have

$$a_1y + b_1\frac{dy}{dt} + c\frac{d}{dt}.\{[X\sqrt{2}\sin(\omega t - \phi_x) + A\epsilon^{\alpha t} + B\epsilon^{\beta t} + C\epsilon^{\gamma t}]\cos\omega t\} = 0$$

that is

$$a_1y + b_1\frac{dy}{dt} + cX\sqrt{2}\,\omega\cos(2\omega t - \phi_x) + c(A\alpha\epsilon^{\alpha t} + B\beta\epsilon^{\beta t} + C\gamma\epsilon^{\gamma t})\cos\omega t$$
$$- c\omega(A\epsilon^{\alpha t} + B\epsilon^{\beta t} + C\epsilon^{\gamma t})\sin\omega t = 0$$

that is

$$\frac{dy}{dt} + \frac{a_1}{b_1}y = -\frac{c}{b_1}X\sqrt{2}\,\omega\cos(2\omega t - \phi_x) + \frac{c}{b_1}.A\epsilon^{\alpha t}\sqrt{\alpha^2 + \omega^2}\sin\left(\omega t - \tan^{-1}\frac{\alpha}{\omega}\right)$$
$$+ \frac{c}{b_1}.B\epsilon^{\beta t}\sqrt{\beta^2 + \omega^2}\sin\left(\omega t - \tan^{-1}\frac{\beta}{\omega}\right)$$
$$+ \frac{c}{b_1}.C\epsilon^{\gamma t}\sqrt{\gamma^2 + \omega^2}\sin\left(\omega t - \tan^{-1}\frac{\gamma}{\omega}\right)$$

which solves to

$$y.\epsilon^{\frac{a_1}{b_1}t} = -\frac{c}{b_1}X\sqrt{2}\,\omega\int\epsilon^{\frac{a_1}{b_1}t}\cos(2\omega t - \phi_x)\,dt$$
$$+ \frac{c}{b_1}A.\sqrt{\alpha^2 + \omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\alpha\right)t}.\sin\left(\omega t - \tan^{-1}\frac{\alpha}{\omega}\right)dt$$
$$+ \frac{c}{b_1}B.\sqrt{\beta^2 + \omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\beta\right)t}.\sin\left(\omega t - \tan^{-1}\frac{\beta}{\omega}\right)dt$$
$$+ \frac{c}{b_1}C.\sqrt{\gamma^2 + \omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\gamma\right)t}.\sin\left(\omega t - \tan^{-1}\frac{\gamma}{\omega}\right)dt$$
$$+ K$$

that is

$$y = \frac{-cX\sqrt{2}\,\omega}{\sqrt{a_1^2 + 4b_1^2\omega^2}} \cos\left(2\omega t - \phi_x - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

$$+ \frac{cA\sqrt{\alpha^2+\omega^2}}{\sqrt{(a_1+b_1\alpha)^2+b_1^2\omega^2}} . \epsilon^{\alpha t} . \sin\left(\omega t - \tan^{-1}\frac{\alpha}{\omega} - \tan^{-1}\frac{b_1\omega}{a_1+b_1\alpha}\right)$$

$$+ \frac{cB\sqrt{\beta^2+\omega^2}}{\sqrt{(a_1+b_1\beta)^2+b_1^2\omega^2}} . \epsilon^{\beta t} . \sin\left(\omega t - \tan^{-1}\frac{\beta}{\omega} - \tan^{-1}\frac{b_1\omega}{a_1+b_1\beta}\right)$$

$$+ \frac{cC\sqrt{\gamma^2+\omega^2}}{\sqrt{(a_1+b_1\gamma)^2+b_1^2\omega^2}} . \epsilon^{\gamma t} . \sin\left(\omega t - \tan^{-1}\frac{\gamma}{\omega} - \tan^{-1}\frac{b_1\omega}{a_1+b_1\gamma}\right)$$

$$+ K\epsilon^{-\frac{a_1}{b_1}t}$$

that is

$$y = \rho X\sqrt{2}\sin\left(2\omega t - \phi_x - \frac{\pi}{2} - \tan^{-1}\frac{2b_1\omega}{a_1}\right) + A_1\epsilon^{\alpha t}\sin(\omega t - \theta_\alpha)$$

$$+ B_1\epsilon^{\beta t}\sin(\omega t - \theta_\beta) + C_1\epsilon^{\gamma t}\sin(\omega t - \theta_\gamma) + K\epsilon^{-\frac{a_1}{b_1}t}$$

where

$$A_1 = \frac{cA\sqrt{\alpha^2+\omega^2}}{\sqrt{(a_1+b_1\alpha)^2+b_1^2\omega^2}}, \quad B_1 = \frac{cB\sqrt{\beta^2+\omega^2}}{\sqrt{(a_1+b_1\beta)^2+b_1^2\omega^2}}, \quad C_1 = \frac{cC\sqrt{\gamma^2+\omega^2}}{\sqrt{(a_1+b_1\gamma)^2+b_1^2\omega^2}}$$

$$\theta_\alpha = \tan^{-1}\frac{a_1\alpha + b_1(\alpha^2+\omega^2)}{a_1\omega}, \qquad \theta_\beta = \tan^{-1}\frac{a_1\beta + b_1(\beta^2+\omega^2)}{a_1\omega},$$

$$\theta_\gamma = \tan^{-1}\frac{a_1\gamma + b_1(\gamma^2+\omega^2)}{a_1\omega}$$

Next putting the complete solution of x in the fundamental equation of z we have

$$a_1 z + b_1\frac{dz}{dt} + c\frac{d}{dt}.\{[X\sqrt{2}\sin(\omega t - \phi_x) + A\epsilon^{\alpha t} + B\epsilon^{\beta t} + C\epsilon^{\gamma t}]\sin\omega t\} = 0$$

that is

$$a_1 z + b_1\frac{dz}{dt} + cX\sqrt{2}\,\omega\sin(2\omega t - \phi_x) + c(A\alpha\epsilon^{\alpha t} + B\beta\epsilon^{\beta t} + C\gamma\epsilon^{\gamma t})\sin\omega t$$

$$+ c\omega(A\epsilon^{\alpha t} + B\epsilon^{\beta t} + C\epsilon^{\gamma t})\cos\omega t = 0$$

that is

$$\frac{dz}{dt}+\frac{a_1}{b_1}z=-\frac{c}{b_1}X\sqrt{2}\,\omega\sin(2\omega t-\phi_x)-\frac{c}{b_1}A\epsilon^{\alpha t}\sqrt{\alpha^2+\omega^2}\cos\left(\omega t-\tan^{-1}\frac{\alpha}{\omega}\right)$$
$$-\frac{c}{b_1}B\epsilon^{\beta t}\sqrt{\beta^2+\omega^2}\cos\left(\omega t-\tan^{-1}\frac{\beta}{\omega}\right)$$
$$-\frac{c}{b_1}C\epsilon^{\gamma t}\sqrt{\gamma^2+\omega^2}\cos\left(\omega t-\tan^{-1}\frac{\gamma}{\omega}\right)$$

which solves to

$$z\epsilon^{\frac{a_1}{b_1}t}=-\frac{c}{b_1}X\sqrt{2}\,\omega\int\epsilon^{\frac{a_1}{b_1}t}\sin(2\omega t-\phi_x)\,dt$$
$$-\frac{c}{b_1}A\sqrt{\alpha^2+\omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\alpha\right)t}.\cos\left(\omega t-\tan^{-1}\frac{\alpha}{\omega}\right)dt$$
$$-\frac{c}{b_1}B\sqrt{\beta^2+\omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\beta\right)t}.\cos\left(\omega t-\tan^{-1}\frac{\beta}{\omega}\right)dt$$
$$-\frac{c}{b_1}C\sqrt{\gamma^2+\omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\gamma\right)t}.\cos\left(\omega t-\tan^{-1}\frac{\gamma}{\omega}\right)dt$$
$$+K_1$$

which, similarly as before, becomes

$$z=\rho X\sqrt{2}\sin\left(2\,\omega t-\phi_x-\pi-\tan^{-1}\frac{2b_1\omega}{a_1}\right)-A_1\epsilon^{\alpha t}\cos(\omega t-\theta_\alpha)$$
$$-B_1\epsilon^{\beta t}\cos(\omega t-\theta_\beta)-C_1\epsilon^{\gamma t}\cos(\omega t-\theta_\gamma)+K_1\epsilon^{-\frac{a_1}{b_1}t}$$

where A_1, B_1, C_1, θ_α, θ_β and θ_γ denote the same things as before.

Now since

$$\tan\theta_\alpha=\frac{a_1\alpha+b_1(\alpha^2+\omega^2)}{a_1\omega}$$

we have

$$\sin\theta_\alpha=\frac{a_1\alpha+b_1(\alpha^2+\omega^2)}{\sqrt{(a_1+b_1\alpha)^2+b_1^2\omega^2}\,.\sqrt{\alpha^2+\omega^2}}$$

and therefore

$$\frac{A_1}{A}\sin\theta_\alpha=\frac{c\,[a_1\alpha+b_1(\alpha^2+\omega^2)]}{(a_1+b_1\alpha)^2+b_1^2\omega^2}$$

But now the equation giving the roots α, β and γ reads

$$c^2\lambda\left[a_1\lambda + b_1(\lambda^2+\omega^2)\right] = bb_1^2\lambda^3 + b_1(ab_1+2a_1b)\lambda^2 + (a_1^2b + 2aa_1b_1 + bb_1^2\omega^2)\lambda + a(a_1^2+b_1^2\omega^2)$$

$$= b_1^2\lambda^2(b\lambda+a) + 2a_1b_1\lambda(b\lambda+a) + (a_1^2+b_1^2\omega^2)(b\lambda+a)$$

$$= (b\lambda+a)\left[(a_1+b_1\lambda)^2 + b_1^2\omega^2\right]$$

so that

$$\frac{c\left[a_1\alpha + b_1(\alpha^2+\omega^2)\right]}{(a_1+b_1\alpha)^2 + b_1^2\omega^2} = \frac{a+b\alpha}{c\alpha}$$

Therefore we have

$$(a+b\alpha)A - c\alpha A_1 \sin\theta_\alpha = 0$$

and similarly from $\tan\theta_\beta$ and $\tan\theta_\gamma$ we have

$$(a+b\beta)B - c\beta B_1\sin\theta_\beta = 0$$

and

$$(a+b\gamma)C - c\gamma C_1\sin\theta_\gamma = 0$$

But now putting the complete solution of x, y and z in the fundamental equation of x and dropping permanent terms, we have

$$(a+b\alpha)A\epsilon^{\alpha t} + (a+b\beta)B\epsilon^{\beta t} + (a+b\gamma)C\epsilon^{\gamma t} - c\frac{d}{dt}\Big(A_1\epsilon^{\alpha t}\sin\theta_\alpha + B_1\epsilon^{\beta t}\sin\theta_\beta + C_1\epsilon^{\gamma t}\sin\theta_\gamma - K\epsilon^{-\frac{a_1}{b_1}t}\cos\omega t - K_1\epsilon^{-\frac{a_1}{b_1}t}\sin\omega t\Big) = 0$$

that is

$$\left[(a+b\alpha)A - c\alpha A_1\sin\theta_\alpha\right]\epsilon^{\alpha t} + \left[(a+b\beta)B - c\beta B_1\sin\theta_\beta\right]\epsilon^{\beta t} + \left[(a+b\gamma)C - c\gamma C_1\sin\theta_\gamma\right]\epsilon^{\gamma t}$$
$$- c\left[K\left(\frac{a_1}{b_1}\cos\omega t + \omega\sin\omega t\right) + K_1\left(\frac{a_1}{b_1}\sin\omega t - \omega\cos\omega t\right)\right]\epsilon^{-\frac{a_1}{b_1}t} = 0$$

Therefore we have

$$\left(K\frac{a_1}{b_1} - K_1\omega\right)\cos\omega t + \left(K\omega + K_1\frac{a_1}{b_1}\right)\sin\omega t = 0$$

so that

$$\left(K\frac{a_1}{b_1} - K_1\omega\right)^2 + \left(K\omega + K_1\frac{a_1}{b_1}\right)^2 = 0$$

that is

$$(K^2+K_1^2)\left(\frac{a_1^2}{b_1^2} + \omega^2\right) = 0$$

so that

$$K=0 \quad \text{and} \quad K_1=0$$

and hence the complete solutions of y and z become

$$y=\rho X\sqrt{2}\sin\left(2\omega t-\phi_x-\frac{\pi}{2}-\tan^{-1}\frac{2b_1\omega}{a_1}\right)$$
$$+A_1\epsilon^{\alpha t}\sin(\omega t-\theta_\alpha)+B_1\epsilon^{\beta t}\sin(\omega t-\theta_\beta)+C_1\epsilon^{\gamma t}\sin(\omega t-\theta_\gamma)$$

$$z=\rho X\sqrt{2}\sin\left(2\omega t-\phi_x-\pi-\tan^{-1}\frac{2b_1\omega}{a_1}\right)$$
$$-A_1\epsilon^{\alpha t}\cos(\omega t-\theta_\alpha)-B_1\epsilon^{\beta t}\cos(\omega t-\theta_\beta)-C_1\epsilon^{\gamma t}\cos(\omega t-\theta_\gamma)$$

Note that the expressions of $\frac{A_1}{A}$, $\tan\theta_\alpha$, $\frac{B_1}{B}$, $\tan\theta_\beta$, $\frac{C_1}{C}$, and $\tan\theta_\gamma$ found before can be checked as follows.

Putting the complete solutions of x and y in the fundamental equation of y and dropping the permanent terms, we have

$$a_1\left[A_1\epsilon^{\alpha t}\sin(\omega t-\theta_\alpha)+B_1\epsilon^{\beta t}\sin(\omega t-\theta_\beta)+C_1\epsilon^{\gamma t}\sin(\omega t-\theta_\gamma)+K\epsilon^{-\frac{a_1}{b_1}t}\right]$$
$$+b_1\frac{d}{dt}\left[A_1\epsilon^{\alpha t}\sin(\omega t-\theta_\alpha)+B_1\epsilon^{\beta t}\sin(\omega t-\theta_\beta)\right.$$
$$\left.+C_1\epsilon^{\gamma t}\sin(\omega t-\theta_\gamma)+K\epsilon^{-\frac{a}{b_1}t}\right]$$
$$+c\frac{d}{dt}\left[(A\epsilon^{\alpha t}+B\epsilon^{\beta t}+C\epsilon^{\gamma t})\cos\omega t\right]=0$$

that is

$$\{A_1[(a_1+b_1\alpha)\sin(\omega t-\theta_\alpha)+b_1\omega\cos(\omega t-\theta_\alpha)]+cA(\alpha\cos\omega t-\omega\sin\omega t)\}\epsilon^{\alpha t}$$
$$+\{B_1[(a_1+b_1\beta)\sin(\omega t-\theta_\beta)+b_1\omega\cos(\omega t-\theta_\beta)]+cB(\beta\cos\omega t-\omega\sin\omega t)\}\epsilon^{\beta t}$$
$$+\{C_1[(a_1+b_1\gamma)\sin(\omega t-\theta_\gamma)+b_1\omega\cos(\omega t-\theta_\gamma)]+cC(\gamma\cos\omega t-\omega\sin\omega t)\}\epsilon^{\gamma t}=0$$

so that

$$[(a_1+b_1\alpha)\cos\theta_\alpha+b_1\omega\sin\theta_\alpha]A_1-c\omega A=0$$
$$[(a_1+b_1\alpha)\sin\theta_\alpha-b_1\omega\cos\theta_\alpha]A_1-c\alpha A=0$$
$$[(a_1+b_1\beta)\cos\theta_\beta+b_1\omega\sin\theta_\beta]B_1-c\omega B=0$$
$$[(a_1+b_1\beta)\sin\theta_\beta-b_1\omega\cos\theta_\beta]B_1-c\beta B=0$$
$$[(a_1+b_1\gamma)\cos\theta_\gamma+b_1\omega\sin\theta_\gamma]C_1-c\omega C=0$$
$$[(a_1+b_1\gamma)\sin\theta_\gamma-b_1\omega\cos\theta_\gamma]C_1-c\gamma C=0$$

that is

$$\frac{A}{A_1} = \frac{(a_1 + b_1 \alpha) \cos \theta_\alpha + b_1 \omega \sin \theta_\alpha}{c\omega} = \frac{(a_1 + b_1 \alpha) \sin \theta_\alpha - b_1 \omega \cos \theta_\alpha}{c\alpha}$$

$$\frac{B}{B_1} = \frac{(a_1 + b_1 \beta) \cos \theta_\beta + b_1 \omega \sin \theta_\beta}{c\omega} = \frac{(a_1 + b_1 \beta) \sin \theta_\beta - b_1 \omega \cos \theta_\beta}{c\beta}$$

$$\frac{C}{C_1} = \frac{(a_1 + b_1 \gamma) \cos \theta_\gamma + b_1 \omega \sin \theta_\gamma}{c\omega} = \frac{(a_1 + b_1 \gamma) \sin \theta_\gamma - b_1 \omega \cos \theta_\gamma}{c\gamma}$$

the first of which gives

$$[(a_1 + b_1 \alpha) \alpha + b_1 \omega^2] \cos \theta_\alpha = [(a_1 + b_1 \alpha) \omega - b_1 \omega \alpha] \sin \theta_\alpha$$

that is

$$\tan \theta_\alpha = \frac{a_1 \alpha + b_1 (\alpha^2 + \omega^2)}{a_1 \omega}$$

so that

$$\sin \theta_\alpha = \frac{a_1 \alpha + b_1 (\alpha^2 + \omega^2)}{\sqrt{(a_1 + b_1 \alpha)^2 + b_1^2 \omega^2} \,.\, \sqrt{\alpha^2 + \omega^2}}$$

and

$$\cos \theta_\alpha = \frac{a_1 \omega}{\sqrt{(a_1 + b_1 \alpha)^2 + b_1^2 \omega^2} \,.\, \sqrt{\alpha^2 + \omega^2}}$$

and hence

$$\frac{A}{A_1} = \frac{(a_1 + b_1 \alpha) a_1 \omega + b_1 \omega [a_1 \alpha + b_1 (\alpha^2 + \omega^2)]}{c\omega \sqrt{(a_1 + b_1 \alpha)^2 + b_1^2 \omega^2} \,.\, \sqrt{\alpha^2 + \omega^2}} = \frac{\sqrt{(a_1 + b_1 \alpha)^2 + b_1^2 \omega^2}}{c \sqrt{\alpha^2 + \omega^2}}$$

similarly from the second we have

$$\tan \theta_\beta = \frac{a_1 \beta + b_1 (\beta^2 + \omega^2)}{a_1 \omega} \quad \text{and} \quad \frac{B}{B_1} = \frac{\sqrt{(a_1 + b_1 \beta)^2 + b_1^2 \omega^2}}{c \sqrt{\beta^2 + \omega^2}}$$

and from the third

$$\tan \theta_\gamma = \frac{a_1 \gamma + b_1 (\gamma^2 + \omega^2)}{a_1 \omega} \quad \text{and} \quad \frac{C}{C_1} = \frac{\sqrt{(a_1 + b_1 \gamma)^2 + b_1^2 \omega^2}}{c \sqrt{\gamma^2 + \omega^2}}$$

The same relations are also obtained by putting the complete solutions of x and z in the fundamental equation of z.

§ 15. Determination of the arbitrary constants A, B and C.

The complete solutions are

$$x = X \sqrt{2} \sin (\omega t - \phi_x) + A \epsilon^{\alpha t} + B \epsilon^{\beta t} + C \epsilon^{\gamma t}$$

$$y = Y \sqrt{2} \sin (2\omega t - \phi_y) + A_1 \epsilon^{\alpha t} \sin (\omega t - \theta_\alpha) + B_1 \epsilon^{\beta t} \sin (\omega t - \theta_\beta) + C_1 \epsilon^{\gamma t} \sin (\omega t - \theta_\gamma)$$

$$z = Z \sqrt{2} \sin (2\omega t - \phi_z) - A_1 \epsilon^{\alpha t} \cos (\omega t - \theta_\alpha) - B_1 \epsilon^{\beta t} \cos (\omega t - \theta_\beta) - C_1 \epsilon^{\gamma t} \cos (\omega t - \theta_\gamma)$$

Hence if the initial conditions be

$$x = x_0, \quad y = y_0 \quad \text{and} \quad z = z_0 \quad \text{at} \quad t = t_0 \quad \text{where} \quad x_0 = X_0\sqrt{2}\sin(\omega t_0 - \phi_{x0})$$

$$y_0 = Y_0\sqrt{2}\sin(2\omega t_0 - \phi_{y0}) \quad \text{and} \quad z_0 = Z_0\sqrt{2}\sin(2\omega t_0 - \phi_{z0})$$

then we have the relations

$$A\epsilon^{\alpha t_0} + B\epsilon^{\beta t_0} + C\epsilon^{\gamma t_0} = -(x_0' - x_0)$$

$$A_1\epsilon^{\alpha t_0}\sin(\omega t_0 - \theta_\alpha) + B_1\epsilon^{\beta t_0}\sin(\omega t_0 - \theta_\beta) + C_1\epsilon^{\gamma t_0}\sin(\omega t_0 - \theta_\gamma) = -(y_0' - y_0)$$

$$A_1\epsilon^{\alpha t_0}\cos(\omega t_0 - \theta_\alpha) + B_1\epsilon^{\beta t_0}\cos(\omega t_0 - \theta_\beta) + C_1\epsilon^{\gamma t_0}\cos(\omega t_0 - \theta_\gamma) = (z_0' - z_0)$$

where

$$x_0' = X\sqrt{2}\sin(\omega t_0 - \phi_x), \qquad y_0' = Y\sqrt{2}\sin(2\omega t_0 - \phi_y)$$

and

$$z_0' = Z\sqrt{2}\sin(2\omega t_0 - \phi_z)$$

Therefore denoting $\frac{A_1}{A}$ by L, $\frac{B_1}{B}$ by M and $\frac{C_1}{C}$ by N, the relations become

$$A\epsilon^{\alpha t_0} + B\epsilon^{\beta t_0} + C\epsilon^{\gamma t_0} = -(x_0' - x_0)$$

$$A\epsilon^{\alpha t_0}L\sin(\omega t_0 - \theta_\alpha) + B\epsilon^{\beta t_0}M\sin(\omega t_0 - \theta_\beta) + C\epsilon^{\gamma t_0}N\sin(\omega t_0 - \theta_\gamma) = -(y_0' - y_0)$$

$$A\epsilon^{\alpha t_0}L\cos(\omega t_0 - \theta_\alpha) + B\epsilon^{\beta t_0}M\cos(\omega t_0 - \theta_\beta) + C\epsilon^{\gamma t_0}N\cos(\omega t_0 - \theta_\gamma) = (z_0' - z_0)$$

which solve to

$$A\epsilon^{\alpha t_0} = \begin{vmatrix} -(x_0' - x_0) & 1 & 1 \\ -(y_0' - y_0) & M\sin(\omega t_0 - \theta_\beta) & N\sin(\omega t_0 - \theta_\gamma) \\ +(z_0' - z_0) & M\cos(\omega t_0 - \theta_\beta) & N\cos(\omega t_0 - \theta_\gamma) \end{vmatrix} \div \Delta$$

where

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ L\sin(\omega t_0 - \theta_\alpha) & M\sin(\omega t_0 - \theta_\beta) & N\sin(\omega t_0 - \theta_\gamma) \\ L\cos(\omega t_0 - \theta_\alpha) & M\cos(\omega t_0 - \theta_\beta) & N\cos(\omega t_0 - \theta_\gamma) \end{vmatrix}$$

$$= MN\sin(\theta_\gamma - \theta_\beta) + NL\sin(\theta_\alpha - \theta_\gamma) + LM\sin(\theta_\beta - \theta_\alpha)$$

But now

$$L\sin\theta_\alpha = \frac{c[a_1\alpha + b_1(\alpha^2 + \omega^2)]}{(a_1 + b_1\alpha)^2 + b_1^2\omega^2} \qquad L\cos\theta_\alpha = \frac{a_1 c\omega}{(a_1 + b_1\alpha)^2 + b_1^2\omega^2}$$

$$M\sin\theta_\beta = \frac{c[a_1\beta + b_1(\beta^2 + \omega^2)]}{(a_1 + b_1\beta)^2 + b_1^2\omega^2} \qquad M\cos\theta_\beta = \frac{a_1 c\omega}{(a_1 + b_1\beta)^2 + b_1^2\omega^2}$$

$$N\sin\theta_\gamma = \frac{c[a_1\gamma + b_1(\gamma^2 + \omega^2)]}{(a_1 + b_1\gamma)^2 + b_1^2\omega^2} \qquad N\cos\theta_\gamma = \frac{a_1 c\omega}{(a_1 + b_1\gamma)^2 + b_1^2\omega^2}$$

Therefore the denominator of $A\epsilon^{\alpha t_0}$ is

$$= \frac{a_1 c^2 \omega [a_1(\gamma-\beta) + b_1(\gamma^2-\beta^2)]}{[(a_1+b_1\gamma)^2+b_1^2\omega^2][(a_1+b_1\beta)^2+b_1^2\omega^2]} + \frac{a_1 c^2\omega [a_1(\alpha-\gamma)+b_1(\alpha^2-\gamma^2)]}{[(a_1+b_1\alpha)^2+b_1^2\omega^2][(a_1+b_1\gamma)^2+b_1^2\omega^2]}$$

$$+ \frac{a_1 c^2\omega [a_1(\beta-\alpha)+b_1(\beta^2-\alpha^2)]}{[(a_1+b_1\beta)^2+b_1^2\omega^2][(a_1+b_1\alpha)^2+b_1^2\omega^2]}$$

$$= a_1 c^2 \omega \frac{\Sigma [a_1(\gamma-\beta)+b_1(\gamma^2-\beta^2)][(a_1+b_1\alpha)^2+b_1^2\omega^2]}{\Pi [(a_1+b_1\alpha)^2+b_1^2\omega^2]}$$

$$= -\frac{a_1^2 b_1^2 c^2 \omega (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{\Pi [(a_1+b_1\alpha)^2+b_1^2\omega^2]}$$

Next the numerator of $A\epsilon^{\alpha t_0}$ is

$$= -(x_0'-x_0) MN \sin(\theta_\gamma-\theta_\beta) - (y_0'-y_0)[M\cos(\omega t_0-\theta_\beta) - N\cos(\omega t_0-\theta_\gamma)]$$

$$- (z_0'-z_0)[M\sin(\omega t_0-\theta_\beta) - N\sin(\omega t_0-\theta_\gamma)]$$

But now

$$x_0' MN\sin(\theta_\gamma-\theta_\beta) + y_0'[M\cos(\omega t_0-\theta_\beta) - N\cos(\omega t_0-\theta_\gamma)]$$

$$+ z_0'[M\sin(\omega t_0-\theta_\beta) - N\sin(\omega t_0-\theta_\gamma)]$$

$$= X\sqrt{2}\, MN\sin(\theta_\gamma-\theta_\beta)\sin(\omega t_0-\phi_x)$$

$$+ Y\sqrt{2}[M\cos(\omega t_0-\theta_\beta) - N\cos(\omega t_0-\theta_\gamma)]\sin(2\omega t_0-\phi_y)$$

$$- Y\sqrt{2}[M\sin(\omega t_0-\theta_\beta) - N\sin(\omega t_0-\theta_\gamma)]\cos(2\omega t_0-\phi_y)$$

$$= X\sqrt{2}\, MN\sin(\theta_\gamma-\theta_\beta)\sin(\omega t_0-\phi_x)$$

$$+ Y\sqrt{2}\, M\sin(\omega t_0-\phi_y+\theta_\beta) - Y\sqrt{2}\, N\sin(\omega t_0-\phi_y+\theta_\gamma)$$

which, when $\phi_x + \frac{\pi}{2} + \tan^{-1}\frac{2b_1\omega}{a_1}$ is put in place of ϕ_y, becomes

$$= X\sqrt{2}\left[MN\sin(\theta_\gamma-\theta_\beta) - \frac{2b_1 c\omega^2}{a_1^2+4b_1^2\omega^2}(M\cos\theta_\beta - N\cos\theta_\gamma)\right.$$

$$\left. + \frac{a_1 c\omega}{a_1^2+4b_1^2\omega^2}(M\sin\theta_\beta - N\sin\theta_\gamma)\right]\sin(\omega t_0-\phi_x)$$

$$- X\sqrt{2}\left[\frac{a_1 c\omega}{a_1^2+4b_1^2\omega^2}(M\cos\theta_\beta - N\cos\theta_\gamma)\right.$$

$$\left. + \frac{2b_1 c\omega^2}{a_1^2+4b_1^2\omega^2}(M\sin\theta_\beta - N\sin\theta_\gamma)\right]\cos(\omega t_0-\phi_x)$$

But

$$MN\sin(\theta_\gamma-\theta_\beta) = \frac{-a_1 c^2\omega(\beta-\gamma)[a_1+b_1(\beta-\gamma)]}{[(a_1+b_1\beta)^2+b_1^2\omega^2][(a_1+b_1\gamma)^2+b_1^2\omega^2]}$$

$$M\cos\theta_\beta - N\cos\theta_\gamma = \frac{-a_1 b_1 c\omega(\beta-\gamma)[2a_1+b_1(\beta+\gamma)]}{\text{"}}$$

and

$$M\sin\theta_\beta - N\sin\theta_\gamma = \frac{c a_1(\beta-\gamma)[a_1^2+a_1 b_1(\beta+\gamma)+b_1^2(\beta\gamma-\omega^2)]}{\text{"}}$$

so that

$$a_1 c\omega\,(M\cos\theta_\beta - N\cos\theta_\gamma) + 2b_1 c\omega^2 (M\sin\theta_\beta - N\sin\theta_\gamma)$$
$$= \frac{a_1 b_1^2 c^2 \omega^2 (\beta-\gamma)\,[a_1(\beta+\gamma) + 2b_1(\beta\gamma-\omega^2)]}{[(a_1+b_1\beta)^2 + b_1^2\omega^2]\,[(a_1+b_1\gamma)^2+b_1^2\omega^2]}$$

and

$$MN\sin(\theta_\gamma-\theta_\beta) - \frac{2b_1 c\omega^2}{a_1^2+4b_1^2\omega^2}(M\cos\theta_\beta - N\cos\theta_\gamma)$$
$$+\frac{a_1 c\omega}{a_1^2+4b_1^2\omega^2}(M\sin\theta_\beta - N\sin\theta_\gamma)$$
$$= \frac{a_1 b_1^2 c^2 \omega(\beta-\gamma)\,[a_1(\beta\gamma-\omega^2) - 2b_1\omega^2(\beta+\gamma)]}{(a_1^2+4b_1^2\omega^2)\,[(a_1+b_1\beta)^2+b_1^2\omega^2]\,[(a_1+b_1\gamma)^2+b_1^2\omega^2]}$$

accordingly

$$\left[MN\sin(\theta_\gamma-\theta_\beta) - \frac{2b_1 c\omega^2}{a_1^2+4b_1^2\omega^2}(M\cos\theta_\beta - N\cos\theta_\gamma)\right.$$
$$\left.+\frac{a_1 c\omega}{a_1^2+4b_1^2\omega^2}(M\sin\theta_\beta - N\sin\theta_\gamma)\right]^2$$
$$+\left[\frac{a_1 c\omega}{a_1^2+4b_1^2\omega^2}(M\cos\theta_\beta - N\cos\theta_\gamma) + \frac{2b_1 c\omega^2}{a_1^2+4b_1^2\omega^2}(M\sin\theta_\beta - N\sin\theta_\gamma)\right]^2$$
$$= \frac{a_1^2 b_1^4 c^4 \omega^2 (\beta-\gamma)^2 (\beta^2+\omega^2)(\gamma^2+\omega^2)}{(a_1^2+4b_1^2\omega^2)\,[(a_1+b_1\beta)^2+b_1^2\omega^2]^2\,[(a_1+b_1\gamma)^2+b_1^2\omega^2]^2}$$

Therefore

$$x_0' MN\sin(\theta_\gamma-\theta_\beta) + y_0'\,[M\cos(\omega t_0-\theta_\beta) - N\cos(\omega t_0-\theta_\gamma)]$$
$$+ z_0'\,[M\sin(\omega t_0-\theta_\beta) - N\sin(\omega t_0-\theta_\gamma)]$$
$$= X\sqrt{2}\,F\sin(\omega t_0 - \phi_x - \phi)$$

where $$F = \frac{a_1 b_1^2 c^2 \omega\,(\beta-\gamma)\sqrt{(\beta^2+\omega^2)(\gamma^2+\omega^2)}}{[(a_1+b_1\beta)^2+b_1^2\omega^2]\,[(a_1+b_1\gamma)^2+b_1^2\omega^2]\sqrt{a_1^2+4b_1^2\omega^2}}$$

and $$\phi = \tan^{-1}\frac{\omega\,[a_1(\beta+\gamma) + 2b_1(\beta\gamma-\omega^2)]}{a_1(\beta\gamma-\omega^2) - 2b_1\omega^2(\beta+\gamma)} = \pi - \phi'$$

where $$\phi' = \tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\beta\gamma-\omega^2}{\omega(\beta+\gamma)}$$

similarly we have

$$x_0\, MN\sin(\theta_\gamma-\theta_\beta) + y_0\,[M\cos(\omega t_0-\theta_\beta) - N\cos(\omega t_0-\theta_\gamma)]$$
$$+ z_0\,[M\sin(\omega t_0-\theta_\beta) - N\sin(\omega t_0-\theta_\gamma)]$$
$$= X_0\sqrt{2}\,F\sin(\omega t_0 - \phi_{x0} - \phi)$$

where F and ϕ denote the same things as before.

Hence the numerator of $A\epsilon^{at_0}$ is

$$-F.[X\sqrt{2}\sin(\omega t_0-\phi_x-\pi+\phi')-X_0\sqrt{2}\sin(\omega t_0-\phi_{x0}-\pi+\phi')]$$
$$=F.[X\sqrt{2}\sin(\omega t_0-\phi_x+\phi')-X_0\sqrt{2}\sin(\omega t_0-\phi_{x0}+\phi')]$$
$$=FX_n\sqrt{2}\sin(\omega t_0-\phi_n+\phi')$$

where

$$X_n=\sqrt{X^2+X_0^2-2XX_0\cos(\phi_x-\phi_{x0})}$$

and

$$\phi_n=\tan^{-1}\frac{X\sin\phi_x-X_0\sin\phi_{x0}}{X\cos\phi_x-X_0\cos\phi_{x0}}$$

Therefore we have

$$A\epsilon^{at_0}=\frac{-X_n\sqrt{2}\,[(a_1+b_1a)^2+b_1^2\omega^2]\sqrt{(\beta^2+\omega^2)(\gamma^2+\omega^2)}}{a_1(\gamma-\alpha)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}$$
$$\times\sin\left(\omega t-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{\beta\gamma-\omega^2}{\omega(\beta+\gamma)}\right)$$

Next

$$B\epsilon^{\beta t_0}=\begin{vmatrix}1 & -(x_0'-x_0) & 1\\ L\sin(\omega t_0-\theta_\alpha) & -(y_0'-y_0) & N\sin(\omega t_0-\theta_\gamma)\\ L\cos(\omega t_0-\theta_\alpha) & +(z_0'-z_0) & N\cos(\omega t_0-\theta_\gamma)\end{vmatrix}\div\Delta$$

where Δ is that Δ which is the denominator of $A\epsilon^{at_0}$

$$=\begin{vmatrix}-(x_0'-x_0) & 1 & 1\\ -(y_0'-y_0) & N\sin(\omega t_0-\theta_\gamma) & L\sin(\omega t_0-\theta_\alpha)\\ +(z_0'-z_0) & N\cos(\omega t_0-\theta_\gamma) & L\cos(\omega t_0-\theta_\alpha)\end{vmatrix}\div\Delta$$

and

$$C\epsilon^{\gamma t_0}=\begin{vmatrix}1 & 1 & -(x_0'-x_0)\\ L\sin(\omega t_0-\theta_\alpha) & M\sin(\omega t_0-\theta_\beta) & -(y_0'-y_0)\\ L\cos(\omega t_0-\theta_\alpha) & M\cos(\omega t_0-\theta_\beta) & +(z_0'-z_0)\end{vmatrix}\div\Delta$$

$$=\begin{vmatrix}-(x_0'-x_0) & 1 & 1\\ -(y_0'-y_0) & L\sin(\omega t_0-\theta_\alpha) & M\sin(\omega t_0-\theta_\beta)\\ +(z_0'-z_0) & L\cos(\omega t_0-\theta_\alpha) & M\cos(\omega t_0-\theta_\beta)\end{vmatrix}\div\Delta$$

so that from $A\epsilon^{at_0}$, by cyclical change of letters α, β and γ, we have

$$B\epsilon^{\beta t_0}=\frac{-X_n\sqrt{2}\,[(a_1+b_1\beta)^2+b_1^2\omega^2]\sqrt{(\gamma^2+\omega^2)(\alpha^2+\omega^2)}}{a_1(\alpha-\beta)(\beta-\gamma)\sqrt{a_1^2+4b_1^2\omega^2}}$$
$$\times\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)}\right)$$

and

$$C\epsilon^{\gamma t_0} = \frac{-X_n\sqrt{2}\left[(a_1+b_1\gamma)^2+b_1^2\omega^2\right]\sqrt{(\alpha^2+\omega^2)(\beta^2+\omega^2)}}{a_1(\beta-\gamma)(\gamma-\alpha)\sqrt{a_1^2+4b_1^2\omega^2}}$$

$$\times \sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)$$

Here note that denoting $A\epsilon^{\alpha t_0}$ by A', $B\epsilon^{\beta t_0}$ by B', $C\epsilon^{\gamma t_0}$ by C', $A_1\epsilon^{\alpha t_0}$ by A_1', $B_1\epsilon^{\beta t_0}$ by B_1' and $C_1\epsilon^{\gamma t_0}$ by C_1' the transient terms of x become

$$A'\epsilon^{\alpha(t-t_0)} + B'\epsilon^{\beta(t-t_0)} + C'\epsilon^{\gamma(t-t_0)}$$

that of y becomes

$$A_1'\epsilon^{\alpha(t-t_0)}\sin(\omega t-\theta_\alpha) + B_1'\epsilon^{\beta(t-t_0)}\sin(\omega t-\theta_\beta) + C_1'\epsilon^{\gamma(t-t_0)}\sin(\omega t-\theta_\gamma)$$

and that of z becomes

$$-A_1'\epsilon^{\alpha(t-t_0)}\cos(\omega t-\theta_\alpha) - B_1'\epsilon^{\beta(t-t_0)}\cos(\omega t-\theta_\beta) - C_1'\epsilon^{\gamma(t-t_0)}\cos(\omega t-\theta_\gamma)$$

where θ_α, θ_β and θ_γ denote the same things as in the previous article and $\frac{A_1'}{A'}$, $\frac{B_1'}{B'}$ and $\frac{C_1'}{C'}$ are equal to $\frac{A_1}{A}$, $\frac{B_1}{B}$ and $\frac{C_1}{C}$ respectively.

§ 16. Evaluation of the roots α, β and γ.

α, β and γ are the roots of

$$b_1(bb_1-c^2)\lambda^3 + (ab_1^2+2a_1bb_1-a_1c^2)\lambda^2 + \left[2aa_1b_1+a_1^2b+b_1(bb_1-c^2)\omega^2\right]\lambda$$
$$+\,a(a_1^2+b_1^2\omega^2)=0$$

which, by dividing by $bb_1^2\omega^2$ and putting

$$\frac{a}{b\omega}=p,\quad \frac{a_1}{b_1\omega}=q,\quad \frac{\lambda}{\omega}=x \quad\text{and}\quad 1-\frac{c^2}{bb_1}=\sigma,$$

becomes

$$\sigma x^3 + (p+q+q\sigma)x^2 + (2pq+q^2+\sigma)x + p(q^2+1)=0$$

Now, since a, b, c, a_1, b_1, ω and σ are all positive, all the coefficients of the cubic equation are positive, so that all real roots of the cubic equation must be negative and therefore, when α, β and γ are all real, the complementary functions of x, y and z are all transient terms, their magnitudes all diminishing with time.

Next, let us take up the case when the given equation has a real root

$-\alpha$ and two imaginary roots $\mu \pm j\nu$ and examine whether μ is positive or negative in that case. The given equation is

$$Ax^3 + Bx^2 + Cx + D = 0$$

where $A = \sigma$, $B = p + q + q\sigma$, $C = 2pq + q^2 + \sigma$ and $D = p(q^2 + 1)$

Therefore we have

$$Ax^3 + Bx^2 + Cx + D \equiv A(x + \alpha)(x^2 - 2\mu x + \mu^2 + \nu^2)$$
$$\equiv A[x^3 + (\alpha - 2\mu)x^2 + (\mu^2 + \nu^2 - 2\mu\alpha)x + \alpha(\mu^2 + \nu^2)]$$

so that

$$A(\alpha - 2\mu) = B, \quad A(\mu^2 + \nu^2 - 2\mu\alpha) = C \text{ and } A\alpha(\mu^2 + \nu^2) = D$$

so that

$$A\left(\frac{D}{A\alpha} - 2\mu\alpha\right) = C \text{ that is } D - 2A\mu\alpha^2 = C\alpha$$

so that

$$D - 2A\mu\alpha^2 = C\left(\frac{B}{A} + 2\mu\right)$$

that is

$$2\mu(A\alpha^2 + C)A = AD - BC$$

so that μ has the same sign as $AD - BC$.

But

$$BC - AD = (p + q + q\sigma)(2pq + q^2 + \sigma) - \sigma p(q^2 + 1)$$
$$= q(p + q)(2p + q + q\sigma) + q\sigma(1 + \sigma)$$
$$> 0$$

Therefore μ is negative, that is, the real term of the two conjugate imaginary roots is negative and this fact proves, as will be seen later, that the complementary functions of x, y and z are all transient terms also when two of the three roots α, β and γ are imaginary.

Now, proceeding to find the roots of the cubic equation, we know that if the cubic equation be expressed in the form

$$Ax^3 + 3B_1x^2 + 3C_1x + D = 0$$

then the roots can be expressed in the form $x = \frac{1}{2}(z - B_1)$ where z is given by

$$z^3 + 3Hz + G = 0$$

where $H = AC_1 - B_1^2$ and $G = 2B_1^3 - 3AB_1C_1 + A^2D$

that is

$$z_1 = P^{\frac{1}{3}} + Q^{\frac{1}{3}}, \quad z_2 = \omega_1 P^{\frac{1}{3}} + \omega_2 Q^{\frac{1}{3}} \quad \text{and} \quad z_3 = \omega_2 P^{\frac{1}{3}} + \omega_1 Q^{\frac{1}{3}}$$

where $$P = -\frac{1}{2}G + \frac{1}{2}\sqrt{G^2 + 4H^3}, \quad Q = -\frac{1}{2}G - \frac{1}{2}\sqrt{G^2 + 4H^3},$$

$$\omega_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{-3} \quad \text{and} \quad \omega_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{-3}$$

and the roots are all real when $G^2 + 4H^3 < 0$ and two of them are imaginary when $G^2 + 4H^3 > 0$.

Proceeding to evaluate $G^2 + 4H^3$ it is

$$(2B_1^3 - 3AB_1C_1 + A^2D)^2 + 4(AC_1 - B_1^2)^2$$

that is $$A^2\Delta$$

where Δ is the discriminant of the cubic equation and is

$$= A^2D^2 - 6AB_1C_1D + 4AC_1^3 + 4B_1^3D - 3B_1^2C_1^2$$
$$= \frac{1}{27} \cdot [4AC^3 - B^2(C^2 - 4BD) - 9AD(2BC - 3AD)]$$

But

$$C^2 - 4BD = -4p^2 - 4pq(q^2\sigma + 1) + (q^2 + \sigma)^2$$

and

$$2BC - 3AD = 4p^2q + p[q^2(6 + \sigma) - \sigma] + 2q(q^2 + \sigma)(1 + \sigma)$$

accordingly

$$\begin{aligned} B^2(C^2 - 4BD) = & -4p^4 - 4p^3q(q^2\sigma + 3 + 2\sigma) - p^2[q^4(8\sigma^2 + 8\sigma - 1) \\ & + 2q^2(6 + 7\sigma + 2\sigma^2) - \sigma^2] - 2pq(1 + \sigma)[q^4(2\sigma^2 + 2\sigma + 1) \\ & + 2q^2 - \sigma^2] + q^2(q^2 + \sigma)^2(1 + \sigma)^2 \end{aligned}$$

and

$$\begin{aligned} 4C^3 - 9D(2BC - 3AD) = & -4p^3q(q^2 + 9) - 3p^2[q^4(2 + 3\sigma) + 2q^2(9 - 8\sigma) - 3\sigma] \\ & - 6pq[q^4(3\sigma - 1) + q^2(3\sigma^2 - 2\sigma + 3) + \sigma(3 - \sigma)] + 4(q^2 + \sigma)^3 \end{aligned}$$

so that

$$\begin{aligned} 27 . \Delta = & A[4C^3 - 9D(2BC - 3AD)] - B^2(C^2 - 4BD) \\ = & 4p^4 + 4p^3q(3 - 7\sigma) - p^2[q^4(1 - \sigma)^2 - 4q^2(3 - 10\sigma + 13\sigma^2) - 8\sigma^2] \\ & - 2pq[q^4(1 - \sigma)^2(1 - 2\sigma) - q^2(1 - 3\sigma)(2 - \sigma + 3\sigma^2) + 2\sigma^2(5 - \sigma)] \\ & - (q^2 + \sigma)^2[q^2(1 - \sigma)^2 - 4\sigma^2] \end{aligned}$$

Now if $q < \frac{1}{100}$ and $\sigma > \frac{1}{3}$ then $\left(\frac{q}{\sigma}\right)^2 < \frac{0{\cdot}9}{1000}$ that is $\left(\frac{q}{\sigma}\right)^2$ is negligible compared with unity and the discriminant becomes

$$27 \,.\, \Delta = 4p^4 + 4p^3 q\,(3 - 7\sigma) + 8p^2\sigma^2 - 4pq\sigma^2\,(5 - \sigma) + 4\sigma^4$$
$$= 4\,[(p^2 + \sigma^2)^2 - pq\,(\overline{5 - \sigma}\,.\,\sigma^2 - p^2\,.\,\overline{3 - 7\sigma})]$$

which is positive when $(5 - \sigma)\,\sigma^2 - p^2\,(3 - 7\sigma) < 0$, that is when

$$p > \sigma\sqrt{\frac{5 - \sigma}{3 - 7\sigma}}$$

But since the value of $\sigma\sqrt{\frac{5 - \sigma}{3 - 7\sigma}}$ is as shown in the following table, we see that in the case of short circuit, etc., p is usually $< \sigma\sqrt{\frac{5 - \sigma}{3 - 7\sigma}}$

σ	0·1	0·15	0·2	0·3	0·4	0·43
$\sigma\sqrt{\frac{5-\sigma}{3-7\sigma}}$	0·146	0·246	0·346	0·686	1·92	∞

Now if p is $< \sigma\sqrt{\frac{5 - \sigma}{3 - 7\sigma}}$ then the discriminant is positive or negative according as $q \lesseqgtr \frac{(p^2 + \sigma^2)^2}{p\,[\sigma^2\,(5 - \sigma) - p^2\,(3 - 7\sigma)]}$.

But the value of $\frac{(p^2 + \sigma^2)^2}{p\,[\sigma^2\,(5 - \sigma) - p^2\,(3 - 7\sigma)]}$ is as shown in the table and curves (figure 4) on the separate sheets.

Therefore we can conclude that the discriminant is usually positive and hence two roots of the given cubic equation are usually imaginary.

Now if $q < \frac{1}{100}$, $p < \frac{3}{100}$ and $\sigma > \frac{1}{3}$, then $\left(\frac{q}{\sigma}\right)^2 < \frac{0{\cdot}9}{1000}$, $\left(\frac{p}{\sigma}\right)^2 < \frac{0{\cdot}81}{100}$ and $\frac{pq}{\sigma^2} < \frac{0{\cdot}27}{100}$; and hence the discriminant becomes

$$27 \,.\, \Delta \doteqdot 4\sigma^4\left[1 - (5 - \sigma)\frac{pq}{\sigma^2}\right] \doteqdot 4\sigma^4$$

and since

$$27G = 2B^3 - 9ABC + 27A^2D = 2p^3 + 6p^2q\,(1 - 2\sigma) + 3p\,[q^2\,(2 - 5\sigma + 5\sigma^2) + 6\sigma^2]$$
$$+ q\,(1 + \sigma)\,[q^2\,(2 - 5\sigma + 2\sigma^2) - 9\sigma^2]$$

we have

$$G \doteqdot \frac{1}{3}\sigma^2[2p - q(1+\sigma)]$$

Therefore

$$\left(-\frac{1}{2}G \pm \frac{1}{2}\sqrt{G^2+4H^3}\right)^{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}\sigma\left[1 \mp \sqrt{3}\,\frac{p-0{\cdot}5q(1+\sigma)}{\sigma}\right]^{\frac{1}{3}}$$

$$\doteqdot \pm\frac{1}{\sqrt{3}}\sigma\left[1 \mp \frac{1}{\sqrt{3}}\,\frac{p-0{\cdot}5q(1+\sigma)}{\sigma}\right]$$

Value of $\dfrac{(p^2+\sigma^2)^2}{p[\sigma^2.(5-\sigma)-p^2.(3-7\sigma)]}$ for

σ \ p	10	1	0·2	0·1	0·03	0·02	0·01
0·02						0·0378	0·01467
0·04	In this blank region $p[\sigma^2(5-\sigma)-p^2(3-7\sigma)]$ becomes negative so that the roots are imaginary irrespective of the value of q.				0·038	0·02925	0·03775
0·06					0·0436	0·0451	0·0768
0·08				0·377	0·0607	0·0758	0·1353
0·10				0·154	0·0845	0·1123	0·2085
0·15			0·488	0·1057	0·157	0·224	0·433
0·20			0·250	0·142	0·293	0·439	0·838
0·30			0·217	0·241	0·653	0·968	1·92
0·40		2·51	0·275	0·393	1·174	1·748	3·49
0·43	1190·	1·66	0·296	0·453	1·361	2·037	Omitted
0·50	19·7	0·96	0·367	0·598	1·867	2·793	
0·60	8·3	0·67	0·472	0·856	2·74	4·11	
0·75	4·5	0·53	0·728	1·35	4·42	6·64	
0·90	3·15	0·5	1·05	2·01	6·60	9·92	
1·00	2·5	0·5	1·30	2·53	8·33	10·25	

so that

$$z_1 = -\frac{2}{3}[p - 0{\cdot}5q(1+\sigma)], \qquad z_2 = \frac{1}{3}[p - 0{\cdot}5q(1+\sigma)] + j\sigma$$

and

$$z_3 = \frac{1}{3}[p - 0{\cdot}5q(1+\sigma)] - j\sigma$$

accordingly

$$\alpha = \frac{\omega}{\sigma}\cdot\left[z_1 - \frac{p+q(1+\sigma)}{3}\right] = -\frac{p}{\sigma}\omega$$

$$\beta = -\frac{\omega}{2\sigma}q(1+\sigma) + j\omega \qquad \text{similarly}$$

$$\gamma = -\frac{\omega}{2\sigma}q(1+\sigma) - j\omega \qquad \text{,,}$$

Here note that we can obtain these values of α, β and γ, when $\left(\frac{p}{\sigma}\right)^2$, $\left(\frac{q}{\sigma}\right)^2$ and $\frac{pq}{\sigma^2}$ are negligible, more easily as follows:—

Since
$$1 + \frac{2pq}{\sigma} + \frac{q^2}{\sigma} \doteqdot 1 \doteqdot 1 + \frac{p}{\sigma}\left(q + \frac{q}{\sigma}\right)$$

we can write the equation

$$\sigma x^3 + (p + q + q\sigma)x^2 + (2pq + q^2 + \sigma)x + p(q^2+1) = 0$$

in the form

$$x^3 + \left(q + \frac{p}{\sigma} + \frac{q}{\sigma}\right)x^2 + \left[1 + \frac{p}{\sigma}\left(q + \frac{q}{\sigma}\right)\right]x + \frac{p}{\sigma} = 0$$

that is

$$\left(x + \frac{p}{\sigma}\right)\left[x^2 + \left(q + \frac{q}{\sigma}\right)x + 1\right] = 0$$

so that

$$x_1 = -\frac{p}{\sigma}$$

$$x_2,\, x_3 = -\frac{1}{2}\left(q + \frac{q}{\sigma}\right) \pm \sqrt{\frac{1}{4}q^2\left(1+\frac{1}{\sigma}\right)^2 - 1} \doteqdot -\frac{1}{2\sigma}\cdot q(1+\sigma) \pm j$$

and $x_1\omega$, $x_2\omega$ and $x_3\omega$ give the same values of α, β and γ as found above.

§ 17. Complete solution of the distortionless alternator with two field windings when the roots β and γ are imaginary.

Let $\beta = \mu + j\nu$ and $\gamma = \mu - j\nu$. Then

$$B\epsilon^{\beta t} + C\epsilon^{\gamma t} = \epsilon^{\mu t}(B\epsilon^{j\nu t} + C\epsilon^{-j\nu t}) = 2\sqrt{BC}\,.\,\epsilon^{\mu t}\sin\left(\nu t + \tan^{-1}\frac{B+C}{j(B-C)}\right)$$

Therefore the complete solution of x is

$$x = X\sqrt{2}\sin(\omega t - \phi_x) + A\epsilon^{\alpha t} + D\epsilon^{\mu t}\sin(\nu t + \psi)$$

where A, D and ψ are the arbitrary constants.

Now putting this complete solution of x in the fundamental equation of y, we have

$$a_1 y + b_1\frac{dy}{dt} + c\frac{d}{dt}\cdot\{[X\sqrt{2}\sin(\omega t - \phi_x) + A\epsilon^{\alpha t} + D\epsilon^{\mu t}\sin(\nu t + \psi)]\cos\omega t\} = 0$$

that is

$$a_1 y + b_1\frac{dy}{dt} = -cX\sqrt{2}\,\omega\cos(2\omega t - \phi_x) - cA\epsilon^{\alpha t}(\alpha\cos\omega t - \omega\sin\omega t)$$

$$-\frac{1}{2}cD\epsilon^{\mu t}[\mu\sin(\overline{\nu+\omega}t + \psi) + (\nu+\omega)\cos(\overline{\nu+\omega}t + \psi)]$$

$$-\frac{1}{2}cD\epsilon^{\mu t}[\mu\sin(\overline{\nu-\omega}t + \psi) + (\nu-\omega)\cos(\overline{\nu-\omega}t + \psi)]$$

that is

$$\frac{dy}{dt} + \frac{a_1}{b_1}y = -\frac{c}{b_1}X\sqrt{2}\,\omega\cos(2\omega t - \phi_x) + \frac{c}{b_1}A\epsilon^{\alpha t}\sqrt{\alpha^2+\omega^2}\sin\left(\omega t - \tan^{-1}\frac{\alpha}{\omega}\right)$$

$$-\frac{1}{2}\frac{c}{b_1}D\epsilon^{\mu t}\sqrt{\mu^2+(\nu+\omega)^2}\sin\left(\overline{\nu+\omega}t + \psi + \tan^{-1}\frac{\nu+\omega}{\mu}\right)$$

$$-\frac{1}{2}\frac{c}{b_1}D\epsilon^{\mu t}\sqrt{\mu^2+(\nu-\omega)^2}\sin\left(\overline{\nu-\omega}t + \psi + \tan^{-1}\frac{\nu-\omega}{\mu}\right)$$

which, similarly as in Art. 14, solves to

$$y = \frac{-cX\sqrt{2}\,\omega}{\sqrt{a_1^2+4b_1^2\omega^2}}\cos\left(2\omega t - \phi_x - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

$$+\frac{cA\sqrt{\alpha^2+\omega^2}}{\sqrt{(a_1+b_1\alpha)^2+b_1^2\omega^2}}\cdot\epsilon^{\alpha t}\cdot\sin\left(\omega t - \tan^{-1}\frac{\alpha}{\omega} - \tan^{-1}\frac{b_1\omega}{a_1+b_1\alpha}\right)$$

$$-\frac{1}{2}\frac{cD\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2}}\cdot\epsilon^{\mu t}$$

$$\times\sin\left(\overline{\nu+\omega}t + \psi + \tan^{-1}\frac{\nu+\omega}{\mu} - \tan^{-1}\frac{b_1(\nu+\omega)}{a_1+b_1\mu}\right)$$

$$-\frac{1}{2}\frac{cD\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2}}\cdot\epsilon^{\mu t}$$

$$\times\sin\left(\overline{\nu-\omega}t + \psi + \tan^{-1}\frac{\nu-\omega}{\mu} - \tan^{-1}\frac{b_1(\nu-\omega)}{a_1+b_1\mu}\right) + K\epsilon^{-\frac{a_1}{b_1}t}$$

that is

$$y = \rho X \sqrt{2} \sin\left(2\omega t - \phi_x - \frac{\pi}{2} - \tan^{-1}\frac{2b_1\omega}{a_1}\right) + A_1\epsilon^{\alpha t}\sin(\omega t - \theta_a)$$

$$- D_1\epsilon^{\mu t}\sin(\overline{\nu+\omega}t + \psi + \psi_1) - D_2\epsilon^{\mu t}\sin(\overline{\nu-\omega}t + \psi + \psi_2) + K\epsilon^{-\frac{a_1}{b_1}t}$$

where

$$A_1 = \frac{cA\sqrt{\alpha^2+\omega^2}}{\sqrt{(a_1+b_1\alpha)^2+b_1^2\omega^2}} \qquad \theta_a = \tan^{-1}\frac{a_1\alpha + b_1(\alpha^2+\omega^2)}{a_1\omega}$$

$$D_1 = \frac{\frac{1}{2}cD\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2}} \qquad \psi_1 = \tan^{-1}\frac{a_1(\nu+\omega)}{\mu(a_1+b_1\mu)+b_1(\nu+\omega)^2}$$

$$D_2 = \frac{\frac{1}{2}cD\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2}} \qquad \psi_2 = \tan^{-1}\frac{a_1(\nu-\omega)}{\mu(a_1+b_1\mu)+b_1(\nu-\omega)^2}$$

Next, putting the complete solution of x in the fundamental equation of z, we have

$$a_1 z + b_1\frac{dz}{dt} = -cX\sqrt{2}\,\omega\sin(2\omega t - \phi_x) - cA\epsilon^{\alpha t}\sqrt{\alpha^2+\omega^2}\cos\left(\omega t - \tan^{-1}\frac{\alpha}{\omega}\right)$$

$$-\frac{1}{2}cD\epsilon^{\mu t}\sqrt{\mu^2+(\nu-\omega)^2}\cos\left(\overline{\nu-\omega}t + \psi + \tan^{-1}\frac{\nu-\omega}{\mu}\right)$$

$$+\frac{1}{2}cD\epsilon^{\mu t}\sqrt{\mu^2+(\nu+\omega)^2}\cos\left(\overline{\nu+\omega}t + \psi + \tan^{-1}\frac{\nu+\omega}{\mu}\right)$$

which like that of y solves to

$$z = \rho X\sqrt{2}\left(2\omega t - \phi_x - \pi - \tan^{-1}\frac{2b_1\omega}{a_1}\right) - A_1\epsilon^{\alpha t}\cos(\omega t - \theta_a)$$

$$+ D_1\epsilon^{\mu t}\cos(\overline{\nu+\omega}t + \psi + \psi_1) - D_2\epsilon^{\mu t}\cos(\overline{\nu-\omega}t + \psi + \psi_2) + K_1\epsilon^{-\frac{a_1}{b_1}t}$$

where A_1, D_1, D_2, θ_a, ψ_1 and ψ_2 denote the same things as those in the complete solution of y.

Note that, since $\epsilon^{\alpha t}$ or $\epsilon^{\mu t}$ is contained in each complementary function and α and μ are negative, as explained in the previous article, the complementary functions consist all of transient terms.

Now putting $B = B'\epsilon^{-\beta t_0}$ and $C = C'\epsilon^{-\gamma t_0}$ where t_0 is that instant when the load is changed, that is, when transient phenomena begin to appear, we have

$$\sqrt{BC} = \sqrt{B'C'}\,.\,\epsilon^{-\frac{1}{2}(\beta+\gamma)t_0} = \sqrt{B'C'}\,.\,\epsilon^{-\mu t_0}$$

$$B + C = \epsilon^{-\mu t_0}.(B'\epsilon^{-j\nu t_0} + C'\epsilon^{j\nu t_0})$$

$$= \epsilon^{-\mu t_0}.[(B'+C')\cos\nu t_0 - j(B'-C')\sin\nu t_0]$$

and similarly,

$$B - C = \epsilon^{-\mu t_0}[(B' - C')\cos\nu t_0 - j(B' + C')\sin\nu t_0]$$

so that
$$\tan^{-1}\frac{B+C}{j(B-C)} = \tan^{-1}\frac{(B'+C')\cos\nu t_0 - j(B'-C')\sin\nu t_0}{(B'-C')\cos\nu t_0 - j(B'+C')\sin\nu t_0}$$

$$= \tan^{-1}\frac{\dfrac{B'+C'}{j(B'-C')} - \tan\nu t_0}{1 + \dfrac{B'+C'}{j(B'-C')}\cdot\tan\nu t_0}$$

$$= \tan^{-1}\frac{B'+C'}{j(B'-C')} - \nu t_0$$

Hence putting also $A = A'\epsilon^{-\alpha t_0}$, the complementary functions of x, y and z become

$$x = X\sqrt{2}\sin(\omega t - \phi_x) + A'\epsilon^{\alpha(t-t_0)} + D'\epsilon^{\mu(t-t_0)}\sin[\nu(t-t_0) + \psi']$$

$$\begin{aligned} y = \rho X\sqrt{2}\sin\left(2\omega t - \phi_x - \frac{\pi}{2} - \tan^{-1}\frac{2b_1\omega}{a_1}\right) &+ A_1'\epsilon^{\alpha(t-t_0)}\sin(\omega t - \theta_a) \\ &- D_1'\epsilon^{\mu(t-t_0)}\sin[\nu(t-t_0) + \omega t + \psi' + \psi_1] \\ &- D_2'\epsilon^{\mu(t-t_0)}\sin[\nu(t-t_0) - \omega t + \psi' + \psi_2] \\ &+ K\epsilon^{-\frac{a_1}{b_1}t} \end{aligned}$$

$$\begin{aligned} z = \rho X\sqrt{2}\left(2\omega t - \phi_x - \pi - \tan^{-1}\frac{2b_1\omega}{a_1}\right) &- A_1'\epsilon^{\alpha(t-t_0)}\cos(\omega t - \theta_a) \\ &+ D_1'\epsilon^{\mu(t-t_0)}\cos[\nu(t-t_0) + \omega t + \psi' + \psi_1] \\ &- D_2'\epsilon^{\mu(t-t_0)}\cos[\nu(t-t_0) - \omega t + \psi' + \psi_2] \\ &+ K_1\epsilon^{-\frac{a_1}{b_1}t} \end{aligned}$$

where

$$D' = 2\sqrt{B'C'} \qquad \psi' = \tan^{-1}\frac{B'+C'}{j(B'-C')}$$

$$A_1' = A'\frac{c\sqrt{\alpha^2+\omega^2}}{\sqrt{(a_1+b_1\alpha)^2 + b_1^2\omega^2}} \qquad \theta_a = \tan^{-1}\frac{a_1\alpha + b_1(\alpha^2+\omega^2)}{a_1\omega}$$

$$D_1' = D'\frac{\frac{1}{2}c\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a_1+b_1\mu)^2 + b_1^2(\nu+\omega)^2}} \qquad \psi_1 = \tan^{-1}\frac{a_1(\nu+\omega)}{\mu(a_1+b_1\mu) + b_1(\nu+\omega)^2}$$

and

$$D_2' = D' \frac{\frac{1}{2} c \sqrt{\mu^2 + (\nu - \omega)^2}}{\sqrt{(a_1 + b_1\mu)^2 + b_1^2 (\nu - \omega)^2}} \qquad \psi_2 = \tan^{-1} \frac{a_1 (\nu - \omega)}{\mu (a_1 + b_1\mu) + b_1 (\nu - \omega)^2}$$

Now, putting the complete solutions of x, y and z in the fundamental equation

$$ax + b\frac{dx}{dt} + c\frac{d}{dt}(y \cos \omega t + z \sin \omega t) = d \sin \omega t$$

we have

$$a[A\epsilon^{\alpha t} + D\epsilon^{\mu t} \sin(\nu t + \psi)] + b\frac{d}{dt}[A\epsilon^{\alpha t} + D\epsilon^{\mu t} \sin(\nu t + \psi)]$$
$$+ c\frac{d}{dt}[A_1\epsilon^{\alpha t} \sin(\omega t - \theta_\alpha) \cos \omega t - A_1\epsilon^{\alpha t} \cos(\omega t - \theta_\alpha) \sin \omega t$$
$$- D_1\epsilon^{\mu t} \sin(\overline{\nu + \omega}t + \psi + \psi_1) \cos \omega t + D_1\epsilon^{\mu t} \cos(\overline{\nu + \omega}t + \psi + \psi_1) \sin \omega t$$
$$- D_2\epsilon^{\mu t} \sin(\overline{\nu - \omega}t + \psi + \psi_2) \cos \omega t - D_2\epsilon^{\mu t} \cos(\overline{\nu - \omega}t + \psi + \psi_2) \sin \omega t$$
$$+ K\epsilon^{-\frac{a_1}{b_1}t} \cos \omega t + K_1\epsilon^{-\frac{a_1}{b_1}t} \sin \omega t] = 0$$

that is

$$(a + b\alpha) A\epsilon^{\alpha t} + (a + b\mu) D\epsilon^{\mu t} \sin(\nu t + \psi) + b\nu D\epsilon^{\mu t} \cos(\nu t + \psi) - c\alpha A_1\epsilon^{\alpha t} \sin \theta_\alpha$$
$$- cD_1\epsilon^{\mu t}[\mu \sin(\nu t + \psi + \psi_1) + \nu \cos(\nu t + \psi + \psi_1)]$$
$$- cD_2\epsilon^{\mu t}[\mu \sin(\nu t + \psi + \psi_2) + \nu \cos(\nu t + \psi + \psi_2)]$$
$$- c\epsilon^{-\frac{a_1}{b_1}t}\left[\left(K\frac{a_1}{b_1} - K_1\omega\right) \cos \omega t + \left(K\omega + K_1\frac{a_1}{b_1}\right) \sin \omega t\right] = 0$$

that is

$$[(a + b\alpha) A - c\alpha A_1 \sin \theta_\alpha]\epsilon^{\alpha t}$$
$$+ D\epsilon^{\mu t}\left[(a + b\mu) - c\left(\mu\frac{D_1}{D} \cos \psi_1 - \nu\frac{D_1}{D} \sin \psi_1 + \mu\frac{D_2}{D} \cos \psi_2\right.\right.$$
$$\left.\left.- \nu\frac{D_2}{D} \sin \psi_2\right)\right] \sin(\nu t + \psi) + D\epsilon^{\mu t}\left[b\nu - c\left(\nu\frac{D_1}{D} \cos \psi_1 + \mu\frac{D_1}{D} \sin \psi_1\right.\right.$$
$$\left.\left.+ \nu\frac{D_2}{D} \cos \psi_2 + \mu\frac{D_2}{D} \sin \psi_2\right)\right] \cos(\nu t + \psi) - c\epsilon^{-\frac{a_1}{b_1}t}\sqrt{(K^2 + K_1^2)\left(\frac{a_1^2}{b_1^2} + \omega^2\right)}$$
$$\times \sin\left(\omega t + \tan^{-1}\frac{K\frac{a_1}{b_1} - K_1\omega}{K\omega - K_1\frac{a_1}{b_1}}\right) = 0$$

so that the conditions sufficient for $K = K_1 = 0$ are

$$(a + b\alpha) A - c\alpha A_1 \sin \theta_\alpha = 0$$

$$(a + b\mu) - c\mu \left(\frac{D_1}{D} \cos \psi_1 + \frac{D_2}{D} \cos \psi_2 \right) + c\nu \left(\frac{D_1}{D} \sin \psi_1 + \frac{D_2}{D} \sin \psi_2 \right) = 0$$

$$b\nu - c\nu \left(\frac{D_1}{D} \cos \psi_1 + \frac{D_2}{D} \cos \psi_2 \right) - c\mu \left(\frac{D_1}{D} \sin \psi_1 + \frac{D_2}{D} \sin \psi_2 \right) = 0$$

the first of which is satisfied as explained in Art. 14.

The two latter conditions give

$$c \left(\frac{D_1}{D} \cos \psi_1 + \frac{D_2}{D} \cos \psi_2 \right) = \frac{a\mu}{\mu^2 + \nu^2} + b$$

$$c \left(\frac{D_1}{D} \sin \psi_1 + \frac{D_2}{D} \sin \psi_2 \right) = \frac{-a\nu}{\mu^2 + \nu^2}$$

But

$$\frac{D_1}{D} \sin \psi_1 + \frac{D_2}{D} \sin \psi_2 = \frac{c}{2} \left(\frac{a_1 (\nu + \omega)}{(a_1 + b_1\mu)^2 + b_1^2 (\nu + \omega)^2} + \frac{a_1 (\nu - \omega)}{(a_1 + b_1\mu)^2 + b_1^2 (\nu - \omega)^2} \right)$$

$$= \frac{c a_1 \nu \left[(a_1 + b_1\mu)^2 + b_1^2 (\nu^2 - \omega^2) \right]}{\left[(a_1 + b_1\mu)^2 + b_1^2 (\nu + \omega)^2 \right] \left[(a_1 + b_1\mu)^2 + b_1^2 (\nu - \omega)^2 \right]}$$

and

$$\frac{D_1}{D} \cos \psi_1 + \frac{D_2}{D} \cos \psi_2 = \frac{c}{2} \left(\frac{\mu (a_1 + b_1\mu) + b_1 (\nu + \omega)^2}{(a_1 + b_1\mu)^2 + b_1^2 (\nu + \omega)^2} + \frac{\mu (a_1 + b_1\mu) + b_1 (\nu - \omega)^2}{(a_1 + b_1\mu)^2 + b_1^2 (\nu - \omega)^2} \right)$$

$$= \frac{\left[\mu (a_1 + b_1\mu) + b_1 (\nu^2 + \omega^2) \right] \left[(a_1 + b_1\mu)^2 + b_1^2 (\nu + \omega)^2 \right] - 4 b_1^3 \nu^2 \omega^2}{\left[(a_1 + b_1\mu)^2 + b_1^2 (\nu + \omega)^2 \right] \left[(a_1 + b_1\mu)^2 + b_1^2 (\nu - \omega)^2 \right]} \cdot c$$

Therefore the two latter conditions give

$$a = \frac{- c^2 a_1 \left[(a_1 + b_1\mu)^2 + b_1^2 (\nu^2 - \omega^2) \right] (\mu^2 + \nu^2)}{\left[(a_1 + b_1\mu)^2 + b_1^2 (\nu + \omega)^2 \right] \left[(a_1 + b_1\mu)^2 + b_1^2 (\nu - \omega)^2 \right]}$$

and

$$b = c^2 \frac{\mu (a_1 + b_1\mu)^3 + \left[a_1\mu + b_1 (\nu^2 + \omega^2) \right] (a_1 + b_1\mu) + 2 a_1 b_1^2 \mu \nu^2 + b_1^3 \mu^2 (\nu^2 + \omega^2) + b_1^3 (\nu^2 - \omega^2)^2}{\left[(a_1 + b_1\mu)^2 + b_1^2 (\nu + \omega)^2 \right] \left[(a_1 + b_1\mu)^2 + b_1^2 (\nu - \omega)^2 \right]}$$

Now, proceeding to examine whether these conditions are satisfied, we have, by putting $\beta, \gamma = \mu \pm j\nu$ in the cubic equation giving the roots α, β and γ,

$$A\mu (\mu^2 - 3\nu^2) + B (\mu^2 - \nu^2) + C\mu + D = 0$$

and

$$A (3\mu^2 - \nu^2) + 2B\mu + C = 0$$

and so also

$$(2A\mu + B)(\mu^2 + \nu^2) - D = 0$$

where $A = b_1(bb_1 - c^2)$ $\quad B = ab_1^2 + 2a_1bb_1 - a_1c^2$

$$C = a_1^2b + 2aa_1b_1 + b_1(bb_1 - c^2)\,\omega^2 \qquad D = a\,(a_1^2 + b_1^2\omega^2)$$

the latter two of which read

$$2b_1(a_1 + b_1\mu)\,a + [(a_1 + 2b_1\mu)^2 - b_1^2(\mu^2 + \nu^2 - \omega^2)]\,b$$
$$= c^2[2(a_1 + b_1\mu)\,\mu + b_1(\mu^2 - \nu^2 + \omega^2)]$$

and

$$[b_1^2(\mu^2 + \nu^2 - \omega^2) - a_1^2]\,a + 2b_1(a_1 + b_1\mu)(\mu^2 + \nu^2)\,b = c^2(a_1 + 2b_1\mu)(\mu^2 + \nu^2)$$

which when solved give the same values of a and b as those in the conditions to be satisfied.

Therefore we can conclude that $K = K_1 = 0$.

Note that, similarly as in Art. 14, the expressions of $\frac{D_1}{D}$, ψ_1, $\frac{D_2}{D}$ and ψ_2 can be checked as follows:—

Putting the complete solutions of x and y in the fundamental equation of y, we have

$$a_1[A_1\epsilon^{at}\sin(\omega t - \theta_a) - D_1\epsilon^{\mu t}\sin(\overline{\nu + \omega}t + \psi + \psi_1) - D_2\epsilon^{\mu t}\sin(\overline{\nu - \omega}t + \psi + \psi_2)]$$
$$+ b_1\{A_1\epsilon^{at}[\alpha\sin(\omega t - \theta_a) + \omega\cos(\omega t - \theta_a)]$$
$$- D_1\epsilon^{\mu t}[\mu\sin(\overline{\nu + \omega}t + \psi + \psi_1) + (\nu + \omega)\cos(\overline{\nu + \omega}t + \psi + \psi_1)]$$
$$- D_2\epsilon^{\mu t}[\mu\sin(\overline{\nu - \omega}t + \psi + \psi_2) + (\nu - \omega)\cos(\overline{\nu - \omega}t + \psi + \psi_2)]\}$$
$$+ c\,\{A\epsilon^{at}(\alpha\cos\omega t - \omega\sin\omega t) + D\epsilon^{\mu t}\mu\sin(\nu t + \psi)\cos\omega t$$
$$+ D\epsilon^{\mu t}[\nu\cos(\nu t + \psi)\cos\omega t - \omega\sin(\nu t + \psi)\sin\omega t]\} = 0$$

that is

$$\epsilon^{at}[(a_1 + b_1\alpha)A_1\cos\theta_a + b_1\omega A_1\sin\theta_a - c\omega A]\sin\omega t$$
$$- \epsilon^{at}[(a_1 + b_1\alpha)A_1\sin\theta_a - b_1\omega A_1\cos\theta_a - c\alpha A]\cos\omega t$$
$$- \epsilon^{\mu t}\left[(a_1 + b_1\mu)D_1\cos\psi_1 - b_1(\nu + \omega)D_1\sin\psi_1 - \frac{1}{2}c\mu D\right]\sin(\overline{\nu + \omega}t + \psi)$$
$$- \epsilon^{\mu t}\left[(a_1 + b_1\mu)D_1\sin\psi_1 + b_1(\nu + \omega)D_1\cos\psi_1 - \frac{1}{2}c(\nu + \omega)D\right]\cos(\overline{\nu + \omega}t + \psi)$$
$$- \epsilon^{\mu t}\left[(a_1 + b_1\mu)D_2\cos\psi_2 - b_1(\nu - \omega)D_2\sin\psi_2 - \frac{1}{2}c\mu D\right]\sin(\overline{\nu - \omega}t + \psi)$$
$$- \epsilon^{\mu t}\left[(a_1 + b_1\mu)D_2\sin\psi_2 + b_1(\nu + \omega)D_2\cos\psi_2 - \frac{1}{2}c(\nu - \omega)D\right]\cos(\overline{\nu - \omega}t + \psi)$$
$$= 0$$

so that

$$\frac{A}{A_1}=\frac{(a_1+b_1\alpha)\cos\theta_\alpha+b_1\omega\sin\theta_\alpha}{c\omega}=\frac{(a_1+b_1\alpha)\sin\theta_\alpha-b_1\omega\cos\theta_\alpha}{c\alpha}$$

$$\frac{D}{D_1}=2\,.\frac{(a_1+b_1\mu)\cos\psi_1-b_1(\nu+\omega)\sin\psi_1}{c\mu}=2\,.\frac{(a_1+b_1\mu)\sin\psi_1+b_1(\nu+\omega)\cos\psi_1}{c(\nu+\omega)}$$

$$\frac{D}{D_2}=2\,.\frac{(a_1+b_1\mu)\cos\psi_2-b_1(\nu-\omega)\sin\psi_2}{c\mu}=2\,.\frac{(a_1+b_1\mu)\sin\psi_2+b_1(\nu-\omega)\cos\psi_2}{c(\nu-\omega)}$$

The first of these three relations gives, as shown in Art. 14, the same values of A_1/A and θ_α as found before.

The second relation gives

$$(a_1+b_1\mu)\frac{D_1}{D}\cos\psi_1-b_1(\nu+\omega)\frac{D_1}{D}\sin\psi_1=\frac{1}{2}c\mu$$

and

$$b_1(\nu+\omega)\frac{D_1}{D}\cos\psi_1+(a_1+b_1\mu)\frac{D_1}{D}\sin\psi_1=\frac{1}{2}c(\nu+\omega)$$

so that

$$\frac{D_1}{D}\cos\psi_1=\frac{c}{2}\cdot\frac{\mu(a_1+b_1\mu)+b_1(\nu+\omega)^2}{(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2}\text{ and }\frac{D_1}{D}\sin\psi_1=\frac{a_1(\nu+\omega)}{(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2}$$

accordingly

$$\psi_1=\tan^{-1}\frac{a_1(\nu+\omega)}{\mu(a_1+b_1\mu)+b_1(\nu+\omega)^2}\text{ and }\frac{D_1}{D}=\frac{\frac{1}{2}c\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2}}$$

and similarly the third relation gives

$$\psi_2=\tan^{-1}\frac{a_1(\nu-\omega)}{\mu(a_1+b_1\mu)+b_1(\nu-\omega)^2}\text{ and }\frac{D_2}{D}=\frac{\frac{1}{2}c\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2}}$$

which are all the same as found before.

Note that, if we put the complete solution of x and z in the fundamental equation of z, we arrive at the same results as above.

§ 18. Another method of obtaining the transient terms when the roots β and γ are imaginary.

β and γ being $=\mu\pm j\nu$ we have, referring to Art. 14,

$$B_1\epsilon^{\beta t}\sin(\omega t-\theta_\beta)+C_1\epsilon^{\gamma t}\sin(\omega t-\theta_\gamma)$$
$$=\epsilon^{\mu t}\,.\left[B_1\epsilon^{j\nu t}\sin(\omega t-\theta_\beta)+C_1\epsilon^{-j\nu t}\sin(\omega t-\theta_\gamma)\right]$$

But

$$B_1\epsilon^{j\nu t}\sin(\omega t-\theta_\beta)+C_1\epsilon^{-j\nu t}\sin(\omega t-\theta_\gamma)$$

$$=B_1\epsilon^{j\nu t}\cdot\frac{1}{2j}\cdot\left(\epsilon^{j(\omega t-\theta_\beta)}-\epsilon^{-j(\omega t-\theta_\beta)}\right)$$

$$+C_1\epsilon^{-j\nu t}\cdot\frac{1}{2j}\cdot\left(\epsilon^{j(\omega t-\theta_\gamma)}-\epsilon^{-j(\omega t-\theta_\gamma)}\right)$$

$$=\frac{1}{2j}\cdot[B_1\epsilon^{-j\theta_\beta}\cdot\epsilon^{j(\nu+\omega)t}-B_1\epsilon^{j\theta_\beta}\cdot\epsilon^{j(\nu-\omega)t}$$

$$+C_1\epsilon^{-j\theta_\gamma}\cdot\epsilon^{-j(\nu-\omega)t}-C_1\epsilon^{j\theta_\gamma}\cdot\epsilon^{-j(\nu+\omega)t}]$$

$$=\frac{1}{2j}\{B_1(\cos\theta_\beta-j\sin\theta_\beta)[\cos(\nu+\omega)t+j\sin(\nu+\omega)t]$$

$$-B_1(\cos\theta_\beta+j\sin\theta_\beta)[\cos(\nu-\omega)t+j\sin(\nu-\omega)t]$$

$$+C_1(\cos\theta_\gamma-j\sin\theta_\gamma)[\cos(\nu-\omega)t-j\sin(\nu-\omega)t]$$

$$-C_1(\cos\theta_\gamma+j\sin\theta_\gamma)[\cos(\nu+\omega)t-j\sin(\nu+\omega)t]\}$$

$$=\left[\frac{1}{2j}(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)-\frac{1}{2}(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)\right]\cos(\nu+\omega)t$$

$$+\left[\frac{1}{2}(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)-\frac{1}{2}j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)\right]\sin(\nu+\omega)t$$

$$-\left[\frac{1}{2j}(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)+\frac{1}{2}(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)\right]\cos(\nu-\omega)t$$

$$-\left[\frac{1}{2}(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)+\frac{1}{2}j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)\right]\sin(\nu-\omega)t$$

But

$$\left[\frac{1}{2j}(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)\mp\frac{1}{2}(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)\right]^2$$

$$+\left[\frac{1}{2}(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)\mp\frac{1}{2}j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)\right]^2$$

$$=B_1C_1(\cos\theta_\beta\cos\theta_\gamma+\sin\theta_\beta\sin\theta_\gamma)\pm jB_1C_1(\cos\theta_\beta\sin\theta_\gamma-\sin\theta_\beta\cos\theta_\gamma)$$

$$=c^2BC.\frac{a_1^2\omega^2+[a_1\beta+b_1(\beta^2+\omega^2)][a_1\gamma+b_1(\gamma^2+\omega^2)]\pm ja_1\omega(\gamma-\beta)[a_1+b_1(\gamma+\beta)]}{[(a_1+b_1\beta)^2+b_1^2\omega^2][(a_1+b_1\gamma)^2+b_1^2\omega^2]}$$

But

$$a_1^2\omega^2+[a_1\beta+b_1(\beta^2+\omega^2)][a_1\gamma+b_1(\gamma^2+\omega^2)]\pm ja_1\omega(\gamma-\beta)[a_1+b_1(\gamma+\beta)]$$

$$=a_1[a_1+b_1(\beta+\gamma)][\omega^2+\beta\gamma\pm j\omega(\gamma-\beta)]+b_1^2[\beta^2\gamma^2+(\beta^2+\gamma^2)\omega^2+\omega^4]$$

$$=a_1(a_1+2b_1\mu)[\mu^2+(\nu\pm\omega)^2]+b_1^2[\mu^2+(\nu+\omega)^2][\mu^2+(\nu-\omega)^2]$$

$$=[\mu^2+(\nu\pm\omega)^2][(a_1+b_1\mu)^2+b_1^2(\nu\mp\omega)^2]$$

and

$$[(a_1+b_1\beta)^2+b_1^2\omega^2][(a_1+b_1\gamma)^2+b_1^2\omega^2]$$
$$=[(a_1+b_1\mu)^2+b_1^2(\omega^2-\nu^2)+j2b_1\nu(a_1+b_1\mu)]$$
$$\times[(a_1+b_1\mu)^2+b_1^2(\omega^2-\nu^2)-j2b_1\nu(a_1+b_1\mu)]$$
$$=[(a_1+b_1\mu)^2+b_1^2(\omega^2-\nu^2)]^2+4b_1^2\nu^2(a_1+b_1\mu)^2$$
$$=[(a_1+b_1\mu)^2+b_1^2(\omega^2+\nu^2)]^2-4b_1^4\nu^2\omega^2$$
$$=[(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2][(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2]$$

so that

$$\left[\frac{1}{2j}(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)\mp\frac{1}{2}(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)\right]^2$$
$$+\left[\frac{1}{2}(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)\mp\frac{1}{2}j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)\right]^2$$
$$=c^2BC\cdot\frac{\mu^2+(\nu\pm\omega)^2}{(a_1+b_1\mu)^2+b_1^2(\nu\pm\omega)^2}$$

and next

$$\frac{\frac{1}{2j}(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)\mp\frac{1}{2}(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)}{\frac{1}{2}(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)\mp\frac{1}{2}j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)}$$
$$=\frac{-j(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)\mp(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)}{(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)\mp j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)}$$
$$=\frac{-cB\dfrac{ja_1\omega\pm a_1\beta\pm b_1(\beta^2+\omega^2)}{(a_1+b_1\beta)^2+b_1^2\omega^2}+cC\dfrac{ja_1\omega\mp a_1\gamma\mp b_1(\gamma^2+\omega^2)}{(a_1+b_1\gamma)^2+b_1^2\omega^2}}{cB\dfrac{a_1\omega\mp ja_1\beta\mp jb_1(\beta^2+\omega^2)}{(a_1+b_1\beta)^2+b_1^2\omega^2}+cC\dfrac{a_1\omega\pm ja_1\gamma\pm jb_1(\gamma^2+\omega^2)}{(a_1+b_1\gamma)^2+b_1^2\omega^2}}$$

But

$$B[ja_1\omega\pm a_1\beta\pm b_1(\beta^2+\omega^2)][(a_1+b_1\gamma)^2+b_1^2\omega^2]$$
$$-C[ja_1\omega\mp a_1\gamma\mp b_1(\gamma^2+\omega^2)][(a_1+b_1\beta)^2+b_1^2\omega^2]$$
$$=(B+C)\{2b_1\nu(a_1+b_1\mu)[a_1\omega\pm\nu(a_1+2b_1\mu)]$$
$$\pm[\mu(a_1+b_1\mu)-b_1(\nu^2-\omega^2)][(a_1+b_1\mu)^2-b_1^2(\nu^2-\omega^2)]\}$$
$$+j(B-C)\{[a_1\omega\pm\nu(a_1+2b_1\mu)][(a_1+b_1\mu)^2-b_1^2(\nu^2-\omega^2)]$$
$$\mp 2b_1\nu(a_1+b_1\mu)[\mu(a_1+b_1\mu)-b_1(\nu^2-\omega^2)]\}$$
$$=\pm(B+C)[\mu(a_1+b_1\mu)+b_1(\nu\pm\omega)^2][(a_1+b_1\mu)^2+b_1^2(\nu\mp\omega)^2]$$
$$\pm j(B-C)a_1(\nu\pm\omega)[(a_1+b_1\mu)^2+b_1^2(\nu\mp\omega)^2]$$

and

$$B[a_1\omega \mp ja_1\beta \mp jb_1(\beta^2+\omega^2)][(a_1+b_1\gamma)^2+b_1^2\omega^2]$$
$$+C[a_1\omega \pm ja_1\gamma \pm jb_1(\gamma^2+\omega^2)][(a_1+b_1\beta)^2+b_1^2\omega^2]$$
$$=(B+C)\{[a_1\omega \mp \nu(a_1+2b_1\mu)][(a_1+b_1\mu)^2-b_1^2(\nu^2-\omega^2)]$$
$$\mp 2b_1\nu(a_1+b_1\mu)[\mu(a_1+b_1\mu)-b_1(\nu^2-\omega^2)]\}$$
$$-j(B-C)\{2b_1\nu(a_1+b_1\mu)[a_1\omega \pm \nu(a_1+2b_1\mu)]$$
$$\pm[\mu(a_1+b_1\mu)-b_1(\nu^2-\omega^2)][(a_1+b_1\mu)^2-b_1^2(\nu^2-\omega^2)]\}$$
$$=\pm(B+C)a_1(\nu\pm\omega)[(a_1+b_1\mu)^2+b_1^2(\nu\mp\omega)^2]$$
$$\mp j(B-C)[\mu(a_1+b_1\mu)+b_1(\nu\pm\omega)^2][(a_1+b_1\mu)^2+b_1^2(\nu\mp\omega)^2]$$

so that

$$\tan^{-1}\frac{\frac{1}{2j}(B_1\cos\theta_\beta - C_1\cos\theta_\gamma)\mp\frac{1}{2}(B_1\sin\theta_\beta + C_1\sin\theta_\gamma)}{\frac{1}{2}(B_1\cos\theta_\beta + C_1\cos\theta_\gamma)\mp\frac{1}{2}j(B_1\sin\theta_\beta - C_1\sin\theta_\gamma)}$$

$$=\tan^{-1}\frac{\mp(B+C)[\mu(a_1+b_1\mu)+b_1(\nu\pm\omega)^2]\mp j(B-C)a_1(\nu\pm\omega)}{\pm(B+C)a_1(\nu\pm\omega)\mp j(B-C)[\mu(a_1+b_1\mu)+b_1(\nu\pm\omega)^2]}$$

$$=\tan^{-1}\frac{\mp\left(\frac{B+C}{j(B-C)}+\frac{a_1(\nu\pm\omega)}{\mu(a_1+b_1\mu)+b_1(\nu\pm\omega)^2}\right)}{\mp\left(1-\frac{B+C}{j(B-C)}\cdot\frac{a_1(\nu\pm\omega)}{\mu(a_1+b_1\mu)+b_1(\nu\pm\omega)^2}\right)}$$

$$=\pi+\tan^{-1}\frac{B+C}{j(B-C)}+\tan^{-1}\frac{a_1(\nu+\omega)}{\mu(a_1+b_1\mu)+b_1(\nu+\omega)^2}$$

or $$\tan^{-1}\frac{B+C}{j(B-C)}+\tan^{-1}\frac{a_1(\nu-\omega)}{\mu(a_1+b_1\mu)+b_1(\nu-\omega)^2}$$

Therefore

$$B_1\epsilon^{\beta t}\sin(\omega t-\theta_\beta)+C_1\epsilon^{\gamma t}\sin(\omega t-\theta_\gamma)$$
$$=-c\sqrt{BC}\frac{\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2}}$$
$$\times\epsilon^{\mu t}\sin\left(\overline{\nu+\omega}\,t+\tan^{-1}\frac{B+C}{j(B-C)}+\tan^{-1}\frac{a_1(\nu+\omega)}{\mu(a_1+b_1\mu)+b_1(\nu+\omega)^2}\right)$$
$$-c\sqrt{BC}\frac{\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2}}$$
$$\times\epsilon^{\mu t}\sin\left(\overline{\nu-\omega}\,t+\tan^{-1}\frac{B+C}{j(B-C)}+\tan^{-1}\frac{a_1(\nu-\omega)}{\mu(a_1+b_1\mu)+b_1(\nu-\omega)^2}\right)$$

Next

$$B_1\epsilon^{\beta t}.\cos(\omega t-\theta_\beta)+C_1\epsilon^{\gamma t}.\cos(\omega t-\theta_\gamma)$$
$$=\epsilon^{\mu t}[B_1\epsilon^{j\nu t}.\cos(\omega t-\theta_\beta)+C_1\epsilon^{-j\nu t}.\cos(\omega t-\theta_\gamma)]$$

But

$$B_1\epsilon^{j\nu t}.\cos(\omega t-\theta_\beta)+C_1\epsilon^{-j\nu t}.\cos(\omega t-\theta_\gamma)$$
$$=B_1\epsilon^{j\nu t}.\frac{1}{2}(\epsilon^{j(\omega t-\theta_\beta)}+\epsilon^{-j(\omega t-\theta_\beta)})+C_1\epsilon^{-j\nu t}.\frac{1}{2}(\epsilon^{j(\omega t-\theta_\gamma)}+\epsilon^{-j(\omega t-\theta_\gamma)})$$
$$=\frac{1}{2}[B_1\epsilon^{-j\theta_\beta}.\epsilon^{j(\nu+\omega)t}+B_1\epsilon^{j\theta_\beta}.\epsilon^{j(\nu-\omega)t}+C_1\epsilon^{-j\theta_\gamma}.\epsilon^{-j(\nu-\omega)t}+C_1\epsilon^{j\theta_\gamma}.\epsilon^{-j(\nu+\omega)t}]$$

which, similarly as before, becomes

$$=\left[\frac{1}{2}(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)-\frac{1}{2}j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)\right]\cos(\nu+\omega)t$$
$$-\left[\frac{1}{2j}(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)-\frac{1}{2}(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)\right]\sin(\nu+\omega)t$$
$$+\left[\frac{1}{2}(B_1\cos\theta_\beta+C_1\cos\theta_\gamma)+\frac{1}{2}j(B_1\sin\theta_\beta-C_1\sin\theta_\gamma)\right]\cos(\nu-\omega)t$$
$$-\left[\frac{1}{2j}(B_1\cos\theta_\beta-C_1\cos\theta_\gamma)+\frac{1}{2}(B_1\sin\theta_\beta+C_1\sin\theta_\gamma)\right]\sin(\nu-\omega)t$$

Hence, similarly as before, we have

$$B_1\epsilon^{\beta t}\cos(\omega t-\theta_\beta)+C_1\epsilon^{\gamma t}\cos(\omega t-\theta_\gamma)$$
$$=-c\sqrt{BC}\frac{\sqrt{\mu^2+(\nu+\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2}}$$
$$\times\epsilon^{\mu t}\cos\left(\overline{\nu+\omega}t+\tan^{-1}\frac{B+C}{j(B-C)}+\tan^{-1}\frac{a_1(\nu+\omega)}{\mu(a_1+b_1\mu)+b_1(\nu+\omega)^2}\right)$$
$$+c\sqrt{BC}\frac{\sqrt{\mu^2+(\nu-\omega)^2}}{\sqrt{(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2}}$$
$$\times\epsilon^{\mu t}\cos\left(\overline{\nu-\omega}t+\tan^{-1}\frac{B+C}{j(B-C)}+\tan^{-1}\frac{a_1(\nu-\omega)}{\mu(a_1+b_1\mu)+b_1(\nu-\omega)^2}\right)$$

Thus the transient terms of y are

$$A_1\epsilon^{\alpha t}\sin(\omega t-\theta_\alpha)-D_1\epsilon^{\mu t}\sin(\overline{\nu+\omega}t+\psi+\psi_1)-D_2\epsilon^{\mu t}\sin(\overline{\nu-\omega}t+\psi+\psi_2)$$

and those of z are

$$-A_1\epsilon^{at}\cos(\omega t-\theta_\beta)+D_1\epsilon^{\mu t}\cos(\overline{\nu+\omega}t+\psi+\psi_1)-D_2\epsilon^{\mu t}\cos(\overline{\nu-\omega}t+\psi+\psi_2)$$

where A_1, D_1, D_2, θ_α, θ_β, ψ, ψ_1 and ψ_2 denote the same things as those given in the previous article.

Thus, by modifying the transient terms when the roots α, β and γ are all real, we have obtained those when the roots β and γ are imaginary, and these are quite the same as those obtained in the previous article.

§ 19. Determination of the arbitrary constants when the roots β and γ are imaginary.

We shall now determine the constants A', D', and ψ'. These constants being determined, the constants A_1', D_1' and D_2' can be obtained from the relations given in Art. 17.

The constant A' is that A' determined in Art. 15, that is

$$A'=\frac{-X_n\sqrt{2}\,[(a_1+b_1\alpha)^2+b_1^2\omega^2]\sqrt{(\beta^2+\omega^2)(\gamma^2+\omega^2)}}{a_1(\gamma-\alpha)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}$$

$$\times\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{\beta\gamma-\omega^2}{\omega(\beta+\gamma)}\right)$$

But

$$(\beta^2+\omega^2)(\gamma^2+\omega^2)=[\mu^2+(\nu+\omega)^2][\mu^2+(\nu-\omega)^2]$$

$$(\alpha-\beta)(\alpha-\gamma)=(\mu-\alpha)^2+\nu^2$$

$$\beta\gamma-\omega^2=\mu^2+\nu^2-\omega^2 \quad \text{and} \quad \beta+\gamma=2\mu$$

Therefore

$$A'=\frac{X_n\sqrt{2}\,[(a_1+b_1\alpha)^2+b_1^2\omega^2]\sqrt{\mu^2+(\nu+\omega)^2}\sqrt{\mu^2+(\nu-\omega)^2}}{a_1[(\mu-\alpha)^2+\nu^2]\sqrt{a_1^2+4b_1^2\omega^2}}$$

$$\times\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{\mu^2+\nu^2-\omega^2}{2\mu\omega}\right)$$

Next, in order to determine D' and ψ', let us make use of the relations

$$D'=2\sqrt{B'C'} \quad \text{and} \quad \psi'=\tan^{-1}\frac{B'+C'}{j(B'-C')}$$

Putting those values of B' and C' determined in Art. 15, we have

$$B'C' = \frac{(X_n\sqrt{2})^2[(a_1+b_1\beta)^2+b_1^2\omega^2][(a_1+b_1\gamma)^2+b_1^2\omega^2](\alpha^2+\omega^2)\sqrt{(\beta^2+\omega^2)(\gamma^2+\omega^2)}}{a_1^2(\beta-\gamma)^2(\gamma-\alpha)(\alpha-\beta)(a_1^2+4b_1^2\omega^2)}$$

$$\times \sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)}\right)$$

$$\times \sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)$$

But

$$[(a_1+b_1\beta)^2+b_1^2\omega^2]\,[(a_1+b_1\gamma)^2+b_1^2\omega^2]$$

$$= [(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2]\,[(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2]$$

$$(\beta^2+\omega^2)(\gamma^2+\omega^2) = [\mu^2+(\nu+\omega)^2]\,[\mu^2+(\nu-\omega)^2]$$

$$(\alpha-\beta)(\alpha-\gamma) = (\mu-\alpha)^2+\nu^2$$

$$(\beta-\gamma)^2 = -4\nu^2$$

$$\sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)}\right)$$

$$\times \sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)$$

$$= \frac{1}{2}\left[\cos\left(\tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)} - \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)\right.$$

$$\left. - \cos\left(2\omega t_0 - 2\phi_n + 2\tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)} + \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)\right]$$

$$\cos\left(\tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)} - \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)$$

$$= \cos\tan^{-1}\frac{(\gamma\alpha-\omega^2)\,\omega(\alpha+\beta) - (\alpha\beta-\omega^2)\,\omega(\gamma+\alpha)}{\omega^2(\alpha+\beta)(\gamma+\alpha) + (\gamma\alpha-\omega^2)(\alpha\beta-\omega^2)}$$

$$= \frac{\omega^2(\gamma+\alpha)(\alpha+\beta) + (\gamma\alpha-\omega^2)(\alpha\beta-\omega^2)}{\sqrt{[\omega^2.(\alpha+\beta)^2+(\alpha\beta-\omega^2)^2][\omega^2(\gamma+\alpha)^2+(\gamma\alpha-\omega^2)^2]}}$$

$$= \frac{\mu^2+\nu^2+\omega^2}{\sqrt{\mu^2+(\nu+\omega)^2}\sqrt{\mu^2+(\nu-\omega)^2}}$$

$$\tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)} + \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)} = \tan^{-1}\frac{(\gamma\alpha-\omega^2)\,\omega(\alpha+\beta) + (\alpha\beta-\omega^2)\,\omega(\gamma+\alpha)}{\omega^2(\gamma+\alpha)(\alpha+\beta) - (\gamma\alpha-\omega^2)(\alpha\beta-\omega^2)}$$

$$= \tan^{-1}\frac{\omega[\alpha^2(\beta+\gamma)+2\alpha\beta\gamma-\omega^2(2\alpha+\beta+\gamma)]}{\omega^2[\alpha^2+\beta\gamma+2\alpha(\beta+\gamma)-\alpha^2\beta\gamma-\omega^4]}$$

$$= \tan^{-1}\frac{2\omega[\alpha(\mu^2+\nu^2-\omega^2)+\mu(\alpha^2-\omega^2)]}{4\omega^2\mu\alpha-(\alpha^2-\omega^2)(\mu^2+\nu^2-\omega^2)}$$

$$= \tan^{-1}\frac{\alpha^2-\omega^2}{2\alpha\omega} + \tan^{-1}\frac{\mu^2+\nu^2-\omega^2}{2\mu\omega}$$

Therefore

$$D' = \frac{X_n\sqrt{2}}{a_1\nu}$$

$$\times \sqrt{\frac{[(a_1+b_1\mu)^2+b_1^2(\nu+\omega)^2][(a_1+b_1\mu)^2+b_1^2(\nu-\omega)^2](\alpha^2+\omega^2)\sqrt{\mu^2+(\nu+\omega)^2}\sqrt{\mu^2+(\nu-\omega)^2}}{2[(\mu-\alpha)^2+\nu^2](a_1^2+4b_1^2\omega^2)}}$$

$$\times \sqrt{\left[\frac{\mu^2+\nu^2+\omega^2}{\sqrt{\mu^2+(\nu+\omega)^2}\sqrt{\mu^2+(\nu-\omega)^2}}\right.}$$

$$\left. - \cos\left(2\omega t_0 - 2\phi_n + 2\tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\alpha^2-\omega^2}{2\alpha\omega} + \tan^{-1}\frac{\mu^2+\nu^2-\omega^2}{2\mu\omega}\right)\right]$$

Next also from Art. 15, we have

$$B' \pm C' = \frac{-X_n\sqrt{2}\sqrt{\alpha^2+\omega^2}}{a_1(\beta-\gamma)\sqrt{a_1^2+4b_1^2\omega^2}}$$

$$\times \left\{\frac{[(a_1+b_1\beta)^2+b_1^2\omega^2]\sqrt{\gamma^2+\omega^2}}{\alpha-\beta}\sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega}\right.\right.$$

$$\left. + \tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)}\right) \pm \frac{[(a_1+b_1\gamma)^2+b_1^2\omega^2]\sqrt{\beta^2+\omega^2}}{\gamma-\alpha}$$

$$\left.\times \sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)\right\}$$

$$= G\sin\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega}\right) + H\cos\left(\omega t_0 - \phi_n + \tan^{-1}\frac{a_1}{2b_1\omega}\right)$$

where

$$G = \frac{-X_n\sqrt{2}\sqrt{\alpha^2+\omega^2}}{a_1(\beta-\gamma)\sqrt{a_1^2+4b_1^2\omega^2}}\left(\frac{[(a_1+b_1\beta)^2+b_1^2\omega^2]\sqrt{\gamma^2+\omega^2}}{\alpha-\beta}\,.\cos\tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)}\right.$$

$$\left.\pm \frac{[(a_1+b_1\gamma)^2+b_1^2\omega^2]\sqrt{\beta^2+\omega^2}}{\gamma-\alpha}\,.\cos\tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)$$

and

$$H = \frac{-X_n\sqrt{2}\sqrt{\alpha^2+\omega^2}}{a_1(\beta-\gamma)\sqrt{a_1^2+4b_1^2\omega^2}}\left(\frac{[(a_1+b_1\beta)^2+b_1^2\omega^2]\sqrt{\gamma^2+\omega^2}}{\alpha-\beta}\,.\sin\tan^{-1}\frac{\gamma\alpha-\omega^2}{\omega(\gamma+\alpha)}\right.$$

$$\left.\pm \frac{[(a_1+b_1\gamma)^2+b_1^2\omega^2]\sqrt{\beta^2+\omega^2}}{\gamma-\alpha}\,.\sin\tan^{-1}\frac{\alpha\beta-\omega^2}{\omega(\alpha+\beta)}\right)$$

But $\omega^2(\gamma+\alpha)^2 + (\gamma\alpha-\omega^2)^2 = (\gamma^2+\omega^2)(\alpha^2+\omega^2)$

and $\omega^2(\alpha+\beta)^2 + (\alpha\beta-\omega^2)^2 = (\alpha^2+\omega^2)(\beta^2+\omega^2)$

so that

$$G = \frac{-X_n\sqrt{2}}{a_1(\beta-\gamma)\sqrt{a_1^2+4b_1^2\omega^2}}\left(\frac{[(a_1+b_1\beta)^2+b_1^2\omega^2]\,\omega\,(\gamma+\alpha)}{\alpha-\beta} \pm \frac{[(a_1+b_1\gamma)^2+b_1^2\omega^2]\,\omega\,(\alpha+\beta)}{\gamma-\alpha}\right)$$

$$= -X_n\sqrt{2}\,\frac{[(a_1+b_1\beta)^2+b_1^2\omega^2]\,\omega\,(\gamma^2-\alpha^2) \pm [(a_1+b_1\gamma)^2+b_1^2\omega^2]\,\omega\,(\alpha^2-\beta^2)}{a_1(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}$$

and similarly

$$H = -X_n\sqrt{2}\,\frac{[(a_1+b_1\beta)^2+b_1^2\omega^2](\gamma\alpha-\omega^2)(\gamma-\alpha) \pm [(a_1+b_1\gamma)^2+b_1^2\omega^2](\alpha\beta-\omega^2)(\alpha-\beta)}{a_1(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}$$

But

$$[(a_1+b_1\beta)^2+b_1^2\omega^2]\,\omega\,(\gamma^2-\alpha^2)+[(a_1+b_1\gamma)^2+b_1^2\omega^2]\,\omega\,(\alpha^2-\beta^2) = 2b_1^2\omega^4(\gamma-\beta)\,Q_1$$

where $$Q_1 = \left[\left(\frac{a_1}{b_1\omega}+\frac{\alpha}{\omega}\right)^2+1\right]\frac{\mu}{\omega}+\frac{a_1}{b_1\omega}\left[\left(\frac{\mu}{\omega}-\frac{\alpha}{\omega}\right)^2+\frac{\nu^2}{\omega^2}\right]$$

$$[(a_1+b_1\beta)^2+b_1^2\omega^2](\gamma\alpha-\omega^2)(\gamma-\alpha)+[(a_1+b_1\gamma)^2+b_1^2\omega^2](\alpha\beta-\omega^2)(\alpha-\beta) = b_1^2\omega^4(\gamma-\beta)\,P_1$$

where $$P_1 = \left[\left(\frac{a_1}{b_1\omega}+\frac{\alpha}{\omega}\right)^2+1\right]\left(\frac{\mu^2}{\omega^2}+\frac{\nu^2}{\omega^2}-1\right)-\frac{a_1^2}{b_1^2\omega^2}\left[\left(\frac{\mu}{\omega}-\frac{\alpha}{\omega}\right)^2+\frac{\nu^2}{\omega^2}\right]$$

$$[(a_1+b_1\beta)^2+b_1^2\omega^2]\,\omega\,(\gamma^2-\alpha^2)-[(a_1+b_1\gamma)^2+b_1^2\omega^2]\,\omega\,(\alpha^2-\beta^2) = 2b_1^2\omega^5 Q_2$$

where $$Q_2 = \left[\left(\frac{a_1}{b_1\omega}+\frac{\mu}{\omega}\right)^2+1\right]\left(\frac{\mu^2}{\omega^2}-\frac{\alpha^2}{\omega^2}\right) - \frac{\nu^2}{\omega^2}\cdot\left[\frac{a_1^2}{b_1^2\omega^2}-\left(\frac{\mu^2}{\omega^2}+\frac{\nu^2}{\omega^2}+\frac{\alpha^2}{\omega^2}-1\right)-\frac{\mu}{\omega}\left(2\frac{a_1}{b_1\omega}+\frac{\mu}{\omega}\right)\right]$$

and

$$[(a_1+b_1\beta)^2+b_1^2\omega^2](\gamma\alpha-\omega^2)(\gamma-\alpha)-[(a_1+b_1\gamma)^2+b_1^2\omega^2](\alpha\beta-\omega^2)(\alpha-\beta) = 2b_1^2\omega^5 P_2$$

where $$P_2 = \left[\left(\frac{a_1}{b_1\omega}+\frac{\mu}{\omega}\right)^2+1\right]\left(\frac{\mu\alpha}{\omega^2}-1\right)\left(\frac{\mu}{\omega}-\frac{\alpha}{\omega}\right) - \frac{\nu^2}{\omega^2}\cdot\frac{\alpha}{\omega}\left[\frac{a_1^2}{b_1^2\omega^2}-\left(\frac{\mu}{\omega^2}+\frac{\nu^2}{\omega^2}-2\right)\right]-\frac{\nu^2}{\omega^2}\left(2\frac{a_1}{b_1\omega}+\frac{\mu}{\omega}\right)\left(\frac{\alpha^2}{\omega^2}-\frac{\mu\alpha}{\omega^2}+1\right)$$

Therefore

$$B' + C' = \frac{-X_n\sqrt{2}\,b_1^2\omega^4(\gamma-\beta)}{a_1(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}$$

$$\times\left[2Q_1\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}\right)+P_1\cos\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}\right)\right]$$

$$= \frac{-X_n\sqrt{2}\,b_1^2\omega^4}{a_1(\alpha-\gamma)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}\sqrt{P_1^2+4Q_1^2}$$

$$\times\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{P_1}{2Q_1}\right)$$

$$= \frac{-X_n\sqrt{2}\,b_1^2\omega^4}{a_1[(\mu-\alpha)^2+\nu^2]\sqrt{a_1^2+4b_1^2\omega^2}}\sqrt{P_1^2+4Q_1^2}$$

$$\times\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{P_1}{2Q_1}\right)$$

and

$$B' - C' = \frac{-X_n\sqrt{2}\,.\,2b_1^2\omega^5}{a_1(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}$$

$$\times\left[Q_2\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}\right)+P_2\cos\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}\right)\right]$$

$$= \frac{X_n\sqrt{2}\,b_1^2\omega^5}{ja_1\nu(\alpha-\gamma)(\alpha-\beta)\sqrt{a_1^2+4b_1^2\omega^2}}\sqrt{P_2^2+Q_2^2}$$

$$\times\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{P_2}{Q_2}\right)$$

$$= \frac{X_n\sqrt{2}\,b_1^2\omega^5}{ja_1\nu[(\mu-\alpha)^2+\nu^2]\sqrt{a_1^2+4b_1^2\omega^2}}\sqrt{P_2^2+Q_2^2}$$

$$\times\sin\left(\omega t_0-\phi_n+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{P_2}{Q_2}\right)$$

so that

$$\psi' = \tan^{-1}\frac{-\nu\,.\,\sqrt{P_1^2+4Q_1^2}\,.\,\sin\left(\omega t_0-\phi_n+\tan^{-1}\dfrac{a_1}{b_1\omega}+\tan^{-1}\dfrac{P_1}{2Q_1}\right)}{\omega\,.\,\sqrt{P_2^2+Q_2^2}\,.\,\sin\left(\omega t_0-\phi_n+\tan^{-1}\dfrac{a_1}{b_1\omega}+\tan^{-1}\dfrac{P_2}{Q_2}\right)}$$

§ 20. Sudden short-circuit of the distortionless alternator with two field windings.

In the preceding articles we have given the complete solution for any sudden change of load. In this article we shall, as a special case, compute the field and armature currents when the distortionless alternator running at no load is suddenly short-circuited.

In the case of short-circuit, as shown in Art. 16, the three roots are

$$\alpha = -\frac{p}{\sigma}\omega, \quad \beta = -\frac{1}{2}(1+\sigma)\frac{q}{\sigma}\omega + j\omega \quad \text{and} \quad \gamma = -\frac{1}{2}(1+\sigma)\frac{q}{\sigma}\omega - j\omega$$

so that

$$(a_1 + b_1\alpha)^2 + b_1^2\omega^2 = b_1^2\omega^2\left[\left(q - \frac{p}{\sigma}\right)^2 + 1\right] \doteqdot b_1^2\omega^2$$

for $\left(\frac{p}{\sigma}\right)^2$, $\left(\frac{q}{\sigma}\right)^2$ and $\frac{2pq}{\sigma^2}$ are usually negligible compared with unity (refer to Art. 16),

$$[\mu^2 + (\nu+\omega)^2][\mu^2 + (\nu-\omega)^2] = \omega^4\left[\frac{1}{4}(1+\sigma)^2\frac{q^2}{\sigma^2} + 4\right]\frac{1}{4}(1+\sigma)^2\frac{q^2}{\sigma^2}$$

$$\doteqdot \omega^4(1+\sigma)^2\left(\frac{q}{\sigma}\right)^2$$

$$(\mu - \alpha)^2 + \nu^2 = \omega^2\left[-\frac{1}{2}(1+\sigma)\frac{q}{\sigma} + \frac{p}{\sigma}\right]^2 + \omega^2 \doteqdot \omega^2$$

$$\mu^2 + \nu^2 - \omega^2 = \frac{1}{4}(1+\sigma)^2\left(\frac{q}{\sigma}\right)^2\omega^2$$

$$2\mu\omega = -(1+\sigma)\,.\,\frac{q}{\sigma}\,.\,\omega^2$$

Therefore denoting by $X_s\sqrt{2}$ and ϕ_{xs} the amplitude and the phase angle of the permanent short-circuit current in the armature, we have (referring to Art. 19)

$$A' = \frac{X_s\sqrt{2}\,b_1^2\omega^4(1+\sigma)\frac{q}{\sigma}}{a_1\omega^2\sqrt{a_1^2 + 4b_1^2\omega^2}}\sin\left(\omega t_0 - \phi_{xs} + \tan^{-1}\frac{a_1}{2b_1\omega} + \pi - \tan^{-1}\frac{1}{4}(1+\sigma)\frac{q}{\sigma}\right)$$

$$\doteqdot -X_s\sqrt{2}\,.\,\frac{1+\sigma}{2\sigma}\,.\,\sin(\omega t_0 - \phi_{xs})$$

$$\doteqdot \frac{-d}{b_i\omega\sigma}\sin\left(\omega t_0 - \frac{\pi}{2}\right)$$

for $\quad X_s\sqrt{2} \doteqdot \dfrac{2d}{b_i\omega(1+\sigma)}$ and $\phi_{xs} \doteqdot \dfrac{\pi}{2}$ (refer to Art. 10).

Next, proceeding to find D' and ψ', we have

$$(a_1 + b_1\mu)^2 + b_1^2(\nu + \omega)^2 = b_1^2\omega^2\left[\left(q - \frac{1+\sigma}{2\sigma}q\right)^2 + 4\right] \doteqdot 4b_1^2\omega^2$$

$$(a_1 + b_1\mu)^2 + b_1^2(\nu - \omega)^2 = b_1^2\omega^2\left(q - \frac{1+\sigma}{2\sigma}q\right)^2 = \frac{1}{4}a_1^2\left(\frac{1}{\sigma} - 1\right)^2$$

$$\alpha^2 + \omega^2 = \left(\frac{p}{\sigma}\right)^2\omega^2 + \omega^2 \doteqdot \omega^2$$

$$[\mu^2 + (\nu+\omega)^2][\mu^2 + (\nu-\omega)^2] = \omega^4(1+\sigma)^2\left(\frac{q}{\sigma}\right)^2$$

$$(\mu - \alpha)^2 + \nu^2 \doteqdot \omega^2$$

$$\mu^2 + \nu^2 + \omega^2 \doteqdot 2\omega^2$$

so that referring to Art. 19,

$$D' = \frac{X_s\sqrt{2}}{a_1\omega}\sqrt{\frac{4b_1^2\omega^2\frac{1}{4}a_1^2\left(\frac{1}{\sigma}-1\right)^2\omega^4(1+\sigma)\frac{q}{\sigma}}{2\omega^2(a_1^2 + 4b_1^2\omega^2)}}$$

$$\times\sqrt{\left[\frac{2\omega^2}{\omega^2(1+\sigma)\frac{q}{\sigma}} - \cos\left(2\omega t_0 - 2\phi_{xs} + 2\tan^{-1}\frac{a_1}{2b_1\omega} + \tan^{-1}\frac{-\omega^2}{-2\frac{p}{\sigma}\omega^2} + \pi - \tan^{-1}\frac{1}{4}(1+\sigma)\frac{q}{\sigma}\right)\right]}$$

$$\doteqdot \frac{X_s\sqrt{2}}{a_1\omega}\sqrt{\frac{b_1^2\omega^2a_1^2\left(\frac{1}{\sigma}-1\right)^2(1+\sigma)q}{8b_1^2\sigma}}\cdot\sqrt{\frac{2\sigma}{(1+\sigma)q}}$$

$$= X_s\sqrt{2}\cdot\frac{1}{2}\left(\frac{1}{\sigma}-1\right)$$

$$= \frac{d}{b_i\omega\sigma}\cdot\frac{1-\sigma}{1+\sigma}$$

Also since

$$Q_1 \doteqdot \frac{\mu}{\omega} + q$$

$$P_1 \doteqdot \frac{\mu^2}{\omega^2} - q^2$$

$$Q_2 \doteqdot \left(\frac{\mu^2}{\omega^2} - \frac{\alpha^2}{\omega^2}\right) - \left[q^2 - \left(\frac{\mu^2}{\omega^2} + \frac{\alpha^2}{\omega^2}\right) - \frac{\mu}{\omega}\left(2q + \frac{\mu}{\omega}\right)\right] = 3\frac{\mu^2}{\omega^2} - q^2 + 2q\frac{\mu}{\omega}$$

$$= \left(\frac{\mu}{\omega} + q\right)\left(3\frac{\mu}{\omega} - q\right)$$

$$P_2 \doteqdot -\left(\frac{\mu}{\omega} - \frac{\alpha}{\omega}\right) - \frac{\alpha}{\omega} - \left(2q + \frac{\mu}{\omega}\right) = -2\left(\frac{\mu}{\omega} + q\right)$$

so that

$$\sqrt{P_1^2+4Q_1^2}=2\left(\frac{\mu}{\omega}+q\right)$$

$$\sqrt{P_2^2+Q_2^2}=2\left(\frac{\mu}{\omega}+q\right)$$

$$\frac{P_1}{2Q_1}=\frac{-\left(\frac{\mu}{\omega}-q\right)}{-2} \quad \text{for } \frac{\mu}{\omega}+q=-\frac{1}{2}(1-\sigma)\frac{q}{\sigma}<0$$

$$\frac{P_2}{Q_2}=\frac{2}{q-3\frac{\mu}{\omega}}$$

we have

$$\begin{aligned}\psi'&=\tan^{-1}\frac{-\sin\left[\omega t_0-\phi_{xs}+\tan^{-1}\frac{a_1}{2b_1\omega}+\pi+\tan^{-1}\frac{1}{2}\left(\frac{\mu}{\omega}-q\right)\right]}{\sin\left(\omega t_0-\phi_{xs}+\tan^{-1}\frac{a_1}{2b_1\omega}+\tan^{-1}\frac{2}{q-3\frac{\mu}{\omega}}\right)}\\&=\tan^{-1}\frac{\sin(\omega t_0-\phi_{xs})}{\cos(\omega t_0-\phi_{xs})}=\omega t_0-\phi_{xs}\\&=\omega t_0-\frac{\pi}{2}\end{aligned}$$

accordingly $$\nu(t-t_0)+\psi'=\omega t-\frac{\pi}{2}$$

Hence, referring to Arts. 10 and 19, the expression of the armature current x_s at sudden short-circuit is

$$\begin{aligned}x_s&=\frac{2d}{b_i\omega(1+\sigma)}\sin\left(\omega t-\frac{\pi}{2}\right)+\frac{-d}{b_i\omega\sigma}\cdot\epsilon^{a(t-t_0)}.\sin\left(\omega t_0-\frac{\pi}{2}\right)\\&\qquad+\frac{d}{b_i\omega\sigma}\cdot\frac{1-\sigma}{1+\sigma}\cdot\epsilon^{\mu(t-t_0)}.\sin\left(\omega t-\frac{\pi}{2}\right)\\&=\frac{-2d}{b_i\omega(1+\sigma)}\cos\omega t+\frac{d}{b_i\omega\sigma}\cdot\epsilon^{a(t-t_0)}.\cos\omega t_0\\&\qquad+\frac{-d}{b_i\omega\sigma}\cdot\frac{1-\sigma}{1+\sigma}\cdot\epsilon^{\mu(t-t_0)}.\cos\omega t\end{aligned}$$

which is obviously zero when $t=t_0$.

This expression of x_s coincides with the result obtained by Mr Boucherot*.

Next, proceeding to calculate the field current at sudden short-circuit, we have

$$A_1' = A' \cdot \frac{c}{b_1} \cdot \sqrt{\frac{\left(\frac{\alpha}{\omega}\right)^2 + 1}{\left(q + \frac{\alpha}{\omega}\right)^2 + 1}} \doteqdot A' \cdot \frac{c}{b_1}$$

$$\theta_a = \tan^{-1} \frac{q \cdot \frac{\alpha}{\omega} + \left(\frac{\alpha}{\omega}\right)^2 + 1}{q} = \tan^{-1} \frac{1}{q} \doteqdot \frac{\pi}{2}$$

$$D_1' = \frac{1}{2} D' \cdot \frac{c}{b_1} \cdot \sqrt{\frac{\left(\frac{\mu}{\omega}\right)^2 + 4}{\left(q + \frac{\mu}{\omega}\right)^2 + 4}} \doteqdot \frac{1}{2} \cdot \frac{c}{b_1} \cdot D'$$

$$D_2' = \frac{1}{2} D' \cdot \frac{c}{b_1} \cdot \sqrt{\frac{\left(\frac{\mu}{\omega}\right)^2}{\left(q + \frac{\mu}{\omega}\right)^2}} = \frac{1}{2} D' \cdot \frac{c}{b_1} \cdot \frac{-\frac{1}{2}(1+\sigma)\frac{q}{\sigma}}{q - \frac{1}{2}(1+\sigma)\frac{q}{\sigma}}$$

$$= \frac{1}{2} D' \cdot \frac{c}{b_1} \cdot \frac{1+\sigma}{1-\sigma}$$

$$\psi_1 = \tan^{-1} \frac{2q}{\frac{\mu}{\omega}\left(q + \frac{\mu}{\omega}\right) + 4} \doteqdot \tan^{-1} \frac{1}{2} q \doteqdot 0$$

$$\psi_2 = 0 \text{ for } \nu = \omega$$

so that

$$\nu(t - t_0) + \omega t + \psi' + \psi_1 = \omega(t - t_0) + \omega t + \omega t_0 - \frac{\pi}{2} = 2\omega t - \frac{\pi}{2}$$

and

$$\nu(t - t_0) - \omega t + \psi' + \psi_2 = -\frac{\pi}{2}$$

---

* See p. 183.

accordingly

$$A_1' \sin(\omega t - \theta_a) = A' \cdot \frac{c}{b_1} \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= -\frac{d}{b_i \omega\sigma} \cdot \frac{c}{b_1} \sin\left(\omega t_0 - \frac{\pi}{2}\right) \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= \frac{-d}{b_i \omega\sigma} \cdot \frac{c}{b_1} \cdot \cos \omega t_0 \cos \omega t$$

$$A_1' \cos(\omega t - \theta_a) = \frac{d}{b_i \omega\sigma} \cdot \frac{c}{b_1} \cdot \cos \omega t_0 \sin \omega t$$

$$D_1' \sin[\nu(t - t_0) + \omega t + \psi' + \psi_1] = \frac{1}{2} D' \cdot \frac{c}{b_1} \sin\left(2\omega t - \frac{\pi}{2}\right)$$

$$= -\frac{d}{2b_i \omega\sigma} \cdot \frac{1-\sigma}{1+\sigma} \cdot \frac{c}{b_1} \cdot \cos 2\omega t$$

$$D_1' \cos[\nu(t - t_0) + \omega t + \psi' + \psi_1] = \frac{d}{2b_i \omega\sigma} \cdot \frac{1-\sigma}{1+\sigma} \cdot \frac{c}{b_1} \sin 2\omega t$$

$$D_2' \sin[\nu(t - t_0) - \omega t + \psi' + \psi_2] = \frac{1}{2} D' \frac{c}{b_1} \cdot \frac{1+\sigma}{1-\sigma} \cdot \sin\left(-\frac{\pi}{2}\right)$$

$$= -\frac{d}{2b_i \omega\sigma} \cdot \frac{c}{b_1}$$

$$D_2' \cos[\nu(t - t_0) - \omega t + \psi' + \psi_2] = \frac{1}{2} D' \frac{c}{b_1} \cdot \frac{1+\sigma}{1-\sigma} \cos\left(-\frac{\pi}{2}\right)$$

$$= 0$$

Hence, referring to Arts. 10 and 19, the expression of the alternating current in the direct field winding is

$$y_s = \frac{d}{b_i \omega(1+\sigma)} \cdot \frac{c}{b_1} \cos 2\omega t + \frac{-d}{b_i \omega\sigma} \cdot \frac{c}{b_1} \cdot \epsilon^{\alpha(t-t_0)} \cdot \cos \omega t_0 \cos \omega t$$

$$+ \frac{d}{2b_i \omega\sigma} \cdot \frac{c}{b_1} \cdot \frac{1-\sigma}{1+\sigma} \cdot \epsilon^{\mu(t-t_0)} \cdot \cos 2\omega t$$

$$+ \frac{d}{2b_i \omega\sigma} \cdot \frac{c}{b_1} \cdot \epsilon^{\mu(t-t_0)}$$

and the expression of the current in the cross field winding is

$$z_s = \frac{d}{b_i \omega(1+\sigma)} \cdot \frac{c}{b_1} \cdot \sin 2\omega t + \frac{-d}{b_i \omega\sigma} \cdot \frac{c}{b_1} \cdot \epsilon^{\alpha(t-t_0)} \cdot \cos \omega t_0 \sin \omega t$$

$$+ \frac{d}{2b_i \omega\sigma} \cdot \frac{c}{b_1} \cdot \frac{1-\sigma}{1+\sigma} \cdot \epsilon^{\mu(t-t_0)} \cdot \sin 2\omega t$$

These expressions of y_s and z_s coincide with the results obtained by Mr Boucherot*.

* See p. 183.

Here note that

$$(y_s)_{t=t_0} = \frac{d}{b_i\omega} \cdot \frac{c}{b_1} \cdot \left[\left(\frac{1}{1+\sigma} + \frac{1}{2\sigma} \cdot \frac{1-\sigma}{1+\sigma} \right) \cos 2\omega t_0 - \frac{1}{2\sigma} (2 \cos^2 \omega t_0 - 1) \right]$$

$$= \frac{d}{b_i\omega} \cdot \frac{c}{b_1} \cdot \frac{1}{2\sigma} \cdot [\cos 2\omega t_0 - (2\cos^2 \omega t_0 - 1)]$$

$$= 0$$

and $$(z_s)_{t=t_0} = \frac{d}{b_i\omega} \cdot \frac{c}{b_1} \cdot \left(\frac{1}{1+\sigma} - \frac{1}{2\sigma} + \frac{1}{2\sigma} \cdot \frac{1-\sigma}{1+\sigma} \right) \sin 2\omega t_0$$

$$= 0$$

also note that $$\frac{cd}{b_i b_1 \omega (1+\sigma)} = I_f \frac{1-\sigma}{1+\sigma}$$

also note that, since $\alpha = -\frac{p}{\sigma}\omega$ and $\mu = -\frac{1}{2}(1+\sigma)\frac{q}{\sigma}\omega$, the smaller the leakage, the greater will be the rate of diminution of the transient currents, and the greater the leakage, the smaller the rate of diminution. Also since $\frac{1-\sigma}{\sigma(1+\sigma)} = \infty$ when $\sigma = 0$ and $\frac{1-\sigma}{\sigma(1+\sigma)} = 0$ when $\sigma = 1$ and there are no maxima and minima of $\frac{1-\sigma}{\sigma(1+\sigma)}$ between $\sigma = 0$ and $\sigma = 1$, we can conclude that the smaller the leakage the greater will be the initial value of all the transient currents, and the greater the leakage the smaller will be the initial values.

§ 21. Direct solution of the sudden short-circuit currents of the distortionless alternator with two field windings.

In the previous article we solved the case of sudden short-circuit as a special case of general sudden change of load. In this article we shall show that the expressions of the sudden short-circuit currents can be obtained directly from their fundamental equations without considering the case of general sudden change of load.

As shown in the beginning of Art. 17, the expression of the sudden short-circuit current x_s in the armature is

$$x_s = X_s\sqrt{2}\sin(\omega t - \phi_{xs}) + A\epsilon^{\alpha t} + D\epsilon^{\mu t}\sin(\omega t + \psi)$$

that is $$x_s = X_s\sqrt{2}\sin(\omega t - \phi_{xs}) + A\epsilon^{\alpha t} + E\epsilon^{\mu t}\sin\omega t + F\epsilon^{\mu t}\cos\omega t$$

Therefore from the fundamental equation

$$\frac{dy}{dt}+\frac{a_1}{b_1}y=-\frac{c}{b_1}\cdot\frac{d}{dt}(x\cos\omega t)$$

we have

$$y_s=-\frac{c}{b_1}\cdot\epsilon^{-\frac{a_1}{b_1}t}\cdot\int\epsilon^{\frac{a_1}{b_1}t}\cdot\frac{d}{dt}\cdot\{[X_s\sqrt{2}\sin(\omega t-\phi_{xs})+A\epsilon^{\alpha t}$$
$$+E\epsilon^{\mu t}\sin\omega t+F\epsilon^{\mu t}\cos\omega t]\cos\omega t\}\,dt+k\epsilon^{-\frac{a_1}{b_1}t}$$

$$=-\frac{c}{b_1}\cdot\epsilon^{-\frac{a_1}{b_1}t}\cdot\Big[X_s\sqrt{2}\,\omega\,.\int\epsilon^{\frac{a_1}{b_1}t}\cos(2\omega t-\phi_{xs})\,dt$$
$$+A\sqrt{\alpha^2+\omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\alpha\right)t}\cos\left(\omega t+\tan^{-1}\frac{\omega}{\alpha}\right)dt$$
$$+\frac{1}{2}E\sqrt{\mu^2+4\omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\mu\right)t}\sin\left(2\omega t+\tan^{-1}\frac{2\omega}{\mu}\right)dt$$
$$+\frac{1}{2}F\int\epsilon^{\left(\frac{a_1}{b_1}+\mu\right)t}\mu\,dt$$
$$+\frac{1}{2}F\sqrt{\mu^2+4\omega^2}\int\epsilon^{\left(\frac{a_1}{b_1}+\mu\right)t}\cos\left(2\omega t+\tan^{-1}\frac{2\omega}{\mu}\right)dt\Big]$$
$$+k\epsilon^{-\frac{a_1}{b_1}t}$$

$$=-\frac{c}{b_1}X_s\sqrt{2}\,\frac{b_1\omega}{\sqrt{a_1^2+4b_1^2\omega^2}}\cos\left(2\omega t-\phi_{xs}-\tan^{-1}\frac{2b_1\omega}{a_1}\right)$$
$$-\frac{c}{b_1}A\,.\,\epsilon^{\alpha t}\frac{b_1\sqrt{\alpha^2+\omega^2}}{\sqrt{(a_1+b_1\alpha)^2+b_1^2\omega^2}}\cos\left(\omega t+\tan^{-1}\frac{\omega}{\alpha}-\tan^{-1}\frac{b_1\omega}{a_1+b_1\alpha}\right)$$
$$-\frac{1}{2}\cdot\frac{c}{b_1}E\,.\,\epsilon^{\mu t}\frac{b_1\sqrt{\mu^2+4\omega^2}}{\sqrt{(a_1+b_1\mu)^2+4b_1^2\omega^2}}\sin\left(2\omega t+\tan^{-1}\frac{2\omega}{\mu}-\tan^{-1}\frac{2b_1\omega}{a_1+b_1\mu}\right)$$
$$-\frac{1}{2}\cdot\frac{c}{b_1}F\,.\,\epsilon^{\mu t}\frac{b_1\mu}{a_1+b_1\mu}$$
$$-\frac{1}{2}\cdot\frac{c}{b_1}F\,.\,\epsilon^{\mu t}\frac{b_1\sqrt{\mu^2+4\omega^2}}{\sqrt{(a_1+b_1\mu)^2+4b_1^2\omega^2}}\cos\left(2\omega t+\tan^{-1}\frac{2\omega}{\mu}-\tan^{-1}\frac{2b_1\omega}{a_1+b_1\mu}\right)$$
$$+k\epsilon^{-\frac{a_1}{b_1}t}$$

$$\doteqdot-\frac{1}{2}\cdot\frac{c}{b_1}X_s\sqrt{2}\sin(2\omega t-\phi_{xs})-\frac{c}{b_1}A\epsilon^{\alpha t}\cos\omega t-\frac{1}{2}\cdot\frac{c}{b_1}E\epsilon^{\mu t}\sin 2\omega t$$
$$-\frac{1}{2}\cdot\frac{c}{b_1}F\epsilon^{\mu t}\cdot\left(\frac{b_1\mu}{a_1+b_1\mu}+\cos 2\omega t\right)+k\epsilon^{-\frac{a_1}{b_1}t}$$

for $$\frac{\omega}{\alpha} \doteqdot \frac{2\omega}{\mu} \doteqdot \frac{2b_1\omega}{a_1} \doteqdot \frac{b_1\omega}{a_1 + b_1\alpha} \doteqdot \frac{2b_1\omega}{a_1 + b_1\mu} \doteqdot \infty$$

But since $\mu = -\frac{1}{2}(1+\sigma)\frac{q}{\sigma}\omega$ and $q = \frac{a_1}{b_1\omega}$ we have

$$\frac{b_1\mu}{a_1 + b_1\mu} = \frac{-a_1(1+\sigma)}{2a_1\sigma - a_1(1+\sigma)} = \frac{1+\sigma}{1-\sigma}$$

so that

$$y_s \doteqdot -\frac{1}{2}\cdot\frac{c}{b_1}\left[X_s\sqrt{2}\sin(2\omega t - \phi_{xs}) + 2A\epsilon^{\alpha t}\cos\omega t + E\epsilon^{\mu t}\sin 2\omega t + F\epsilon^{\mu t}\left(\frac{1+\sigma}{1-\sigma} + \cos 2\omega t\right)\right] + k\epsilon^{-\frac{a_1}{b_1}t}$$

Next, from the fundamental equation

$$\frac{dz}{dt} + \frac{a_1}{b_1}z = -\frac{c}{b_1}\cdot\frac{d}{dt}(x\sin\omega t)$$

we have

$$z_s = -\frac{c}{b_1}\epsilon^{-\frac{a_1}{b_1}t}\int \epsilon^{\frac{a_1}{b_1}t}\frac{d}{dt}\{[X_s\sqrt{2}\sin(\omega t - \phi_{xs}) + A\epsilon^{\alpha t} + E\epsilon^{\mu t}\sin\omega t + F\epsilon^{\mu t}\cos\omega t]\sin\omega t\}\,dt + k_1\epsilon^{-\frac{a_1}{b_1}t}$$

$$= -\frac{c}{b_1}\epsilon^{-\frac{a_1}{b_1}t}\Bigg[X_s\sqrt{2}\,\omega\int \epsilon^{\frac{a_1}{b_1}t}\sin(2\omega t - \phi_{xs})\,dt + A\sqrt{\alpha^2+\omega^2}\int \epsilon^{\left(\frac{a_1}{b_1}+\alpha\right)t}\sin\left(\omega t + \tan^{-1}\frac{\omega}{\alpha}\right)dt + \frac{1}{2}E\int \epsilon^{\left(\frac{a_1}{b_1}+\mu\right)t}\mu\,dt$$
$$- \frac{1}{2}E\sqrt{\mu^2+4\omega^2}\int \epsilon^{\left(\frac{a_1}{b_1}+\mu\right)t}\cos\left(2\omega t + \tan^{-1}\frac{2\omega}{\mu}\right)dt + \frac{1}{2}F\sqrt{\mu^2+4\omega^2}\int \epsilon^{\left(\frac{a_1}{b_1}+\mu\right)t}\sin\left(2\omega t + \tan^{-1}\frac{2\omega}{\mu}\right)dt\Bigg] + k_1\epsilon^{-\frac{a_1}{b_1}t}$$

$$= -\frac{c}{b_1} X_s \sqrt{2} \frac{b_1 \omega}{\sqrt{a_1^2 + 4b_1^2 \omega^2}} \sin\left(2\omega t - \phi_{xs} - \tan^{-1}\frac{2b_1\omega}{a_1}\right)$$

$$- \frac{c}{b_1} A \,.\, \epsilon^{\alpha t} \frac{b_1 \sqrt{\alpha^2 + \omega^2}}{\sqrt{(a_1 + b_1\alpha)^2 + b_1^2 \omega^2}} \sin\left(\omega t + \tan^{-1}\frac{\omega}{\alpha} - \tan^{-1}\frac{b_1\omega}{a_1 + b_1\alpha}\right)$$

$$- \frac{1}{2} \cdot \frac{c}{b_1} E \,.\, \epsilon^{\mu t} \frac{b_1 \mu}{a_1 + b_1 \mu}$$

$$+ \frac{1}{2} \cdot \frac{c}{b_1} E \,.\, \epsilon^{\mu t} \frac{b_1 \sqrt{\mu^2 + 4\omega^2}}{\sqrt{(a_1 + b_1\mu)^2 + 4b_1^2\omega^2}} \times \cos\left(2\omega t + \tan^{-1}\frac{2\omega}{\mu} - \tan^{-1}\frac{2b_1\omega}{a_1 + b_1\mu}\right)$$

$$- \frac{1}{2} \cdot \frac{c}{b_1} F \,.\, \epsilon^{\mu t} \frac{b_1 \sqrt{\mu^2 + 4\omega^2}}{\sqrt{(a_1 + b_1\mu)^2 + 4b_1^2\omega^2}} \times \sin\left(2\omega t + \tan^{-1}\frac{2\omega}{\mu} - \tan^{-1}\frac{2b_1\omega}{a_1 + b_1\mu}\right)$$

$$+ k_1 \epsilon^{-\frac{a_1}{b_1} t}$$

$$\doteqdot \frac{1}{2} \cdot \frac{c}{b_1} X_s \sqrt{2} \cos(2\omega t - \phi_{xs}) - \frac{c}{b_1} A \epsilon^{\alpha t} \sin \omega t - \frac{1}{2} \cdot \frac{c}{b_1} E \epsilon^{\mu t} \left(\frac{b_1 \mu}{a_1 + b_1 \mu} - \cos 2\omega t\right) - \frac{1}{2} \cdot \frac{c}{b_1} F \epsilon^{\mu t} \sin 2\omega t + k_1 \epsilon^{-\frac{a_1}{b_1} t}$$

$$= \frac{1}{2} \cdot \frac{c}{b_1} \left[X_s \sqrt{2} \cos(2\omega t - \phi_{xs}) - 2A \epsilon^{\alpha t} \sin \omega t - E \epsilon^{\mu t} \left(\frac{1 + \sigma}{1 - \sigma} - \cos 2\omega t\right) - F \epsilon^{\mu t} \sin 2\omega t \right] + k_1 \epsilon^{-\frac{a_1}{b_1} t}$$

Now, dropping the transient terms, we have

$$y_s \cos \omega t + z_s \sin \omega t = -\frac{1}{2} \cdot \frac{c}{b_1} X_s \sqrt{2} \left[\sin(2\omega t - \phi_{xs}) \cos \omega t - \cos(2\omega t - \phi_{xs}) \sin \omega t\right]$$

$$= -\frac{1}{2} \cdot \frac{c}{b_1} X_s \sqrt{2} \sin(\omega t - \phi_{xs})$$

so that the fundamental equation

$$a_i x + b_i \frac{dx}{dt} + c \frac{d}{dt}(y \cos \omega t + z \sin \omega t) = d \sin \omega t$$

becomes

$$a_i X_s \sqrt{2} \sin(\omega t - \phi_{xs}) + b_i \omega X_s \sqrt{2} \cos(\omega t - \phi_{xs}) - \frac{c^2}{2b_1} X_s \sqrt{2}\, \omega \cos(\omega t - \phi_{xs}) \equiv d \sin \omega t$$

that is

$$a_i X_s \sqrt{2} \sin(\omega t - \phi_{xs}) + b_i \omega \left(1 - \frac{c^2}{2b_i b_1}\right) X_s \sqrt{2} \cos(\omega t - \phi_{xs}) \equiv d \sin \omega t$$

so that

$$a_i X_s \sqrt{2} = d \cos \phi_{xs} \text{ and } b_i \omega \left(1 - \frac{c^2}{2b_i b_1}\right) X_s \sqrt{2} = d \sin \phi_{xs}$$

so that

$$X_s \sqrt{2} = \frac{d}{\sqrt{a_i^2 + b_i^2 \left(1 - \frac{c^2}{2b_i b_1}\right)^2 \omega^2}} = \frac{d}{\sqrt{a_i^2 + \frac{1}{4} b_i^2 (1+\sigma)^2 \omega^2}} \doteqdot \frac{2d}{b_i \omega (1+\sigma)}$$

and

$$\phi_{xs} = \tan^{-1} \frac{b_i \omega \left(1 - \frac{c^2}{2b_i b_1}\right)}{a_i} = \tan^{-1} \frac{b_i (1+\sigma) \omega}{2a_i} \doteqdot \frac{\pi}{2}$$

Thus, if we drop the terms containing k and k_1, then the expressions of x_s, y_s and z_s become

$$x_s = -X_s \sqrt{2} \cos \omega t + A\epsilon^{at} + E\epsilon^{\mu t} \sin \omega t + F\epsilon^{\mu t} \cos \omega t$$

$$y_s = \frac{1}{2} \cdot \frac{c}{b_1} \cdot [X_s \sqrt{2} \cos 2\omega t - 2A\epsilon^{at} \cos \omega t - E\epsilon^{\mu t} \sin 2\omega t - F\epsilon^{\mu t} (k + \cos 2\omega t)]$$

$$z_s = \frac{1}{2} \cdot \frac{c}{b_1} \cdot [X_s \sqrt{2} \sin 2\omega t - 2A\epsilon^{at} \sin \omega t - E\epsilon^{\mu t} (k - \cos 2\omega t) - F\epsilon^{\mu t} \sin 2\omega t]$$

where $$X_s \sqrt{2} = \frac{2d}{b_i \omega (1+\sigma)} \quad \text{and} \quad k = \frac{1+\sigma}{1-\sigma}$$

Now, proceeding to determine the arbitrary constants A, E and F, if the initial condition be $x_s = y_s = z_s = 0$ at $t = t_0$ then we have the relations

$$A\epsilon^{at_0} + E\epsilon^{\mu t_0} \sin \omega t_0 + F\epsilon^{\mu t_0} \cos \omega t_0 = X_s \sqrt{2} \cos \omega t_0$$

$$2A\epsilon^{at_0} \cos \omega t_0 + E\epsilon^{\mu t_0} \sin 2\omega t_0 + F\epsilon^{\mu t_0} (k + \cos 2\omega t_0) = X_s \sqrt{2} \cos 2\omega t_0$$

$$2A\epsilon^{at_0} \sin \omega t_0 + E\epsilon^{\mu t_0} (k - \cos 2\omega t_0) + F\epsilon^{\mu t_0} \sin 2\omega t_0 = X_s \sqrt{2} \sin 2\omega t_0$$

the first and third of which give

$$E\epsilon^{\mu t_0}(1-k) = 0 \quad \text{so that} \quad E = 0$$

The first and second relations give

$$F\epsilon^{\mu t_0}(1-k) = X_s \sqrt{2}$$

so that $$F = \frac{2d}{b_i \omega (1+\sigma)} \cdot \frac{1-\sigma}{-2\sigma} \cdot \epsilon^{-\mu t_0} = \frac{-d(1-\sigma)}{b_i \omega (1+\sigma) \sigma} \cdot \epsilon^{-\mu t_0}$$

and hence the first relation gives

$$A = \left(X_s\sqrt{2}\cos\omega t_0 - \frac{X_s\sqrt{2}}{1-k}\cos\omega t_0\right)\epsilon^{-at_0} = \frac{1+\sigma}{2\sigma}X_s\sqrt{2}\,\epsilon^{-at_0}\cos\omega t_0$$

The constants A, E and F have now been found and the expressions of x_s, y_s and z_s become

$$x_s = \frac{-2d}{b_i\omega(1+\sigma)}.\cos\omega t + \frac{d}{b_i\omega\sigma}.\epsilon^{\alpha(t-t_0)}.\cos\omega t_0 + \frac{-d(1-\sigma)}{b_i\omega(1+\sigma)\sigma}.\epsilon^{\mu(t-t_0)}.\cos\omega t$$

$$y_s = \frac{cd}{b_ib_1\omega(1+\sigma)}.\cos 2\omega t + \frac{-cd}{b_ib_1\omega\sigma}.\epsilon^{\alpha(t-t_0)}.\cos\omega t_0\cos\omega t + \frac{cd(1-\sigma)}{2b_ib_1\omega(1+\sigma)\sigma}.\epsilon^{\mu(t-t_0)}.\cos 2\omega t + \frac{cd}{2b_ib_1\omega\sigma}.\epsilon^{\mu(t-t_0)}$$

$$z_s = \frac{cd}{b_ib_1\omega(1+\sigma)}\sin 2\omega t + \frac{-cd}{b_ib_1\omega\sigma}.\epsilon^{\alpha(t-t_0)}\cos\omega t_0\sin\omega t + \frac{cd(1-\sigma)}{2b_ib_1\omega(1+\sigma)\sigma}.\epsilon^{\mu(t-t_0)}\sin 2\omega t$$

which are just the same as those obtained before (see Art. 20).

§ 22. Maximum sudden short-circuit currents of the distortionless alternator with two field windings and comparison of them with those of other alternators.

Putting $\frac{\alpha}{\omega} = \frac{\mu}{\omega} \doteqdot 0$ so that $\epsilon^{\alpha(t-t_0)} \doteqdot \epsilon^{\mu(t-t_0)} \doteqdot 1$

the expressions of the sudden short-circuit currents become

$$x_s = \frac{-2d}{b_i\omega(1+\sigma)}\left(\cos\omega t - \frac{1+\sigma}{2\sigma}\cos\omega t_0 + \frac{1-\sigma}{2\sigma}\cos\omega t\right)$$

$$= \frac{-d}{b_i\omega\sigma}(\cos\omega t - \cos\omega t_0)$$

$$y_s = \frac{cd}{b_ib_1\omega(1+\sigma)}\left(\cos 2\omega t - \frac{1+\sigma}{\sigma}\cos\omega t_0\cos\omega t + \frac{1-\sigma}{2\sigma}\cos 2\omega t + \frac{1+\sigma}{2\sigma}\right)$$

$$= \frac{cd}{2b_ib_1\omega\sigma}(\cos 2\omega t - 2\cos\omega t_0\cos\omega t + 1)$$

$$z_s = \frac{cd}{b_ib_1\omega(1+\sigma)}\left(\sin 2\omega t - \frac{1+\sigma}{\sigma}\cos\omega t_0\sin\omega t + \frac{1-\sigma}{2\sigma}\sin 2\omega t\right)$$

$$= \frac{cd}{2b_ib_1\omega\sigma}(\sin 2\omega t - 2\cos\omega t_0\sin\omega t)$$

so that denoting $\frac{-d}{b_i\omega\sigma}$ by M, $\frac{cd}{2b_i b_1\omega\sigma}$ by N, ωt by α and ωt_0 by β, we have

$$x_s = M(\cos\alpha - \cos\beta)$$

$$y_s = N(\cos 2\alpha - 2\cos\alpha\cos\beta + 1)$$

$$z_s = N(\sin 2\alpha - 2\sin\alpha\cos\beta)$$

Now $\frac{\partial x_s}{\partial\alpha} = 0$ gives $\sin\alpha = 0$ and $\frac{\partial x_s}{\partial\beta} = 0$ gives $\sin\beta = 0$ so that we have

$$(x_s)_{\text{max.}} = \pm\, 2M = \frac{\pm\, 2d}{b_i\omega\sigma}$$

Next $\frac{\partial y_s}{\partial\alpha} = 0$ gives $(2\cos\alpha - \cos\beta)\sin\alpha = 0$ and $\frac{\partial y_s}{\partial\beta} = 0$ gives $\cos\alpha\sin\beta = 0$.

Accordingly we have, when $\sin\beta = 0$ and $\sin\alpha = 0$ are taken up,

$$(y_s)_{\text{max.}} = 4N = \frac{2cd}{b_i b_1\omega\sigma} = 2\,\frac{1-\sigma}{\sigma}\,I_f$$

and, when $\sin\beta = 0$ and $\cos\alpha = \frac{1}{2}$ are taken up,

$$(y_s)_{\text{max.}} = -\frac{1}{2}N = -\frac{cd}{4b_i b_1\omega\sigma} = -\frac{1}{4}\cdot\frac{1-\sigma}{\sigma}\,I_f$$

and, when $\cos\alpha = 0$ and $\cos\beta = 0$ are taken up,

$$(y_s)_{\text{max.}} = 0$$

Obviously $(y_s)_{\text{max. max.}}$ is $= 2\,\frac{1-\sigma}{\sigma}\,I_f$ and hence $(y_s + I_f)_{\text{max. max.}}$ is $= I_f\cdot\left(\frac{2}{\sigma} - 1\right)$. Note that y_s, that is $N(\cos 2\alpha - 2\cos\alpha\cos\beta + 1)$, becomes $N(\cos 2\alpha - 2\cos\alpha - 1)$ when $\beta = 0$ and $N(\cos 2\alpha + 2\cos\alpha + 1)$ when $\beta = \pi$. It can however be expressed in the form $N[\cos 2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1]$ both when $\beta = 0$ and $\beta = \pi$. Figure 5 shows this $N[\cos 2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1]$.

Next $\frac{\partial z_s}{\partial\alpha} = 0$ gives $\cos 2\alpha - \cos\beta\cos\alpha = 0$ and $\frac{\partial z_s}{\partial\beta} = 0$ gives $\sin\alpha\sin\beta = 0$.

Accordingly we have, when $\sin\beta = 0$ and $\cos\alpha = \pm 1$ are taken up,

$$(z_s)_{\text{max.}} = 0$$

and, when $\sin\beta = 0$ and $\cos\alpha = \pm\frac{1}{2}$ are taken up,

$$(z_s)_{\text{max.}} = \pm\frac{3\sqrt{3}}{2}\cdot\frac{cd}{2b_ib_1\omega\sigma} = \pm\frac{3\sqrt{3}}{2}\cdot\frac{1-\sigma}{2\sigma}\cdot I_f$$

and, when $\sin\alpha = 0$ and $\cos\beta = \pm 1$ are taken up,

$$(z_s)_{\text{max.}} = 0$$

Obviously

$$(z_s)_{\text{max. max.}} \text{ is} = \pm\frac{3\sqrt{3}}{2}\cdot\frac{1-\sigma}{2\sigma}\cdot I_f \text{ that is } \frac{3\sqrt{3}(1-\sigma)}{4\sigma}\cdot I_f$$

Note that z_s, that is $N(\sin 2\alpha - 2\sin\alpha\cos\beta)$, can be expressed in the form $N[\sin 2(\alpha-\beta) - 2\sin(\alpha-\beta)]$ both when $\beta = 0$ or π. Figure 6 shows this $N[\sin 2(\alpha-\beta) - 2\sin(\alpha-\beta)]$.

Now putting $\sin\beta = 0$ so that $\beta = 0$ or π in the expressions of x_s, y_s and z_s, we have

$$x_s = \frac{\mp 2d}{b_i\omega(1+\sigma)}\left[\cos\omega(t-t_0) - \frac{1+\sigma}{2\sigma}\epsilon^{\alpha(t-t_0)} + \frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\omega(t-t_0)\right]$$

$$y_s = \frac{cd}{b_ib_1\omega(1+\sigma)}\left[\cos 2\omega(t-t_0) - \frac{1+\sigma}{\sigma}\epsilon^{\alpha(t-t_0)}\cos\omega(t-t_0)\right.$$
$$\left. + \frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos 2\omega(t-t_0) + \frac{1+\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\right]$$

$$z_s = \frac{cd}{b_ib_1\omega(1+\sigma)}\left[\sin 2\omega(t-t_0) - \frac{1+\sigma}{\sigma}\epsilon^{\alpha(t-t_0)}\sin\omega(t-t_0)\right.$$
$$\left. + \frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\sin 2\omega(t-t_0)\right]$$

Curves showing these x_s, y_s and z_s with $\sigma = 0{\cdot}5$, $\frac{a_1}{b_1\omega} = \frac{1}{100}$ and $\frac{a_i}{b_i\omega} = \frac{3}{100}$ so that $\frac{\alpha}{\omega} = \frac{-6}{100}$ and $\frac{\mu}{\omega} = \frac{-1{\cdot}5}{100}$ are given in figures 7, 8 and 9 respectively. These curves show approximately the manner in which the instantaneous values of the maximum sudden short-circuit currents change with time starting at $t = t_0$.

Now, considering the distortionless alternator with single field winding, the complete solution of its armature current i_s at sudden short-circuit is

$$i_s = \frac{d}{\sqrt{a_i^2 + b_i^2\omega^2}}\left[\sin\left(\omega t - \tan^{-1}\frac{b_i\omega}{a_i}\right) - \epsilon^{-\frac{a_i}{b_i}(t-t_0)} . \sin\left(\omega t_0 - \tan^{-1}\frac{b_i\omega}{a_i}\right)\right]$$

$$\doteqdot \frac{-d}{b_i\omega}\left(\cos\omega t - \epsilon^{-\frac{a_i}{b_i}(t-t_0)}\cos\omega t_0\right)$$

all the letters in which denote the same things as before.

This i_s is maximum when $\frac{\partial i_s}{\partial t_0} = 0$ that is when $\frac{a_i}{b_i}\cos\omega t_0 + \omega\sin\omega t_0 = 0$

that is when $\tan\omega t_0 = -\frac{a_i}{b_i\omega} = 0$ and its maximum value is

$$(i_s)_{\text{max.}} = \frac{\mp d}{b_i\omega} . \left[\cos\omega(t-t_0) - \epsilon^{-\frac{a_i}{b_i\omega}(\omega t - \omega t_0)}\right]$$

the maximum value of which is

$$\doteqdot \frac{\mp d}{b_i\omega}\left(1 + \epsilon^{-\frac{a_i}{b_i\omega} . \pi}\right)$$

which is

$$\doteqdot \mp\frac{2d}{b_i\omega}$$

The curve showing $(i_s)_{\text{max.}}$ with $\frac{a_i}{b_i\omega} = \frac{1}{25}$ is given in figure 10.

Note that this case of the distortionless alternator with single field winding is none other than a case of a simple inductance with finely laminated iron and no alternating current flows in the field winding.

The maximum sudden short-circuit currents of the ordinary alternator with non-salient and laminated poles, as shown by Mr Boucherot* and as will be explained later in the theory of ordinary single phase generator, are

$$\text{that in the armature winding} = \frac{\pm 2d}{b_i\omega\sigma}$$

and

$$\text{that in the field winding} = 2 . \frac{1-\sigma}{\sigma} . I_f$$

(including the exciting current I_f)

* See p. 183.

Note that the dispersion coefficient σ is

$$\sigma = 1 - \frac{1}{\nu \cdot \nu_f} = 1 - \frac{1}{(1+\tau)(1+\tau_f)} \doteqdot 1 - (1-\tau)(1-\tau_f)$$

$$\doteqdot \tau + \tau_f \quad \text{(neglecting } \tau \cdot \tau_f)$$

where τ is the ratio of the leakage flux to the difference of the total and the leakage flux produced by the armature current and τ_f the same ratio of the fluxes produced by the current in the field circuit; accordingly the maximum sudden short-circuit current in the armature winding of the ordinary single phase generator is $\frac{\pm 2d}{b_i \omega (\tau + \tau_f)}$ which shows that Mr Berg* gives too big a figure of the maximum sudden short-circuit current in the ordinary single phase armature, considering only the leakage reactance in the armature.

Comparing the maximum sudden short-circuit currents of the distortionless and ordinary alternators, those in the armature winding have the ratio:

Distortionless alternator with single field winding	Distortionless alternator with two field windings	Ordinary alternator
1	$\frac{1}{\sigma}$	$\frac{1}{\sigma}$

and those in the field windings have the ratio: (exciting current I_f not included)

Distortionless alternator with single field winding	Distortionless alternator with two field windings	Ordinary alternator
0	$2 \cdot \frac{1-\sigma}{\sigma} I_f$ or $\frac{3\sqrt{3}}{\sigma} \cdot \frac{1-\sigma}{\sigma} I_f$	$2 \cdot \frac{1-\sigma}{\sigma} I_f$

* Berg.—Electrical Engineering, Vol. I.

Thus with respect to the maximum sudden short-circuit currents, the distortionless alternator with single field winding is the best one. This comparison is however made of the maximum sudden short-circuit currents when $\frac{\alpha}{\omega}$ and $\frac{\mu}{\omega}$ are assumed to be zero. Measuring $(x_s)_{\max.}$ from figure 7, it is $2{\cdot}7 \frac{2d}{b_i\omega(1+\sigma)}$ that is $2{\cdot}7 \frac{2}{1{\cdot}5} \cdot \frac{d}{b_i\omega}$ that is $3{\cdot}6 \frac{d}{b_i\omega}$ while figure 10 gives

$$(x_s)_{\max.} = 1{\cdot}88 \frac{d}{b_i\omega}$$

Thus the ratio of the maximum sudden short-circuit currents in the armature circuits of the two distortionless alternators is

$$\frac{3{\cdot}6}{1{\cdot}88} \cdot \frac{3}{4} = 1{\cdot}44$$

Note that this $\frac{3}{4}$ is the ratio of $\frac{a_i}{b_i\omega}$ in the two alternators, that is that of $\frac{1}{b_i\omega}$ when a_i is assumed constant. We have put $\frac{a_i}{b_i\omega} = \frac{1}{25}$ in the distortionless alternator with single field winding in order to compare it with the distortionless alternator with two field windings having the same magnitude of the voltage regulation. If we assume $\frac{a_i}{b_i\omega} = \frac{3}{100}$ in the distortionless alternator with single field winding, then the maximum sudden short-circuit current becomes

$$(x_s)_{\max.} \doteqdot \frac{d}{b_i\omega}\left(1 + \epsilon^{-\frac{3}{100}\pi}\right) = 1{\cdot}91 \frac{d}{b_i\omega}$$

accordingly the ratio of $(x_s)_{\max.}$ in the two alternators with two and single field windings becomes $\frac{3{\cdot}6}{1{\cdot}91} = 1{\cdot}98$.

As to the maximum sudden short-circuit currents in the field windings of the distortionless alternator with two field windings, we have the following values with the same $\frac{a_1}{b_1\omega}$, $\frac{a_i}{b_i\omega}$ and σ as given before:

	When both $\frac{\alpha}{\omega}$ and $\frac{\mu}{\omega}$ are assumed$\doteqdot 0$	From Figs. 8 and 9
$(y_s)_{\max.}$	$2 \cdot \frac{1-\sigma}{\sigma} \cdot I_f = 2I_f$	$5{\cdot}3 \frac{1-\sigma}{1+\sigma} \cdot I_f = 1{\cdot}8 I_f$
$(z_s)_{\max.}$	$\frac{3\sqrt{3}}{4} \cdot \frac{1-\sigma}{\sigma} \cdot I_f = 1{\cdot}3 I_f$	$3{\cdot}6 \frac{1-\sigma}{1+\sigma} \cdot I_f = 1{\cdot}2 I_f$

§ 23. Sudden short-circuit currents of the distortionless alternator with two field windings when both fields are excited.

As said in Art. 7, the necessary modification of the fundamental equations, when both fields are excited, is only the replacement of the induced E.M.F. $d \sin \omega t$ by $d\sqrt{2} \sin\left(\omega t - \frac{\pi}{4}\right)$ so that there will be no change in the expressions of the transient terms of x, y and z. The arbitrary constants however will be changed.

We have in this case, referring to Arts. 17 and 20,

$$A' \doteqdot \frac{-d\sqrt{2}}{b_i \omega \sigma} \sin\left(\omega t_0 - \frac{\pi}{4} - \phi_{xs}\right)$$

$$D' = \frac{d\sqrt{2}}{b_i \omega \sigma} \cdot \frac{1-\sigma}{1+\sigma}$$

$$\psi' = \omega t_0 - \frac{\pi}{4} - \phi_{xs}$$

$$\psi_1 = \psi_2 = 0$$

$$\nu(t - t_0) + \omega t + \psi' + \psi_1 = 2\omega t - \frac{\pi}{4} - \phi_{xs}$$

$$\nu(t - t_0) - \omega t + \psi' + \psi_2 = -\frac{\pi}{4} - \phi_{xs}$$

while $\frac{A_1'}{A'}$, $\frac{D_1'}{D'}$, $\frac{D_2'}{D'}$ and θ_a remain the same as before.

Therefore the solution of the armature current x_s becomes

$$x_s \doteqdot \frac{2d\sqrt{2}}{b_i\omega(1+\sigma)}\sin\left(\omega t-\frac{\pi}{4}-\phi_{xs}\right)+\frac{-d\sqrt{2}}{b_i\omega\sigma}\cdot\epsilon^{a(t-t_0)}\sin\left(\omega t_0-\frac{\pi}{4}-\phi_{xs}\right)$$
$$+\frac{d\sqrt{2}}{b_i\omega\sigma}\cdot\frac{1-\sigma}{1+\sigma}\cdot\epsilon^{\mu(t-t_0)}\sin\left(\omega t-\frac{\pi}{4}-\phi_{xs}\right)$$
$$\doteqdot \frac{-2d\sqrt{2}}{b_i\omega(1+\sigma)}\cdot\left[\cos\left(\omega t-\frac{\pi}{4}\right)-\frac{1+\sigma}{2\sigma}\cdot\epsilon^{a(t-t_0)}\cdot\cos\left(\omega t_0-\frac{\pi}{4}\right)\right.$$
$$\left.+\frac{1-\sigma}{2\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\cos\left(\omega t-\frac{\pi}{4}\right)\right]$$

and those of the field currents y_s and z_s become

$$y_s \doteqdot \frac{-cd\sqrt{2}}{b_ib_1\omega(1+\sigma)}\cdot\sin\left(2\omega t-\frac{\pi}{4}-\phi_{xs}\right)$$
$$+\frac{cd\sqrt{2}}{b_ib_1\omega\sigma}\cdot\epsilon^{a(t-t_0)}\cdot\sin\left(\omega t_0-\frac{\pi}{4}-\phi_{xs}\right)\cos\omega t$$
$$+\frac{-cd\sqrt{2}}{2b_ib_1\omega\sigma}\cdot\frac{1-\sigma}{1+\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\sin\left(2\omega t-\frac{\pi}{4}-\phi_{xs}\right)$$
$$+\frac{cd\sqrt{2}}{2b_ib_1\omega\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\sin\left(\frac{\pi}{4}+\phi_{xs}\right)$$
$$\doteqdot \frac{cd\sqrt{2}}{b_ib_1\omega(1+\sigma)}\cdot\left[\cos\left(2\omega t-\frac{\pi}{4}\right)-\frac{1+\sigma}{\sigma}\cdot\epsilon^{a(t-t_0)}\cdot\cos\left(\omega t_0-\frac{\pi}{4}\right)\cos\omega t\right.$$
$$\left.+\frac{1-\sigma}{2\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\cos\left(2\omega t-\frac{\pi}{4}\right)+\frac{1+\sigma}{2\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\cos\frac{\pi}{4}\right]$$

$$z_s \doteqdot \frac{cd\sqrt{2}}{b_ib_1\omega(1+\sigma)}\cos\left(2\omega t-\frac{\pi}{4}-\phi_{xs}\right)$$
$$+\frac{cd\sqrt{2}}{b_ib_1\omega\sigma}\cdot\epsilon^{a(t-t_0)}\cdot\sin\left(\omega t_0-\frac{\pi}{4}-\phi_{xs}\right)\sin\omega t$$
$$+\frac{cd\sqrt{2}}{2b_ib_1\omega\sigma}\cdot\frac{1-\sigma}{1+\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\cos\left(2\omega t-\frac{\pi}{4}-\phi_{xs}\right)$$
$$+\frac{-cd\sqrt{2}}{2b_ib_1\omega\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\cos\left(\frac{\pi}{4}+\phi_{xs}\right)$$
$$\doteqdot \frac{cd\sqrt{2}}{b_ib_1\omega(1+\sigma)}\cdot\left[\sin\left(2\omega t-\frac{\pi}{4}\right)-\frac{1+\sigma}{\sigma}\cdot\epsilon^{a(t-t_0)}\cdot\cos\left(\omega t_0-\frac{\pi}{4}\right)\sin\omega t\right.$$
$$\left.+\frac{1-\sigma}{2\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\sin\left(2\omega t-\frac{\pi}{4}\right)+\frac{1+\sigma}{2\sigma}\cdot\epsilon^{\mu(t-t_0)}\cdot\sin\frac{\pi}{4}\right]$$

Now proceeding to consider the maximum values of x_s, y_s and z_s, assume $\alpha/\omega \doteqdot \mu/\omega \doteqdot 0$. Then we have

$$x_s = \frac{-d\sqrt{2}}{b_i \omega\sigma} \cdot \left[\cos\left(\omega t - \frac{\pi}{4}\right) - \cos\left(\omega t_0 - \frac{\pi}{4}\right)\right]$$

$$y_s = \frac{cd\sqrt{2}}{2b_i b_1 \omega\sigma} \cdot \left[\cos\left(2\omega t - \frac{\pi}{4}\right) - 2\cos\left(\omega t_0 - \frac{\pi}{4}\right)\cos\omega t + \frac{1}{\sqrt{2}}\right]$$

$$z_s = \frac{cd\sqrt{2}}{2b_i b_1 \omega\sigma} \cdot \left[\sin\left(2\omega t - \frac{\pi}{4}\right) - 2\cos\left(\omega t_0 - \frac{\pi}{4}\right)\sin\omega t + \frac{1}{\sqrt{2}}\right]$$

Now all of these x_s, y_s and z_s are maximum when $\sin\left(\omega t_0 - \frac{\pi}{4}\right) = 0$ that is when $\omega t_0 - \frac{\pi}{4} = 0$ or π and the expressions of $(x_s)_{\text{max.}}$, $(y_s)_{\text{max.}}$ and $(z_s)_{\text{max.}}$ are

$$(x_s)_{\text{max.}} = \frac{\mp d\sqrt{2}}{b_i \omega\sigma} \cdot [\cos\omega(t - t_0) - 1]$$

$$(y_s)_{\text{max.}} = \frac{cd\sqrt{2}}{2b_i b_1 \omega\sigma} \cdot \left\{\cos\left[2\omega(t - t_0) + \frac{\pi}{4}\right] - 2\cos\left[\omega(t - t_0) + \frac{\pi}{4}\right] + \frac{1}{\sqrt{2}}\right\}$$

$$(z_s)_{\text{max.}} = \frac{cd\sqrt{2}}{2b_i b_1 \omega\sigma} \cdot \left\{\sin\left[2\omega(t - t_0) + \frac{\pi}{4}\right] - 2\sin\left[\omega(t - t_0) + \frac{\pi}{4}\right] + \frac{1}{\sqrt{2}}\right\}$$

that is

$$(x_s)_{\text{max.}} = M(\cos\alpha - 1)$$

$$(y_s)_{\text{max.}} = N\left[\cos\left(2\alpha + \frac{\pi}{4}\right) - 2\cos\left(\alpha + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\right]$$

$$(z_s)_{\text{max.}} = N\left[\sin\left(2\alpha + \frac{\pi}{4}\right) - 2\sin\left(\alpha + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\right]$$

where $M = \frac{\mp d\sqrt{2}}{b_i \omega\sigma}$, $N = \frac{cd\sqrt{2}}{2b_i b_1 \omega\sigma} = \frac{1-\sigma}{2\sigma} I_f \sqrt{2}$ and $\alpha = \omega(t - t_0)$

Curves showing these $(y_s)_{\text{max.}}$ and $(z_s)_{\text{max.}}$ with $\sigma = 0{\cdot}5$ are given in figures 11 and 12 respectively.

Now, proceeding to find $(x_s)_{max.\,max.}$, $(y_s)_{max.\,max.}$ and $(z_s)_{max.\,max.}$ we have: $(x_s)_{max.}$ is maximum when $\sin\alpha = 0$ and $(x_s)_{max.\,max.}$ is

$$(x_s)_{max.\,max.} = \frac{\mp 2d\sqrt{2}}{b_i\omega\sigma}$$

$(y_s)_{max.}$ is maximum or minimum when $\sin\left(2\alpha + \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{4}\right)$ that is when $\alpha = 0$ or $\frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$ or $3\frac{\pi}{2}$, and $(y_s)_{max.}$ becomes zero when $\alpha = 0$, $-0{\cdot}0686N$ when $\alpha = \frac{\pi}{6}$, $3{\cdot}605N$ when $\alpha = \pi - \frac{\pi}{6}$, and $-1{\cdot}414N$ when $\alpha = 3\frac{\pi}{2}$

so that $(y_s)_{max.\,max.}$ is $= 3{\cdot}605N = 1{\cdot}8025\,\frac{1-\sigma}{\sigma} I_f\sqrt{2}$.

$(z_s)_{max.}$ is maximum or minimum when $\cos\left(2\alpha + \frac{\pi}{4}\right) = \cos\left(\alpha + \frac{\pi}{4}\right)$ that is when $\alpha = 0$ or $\frac{\pi}{2}$ or $\pi + \frac{\pi}{6}$ or $2\pi - \frac{\pi}{6}$, and $(z_s)_{max.}$ becomes zero when $\alpha = 0$, $-1{\cdot}414N$ when $\alpha = \frac{\pi}{2}$, $3{\cdot}605N$ when $\alpha = \pi + \frac{\pi}{6}$, and $-0{\cdot}0686N$ when $\alpha = 2\pi - \frac{\pi}{6}$

so that $(z_s)_{max.\,max.}$ is $= 3{\cdot}605N = 1{\cdot}8025\,\frac{1-\sigma}{\sigma} I_f\sqrt{2}$.

Note that these $(y_s)_{max.}$ and $(z_s)_{max.}$ do not include the exciting current I_f.

Also note that, both field windings being excited, $\frac{1}{\sqrt{2}}$ times that exciting current when one field winding only is excited will do for the same amount of the induced E.M.F.

Thus we see that, exciting both field windings, the maximum sudden short-circuit current in the direct field winding is reduced in the ratio $2:1{\cdot}8025$. That in the armature winding remains the same and that in the cross field winding is increased in the ratio $1{\cdot}3:1{\cdot}8025$.

§ 24. E.m.f. induced in open phases at short-circuit of one phase when the armature is wound in two or three phases.

In the distortionless alternator with two field windings and wound in three phases, the E.M.F. induced in the open phases at permanent short-circuit of one phase are

$$\doteqdot -c\frac{d}{dt}\left[I_f \cos\left(\omega t \pm \frac{2\pi}{3}\right)\right]$$

$$-c\cdot\frac{cd}{b_i b_1 \omega(1+\sigma)}\cdot\frac{d}{dt}\left[\cos 2\omega t \cos\left(\omega t \pm \frac{2\pi}{3}\right) + \sin 2\omega t \sin\left(\omega t \pm \frac{2\pi}{3}\right)\right]$$

$$-\left(-\frac{1}{2}\right) b_i \cdot \frac{-2d}{b_i \omega(1+\sigma)}\cdot\frac{d}{dt}(\cos\omega t)$$

$$= cI_f\omega \sin\left(\omega t \pm \frac{2\pi}{3}\right) - \frac{c^2 d}{b_i b_1 \omega(1+\sigma)}\cdot\frac{d}{dt}\left[\cos\left(\omega t \mp \frac{2\pi}{3}\right)\right] + \frac{d}{1+\sigma}\sin\omega t$$

$$= d\sin\left(\omega t \pm \frac{2\pi}{3}\right) + d\frac{1-\sigma}{1+\sigma}\sin\left(\omega t \mp \frac{2\pi}{3}\right) + \frac{d}{1+\sigma}\sin\omega t$$

$$= d\cdot\left(\frac{1}{1+\sigma} - \frac{1}{2} - \frac{1}{2}\cdot\frac{1-\sigma}{1+\sigma}\right)\sin\omega t \pm \frac{\sqrt{3}}{2}d\cdot\left(1 - \frac{1-\sigma}{1+\sigma}\right)\cos\omega t$$

$$= \pm\sqrt{3}\,d\cdot\frac{\sigma}{1+\sigma}\cdot\cos\omega t$$

which is $\sqrt{3}\cdot\frac{\sigma}{1+\sigma}$ times those at open circuit in amplitude.

Next, those induced in the open phases at sudden short-circuit are

$$\doteqdot -c\frac{d}{dt}\left[I_f \cos\left(\omega t \pm \frac{2\pi}{3}\right)\right]$$

$$-c\cdot\frac{cd}{b_i b_1 \omega(1+\sigma)}\cdot\frac{d}{dt}\left[\cos\left(\omega t \mp \frac{2\pi}{3}\right) - \frac{1+\sigma}{\sigma}\cdot\epsilon^{\alpha(t-t_0)}\cos\omega t_0 \cos\left(\mp\frac{2\pi}{3}\right)\right.$$

$$+\frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\left(\omega t \mp \frac{2\pi}{3}\right)$$

$$\left.+\frac{1+\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\left(\omega t \pm \frac{2\pi}{3}\right)\right]$$

$$-\frac{d}{\omega(1+\sigma)}\cdot\frac{d}{dt}\left(\cos\omega t - \frac{1+\sigma}{2\sigma}\epsilon^{\alpha(t-t_0)}\cos\omega t_0 + \frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\omega t\right)$$

$$= d \sin\left(\omega t \pm \frac{2\pi}{3}\right) + \frac{d}{\omega} \cdot \frac{1-\sigma}{1+\sigma}$$

$$\times \left\{\omega \sin\left(\omega t \mp \frac{2\pi}{3}\right) + \frac{1+\sigma}{\sigma} . \alpha . \epsilon^{\alpha(t-t_0)} \cos \omega t_0 \cos\left(\mp \frac{2\pi}{3}\right)\right.$$

$$- \frac{1-\sigma}{2\sigma} . \epsilon^{\mu(t-t_0)} \left[\mu \cos\left(\omega t \mp \frac{2\pi}{3}\right) - \omega \sin\left(\omega t \mp \frac{2\pi}{3}\right)\right]$$

$$\left. - \frac{1+\sigma}{2\sigma} . \epsilon^{\mu(t-t_0)} \left[\mu \cos\left(\omega t \pm \frac{2\pi}{3}\right) - \omega \sin\left(\omega t \pm \frac{2\pi}{3}\right)\right]\right\}$$

$$+ \frac{d}{\omega(1+\sigma)} \cdot \left[\omega \sin \omega t + \frac{1+\sigma}{2\sigma} . \alpha . \epsilon^{\alpha(t-t_0)} \cos \omega t_0\right.$$

$$\left. - \frac{1-\sigma}{2\sigma} \epsilon^{\mu(t-t_0)} (\mu \cos \omega t - \omega \sin \omega t)\right]$$

which, when $\frac{\alpha}{\omega}$ and $\frac{\mu}{\omega}$ are $\doteqdot 0$, become

$$= d \sin\left(\omega t \pm \frac{2\pi}{3}\right) + d . \frac{1-\sigma}{1+\sigma} \cdot \left[\sin\left(\omega t \mp \frac{2\pi}{3}\right) + \frac{1-\sigma}{2\sigma} \sin\left(\omega t \mp \frac{2\pi}{3}\right)\right.$$

$$\left. + \frac{1+\sigma}{2\sigma} \sin\left(\omega t \pm \frac{2\pi}{3}\right)\right]$$

$$+ \frac{d}{1+\sigma} \cdot \left(\sin \omega t + \frac{1-\sigma}{2\sigma} \sin \omega t\right)$$

$$= d \sin\left(\omega t \pm \frac{2\pi}{3}\right) + d . \frac{1-\sigma}{2\sigma} \left[\sin\left(\omega t \mp \frac{2\pi}{3}\right) + \sin\left(\omega t \pm \frac{2\pi}{3}\right)\right] + \frac{d}{2\sigma} \sin \omega t$$

$$= \frac{d(1+\sigma)}{2\sigma} \sin\left(\omega t \pm \frac{2\pi}{3}\right) + \frac{d(1-\sigma)}{2\sigma} \sin\left(\omega t \mp \frac{2\pi}{3}\right) + \frac{d}{2\sigma} \sin \omega t$$

$$= \pm \frac{\sqrt{3}}{2} d \cos \omega t$$

which are $\frac{\sqrt{3}}{2}$ times those at open circuit in amplitude.

Now, if both field windings be excited, then the E.M.F. induced in the open phases at permanent short-circuit are, similarly as before,

$$d \sqrt{2} . \frac{\sqrt{3}\sigma}{1+\sigma} \cos\left(\omega t - \frac{\pi}{4}\right)$$

Next, those induced at sudden short-circuit when both field windings are excited are

$$= -c\frac{d}{dt}\left[I_f\sqrt{2}\cos\left(\omega t-\frac{\pi}{4}\pm\frac{2\pi}{3}\right)\right]$$

$$-c\cdot\frac{cd\sqrt{2}}{b_i b_1\omega(1+\sigma)}\cdot\frac{d}{dt}\left[\cos\left(\omega t-\frac{\pi}{4}\mp\frac{2\pi}{3}\right)\right.$$

$$+\frac{1+\sigma}{\sigma}\epsilon^{\alpha(t-t_0)}\cos\left(\omega t_0-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\mp\frac{2\pi}{3}\right)$$

$$+\frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\left(\omega t-\frac{\pi}{4}\mp\frac{2\pi}{3}\right)$$

$$\left.+\frac{1+\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\left(\omega t-\frac{\pi}{4}\pm\frac{2\pi}{3}\right)\right]$$

$$-\frac{d\sqrt{2}}{\omega(1+\sigma)}\cdot\frac{d}{dt}\left[\cos\left(\omega t-\frac{\pi}{4}\right)-\frac{1+\sigma}{2\sigma}\epsilon^{\alpha(t-t_0)}\cos\left(\omega t_0-\frac{\pi}{4}\right)\right.$$

$$\left.+\frac{1-\sigma}{2\sigma}\epsilon^{\mu(t-t_0)}\cos\left(\omega t-\frac{\pi}{4}\right)\right]$$

which, similarly as before, becomes

$$=\pm d\sqrt{2}.\frac{\sqrt{3}}{2}\cos\left(\omega t-\frac{\pi}{4}\right)$$

Next in the distortionless alternator with single field winding, wound in three phases, the E.M.F. induced in the open phases at permanent short-circuit are

$$= d\sin\left(\omega t\pm\frac{2\pi}{3}\right)-\left[\cos\left(\pm\frac{2\pi}{3}\right)\right]b_i\frac{-d}{b_i\omega}\cdot\frac{d}{dt}(\cos\omega t)$$

$$= d\sin\left(\omega t\pm\frac{2\pi}{3}\right)+\frac{1}{2}d\sin\omega t$$

$$=\pm d\frac{\sqrt{3}}{2}\cos\omega t$$

Those at sudden short-circuit are

$$= d \sin\left(\omega t \pm \frac{2\pi}{3}\right) - \left[\cos\left(\pm \frac{2\pi}{3}\right)\right] b_i \frac{-d}{b_i \omega} \cdot \frac{d}{dt}\left(\cos \omega t - \epsilon^{-\frac{a_i}{b_i}(t - t_0)} \cos \omega t_0\right)$$

which when $\dfrac{a_i}{b_i \omega} \doteqdot 0$ becomes

$$\doteqdot d \sin\left(\omega t \pm \frac{2\pi}{3}\right) + \frac{1}{2} d \sin \omega t$$

which are the same as those at permanent short-circuit.

In the alternators wound in two phases there is no E.M.F. in the open phase induced by the current in the short-circuited phase; so that in this case we have the E.M.F. in the open phase as follows:

(1) Distortionless alternator with two field windings when one field winding only is excited.

(*a*) The E.M.F. induced at permanent short-circuit

$$= d \sin\left(\omega t \pm \frac{\pi}{2}\right) + d \frac{1 - \sigma}{1 + \sigma} \sin\left(\omega t \mp \frac{\pi}{2}\right)$$

$$= \pm d \cos \omega t \mp d \frac{1 - \sigma}{1 + \sigma} \cos \omega t$$

$$= \pm d \frac{2\sigma}{1 + \sigma} \cos \omega t$$

which coincides with the result obtained by Mr Boucherot.

(*b*) The E.M.F. induced at sudden short-circuit

$$= d \sin\left(\omega t \pm \frac{\pi}{2}\right) + d \frac{1 - \sigma}{2\sigma} \left[\sin\left(\omega t \mp \frac{\pi}{2}\right) + \sin\left(\omega t \pm \frac{\pi}{2}\right)\right]$$

$$= \frac{d}{2\sigma} \left[(1 + \sigma) \sin\left(\omega t \pm \frac{\pi}{2}\right) + (1 - \sigma) \sin\left(\omega t \mp \frac{\pi}{2}\right)\right]$$

$$= \pm d \cos \omega t$$

which also coincides with the result obtained by Mr Boucherot.

(2) Distortionless alternator with two field windings, when both field windings are excited

(*a*) The E.M.F. induced at permanent short-circuit

$$= \pm d\sqrt{2}\,\frac{2\sigma}{1+\sigma}\cos\left(\omega t - \frac{\pi}{4}\right)$$

(*b*) The E.M.F. induced at sudden short-circuit

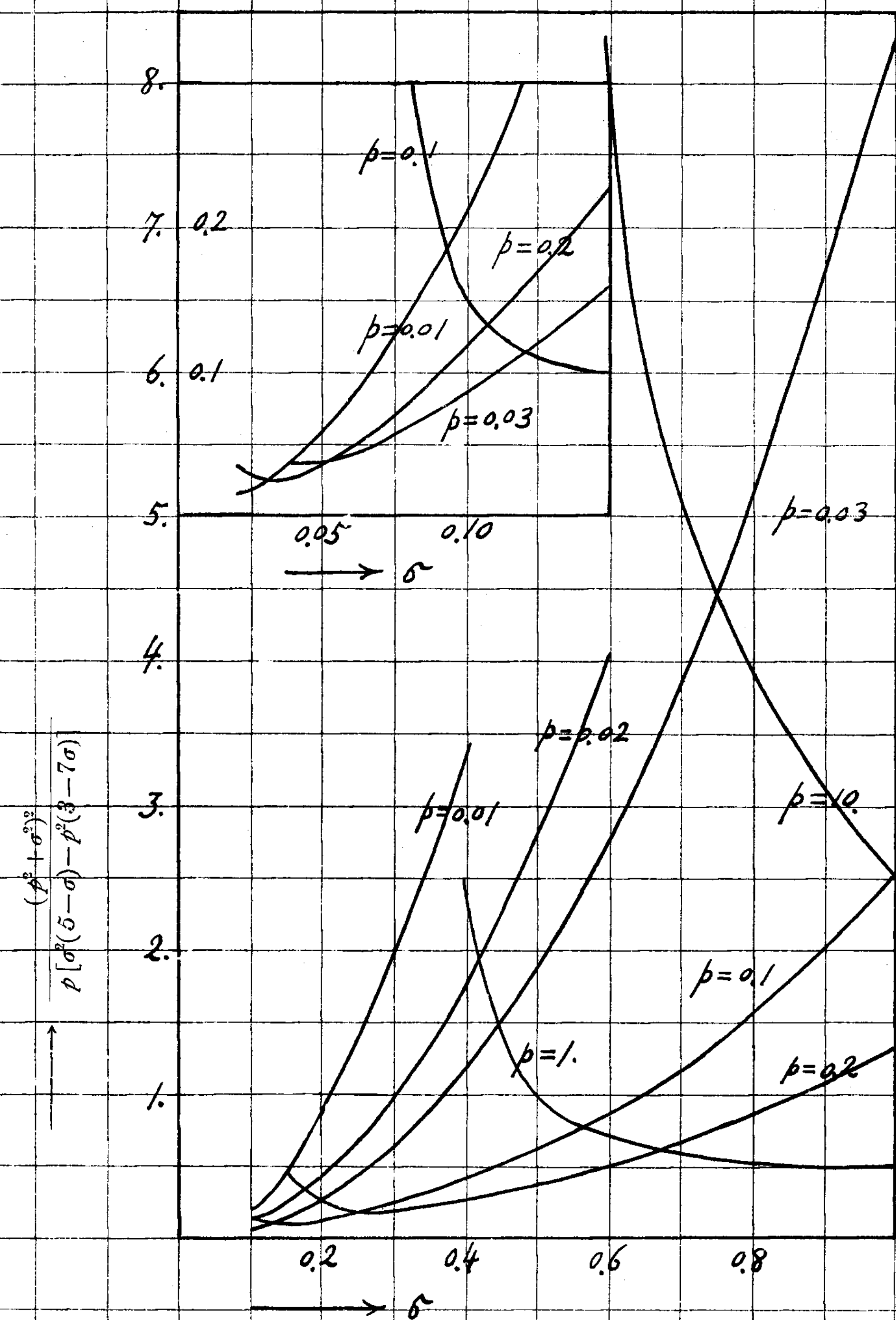
$$= \pm d\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

(3) Distortionless alternator with single field winding.
In this case the E.M.F. induced at both permanent and sudden short-circuit is

$$= \pm d \cos \omega t$$

In conclusion the author wishes to express his sincere thanks to Mr T. Otake for his suggestions in simplifying deductions.

Figure. 4.



8.
7.
6.
5.
4.
3.
2.
1.
0.2
0.1
0.05
0.10
σ
0.2
0.4
0.6
0.8
σ
p=0.1
p=0.2
p=0.01
p=0.03
p=0.03
p=0.02
p=0.01
p=10.
p=0.1
p=1.
p=0.2

Figure. 5

$y = \cos 2\alpha - \cos 2\,\alpha + 1$

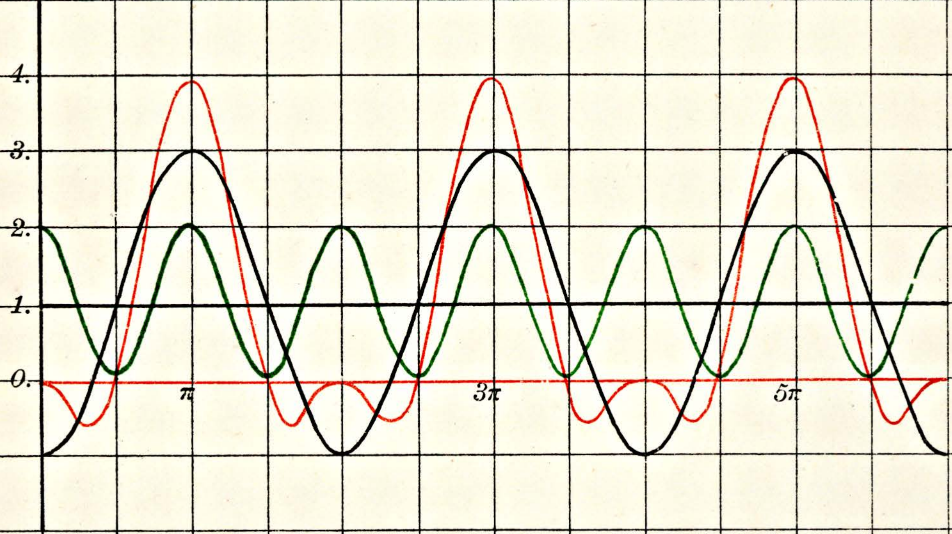


Figure. 6

$z = \sin 2\alpha - 2\sin\alpha$

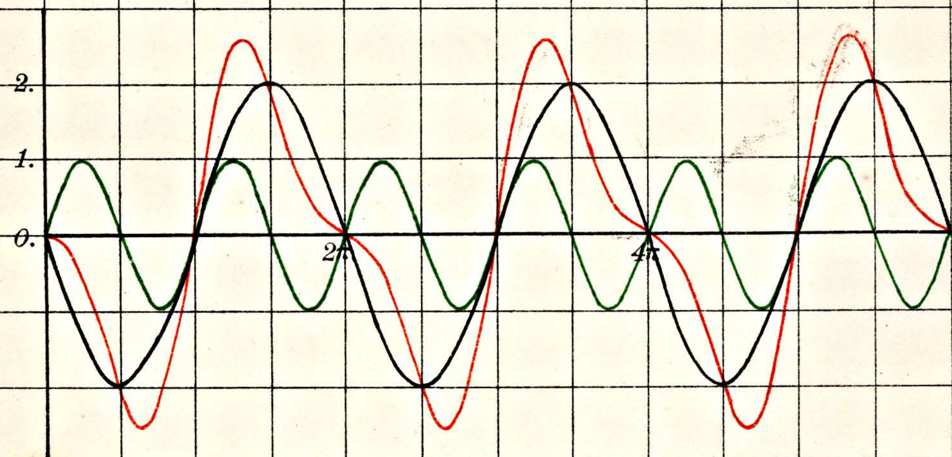
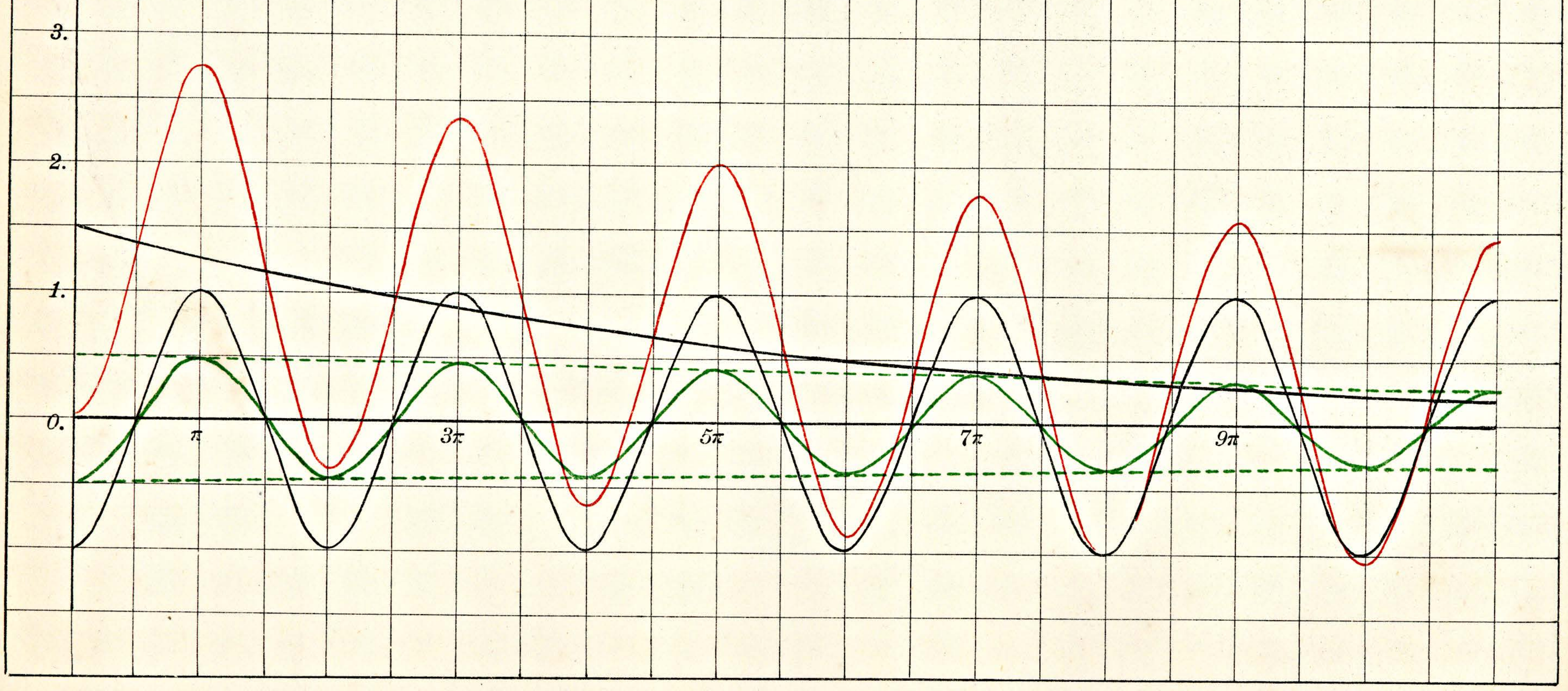


Figure. 7

Maximum armature current at sudden short-circuit.

$$x_s = \frac{\mp 2d}{b_i\omega(1+\sigma)}\left[\cos\omega\,(t-t_0) - \frac{1+\sigma}{2\sigma}\cdot \varepsilon^{\alpha(t-t_0)} + \frac{1-\sigma}{2\sigma}\cdot \varepsilon^{\mu(t-t_0)}\cos\omega\,(t-t_0)\right]$$

$$\frac{a_1}{b_1\omega} = \frac{1}{100} \qquad \frac{a_i}{b_i\omega} = \frac{3}{100} \qquad \sigma = 0.5$$



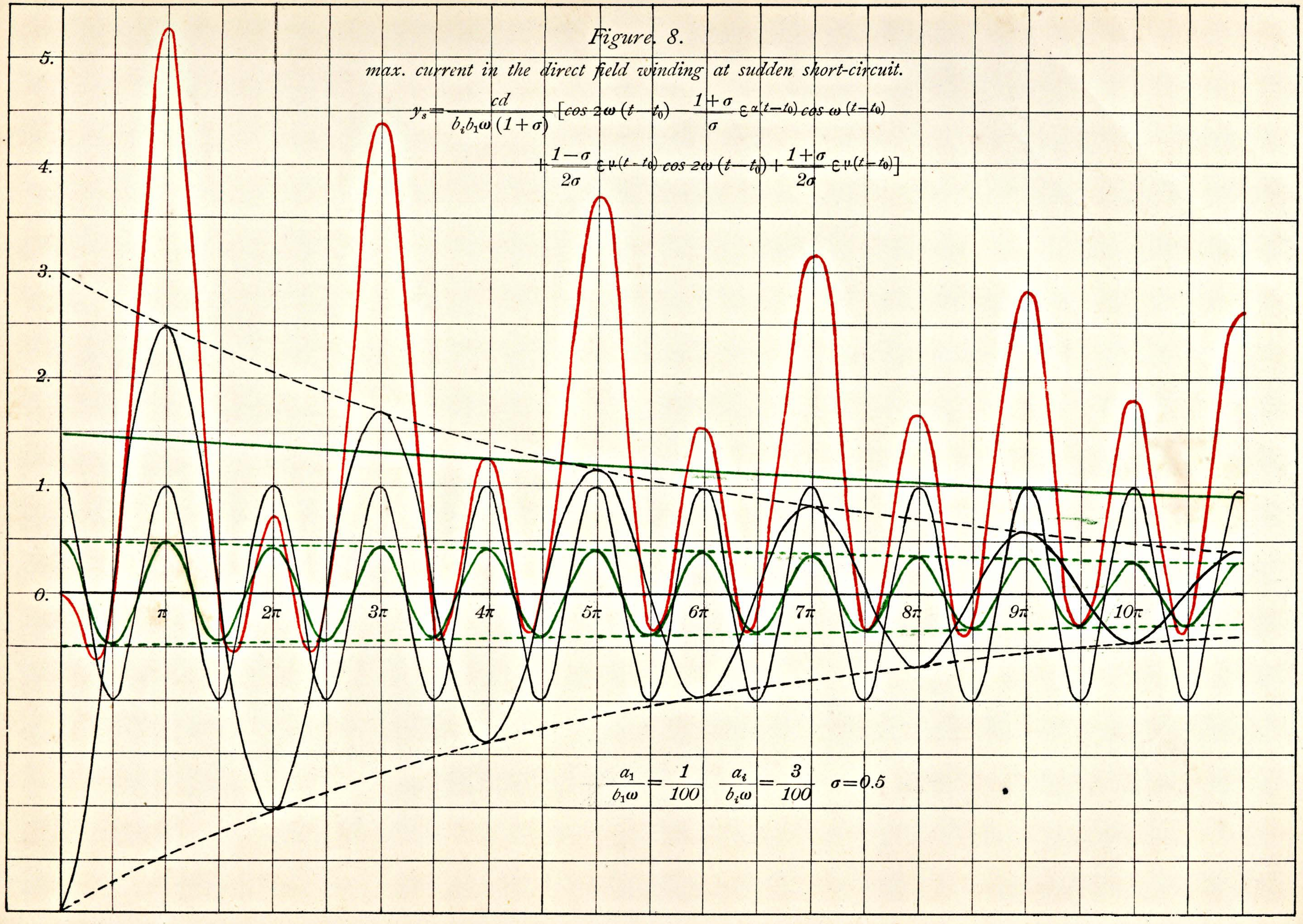


Figure. 8.

max. current in the direct field winding at sudden short-circuit.

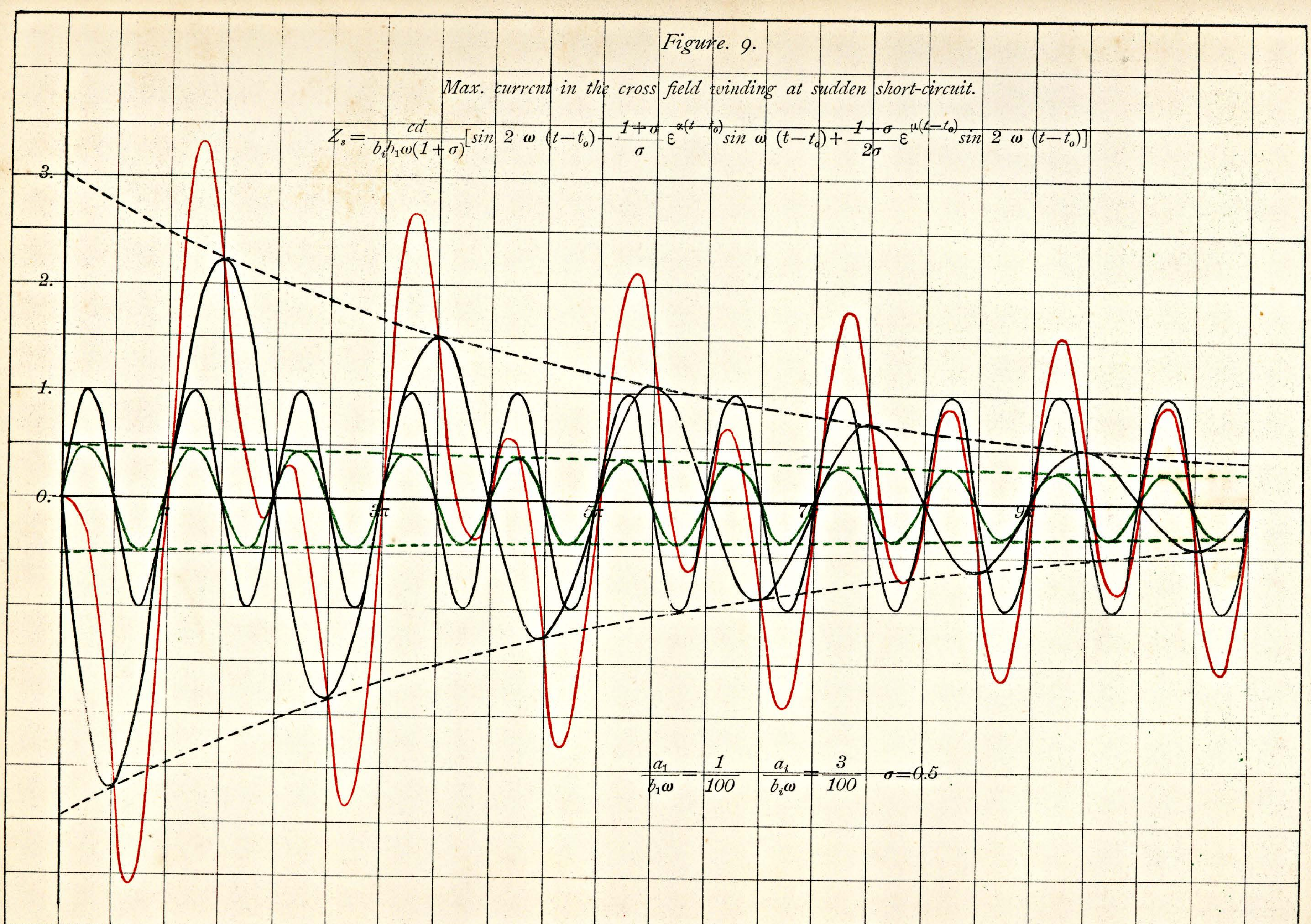


Figure. 9.

Max. current in the cross field winding at sudden short-circuit.

Figure. 10.

Max. sudden short-circuit current in the armature of the distortionless alternator with 1 field winding.

$$i_s = \frac{\mp d}{b_i\omega}\left[\cos \omega\,(t-t_0) - \varepsilon^{-\frac{a_i}{b_i\omega}\cdot\omega(t-t_0)}\right]$$

$$\text{with } \frac{a_i}{b_i\omega} = \frac{1}{25}$$

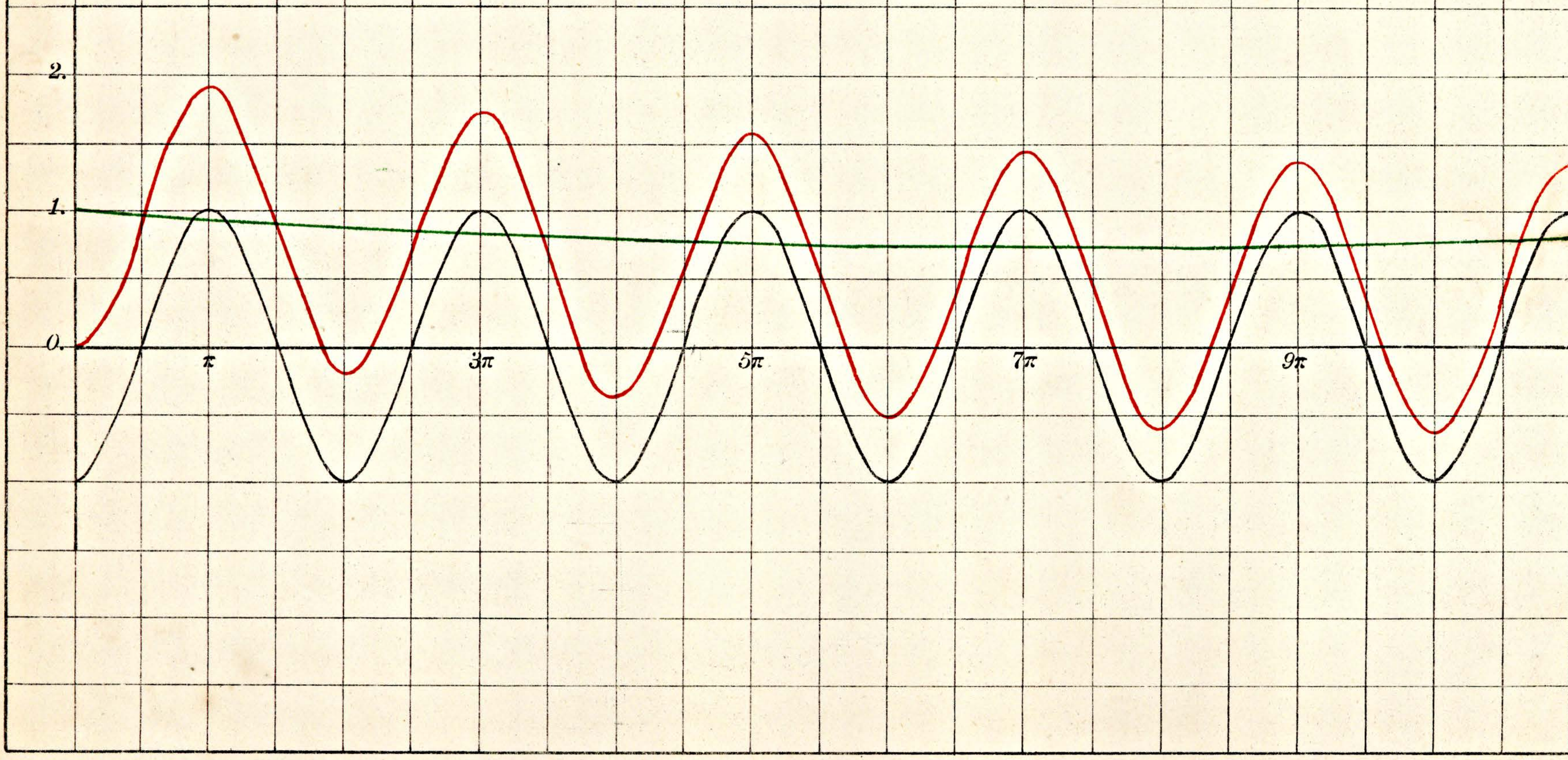
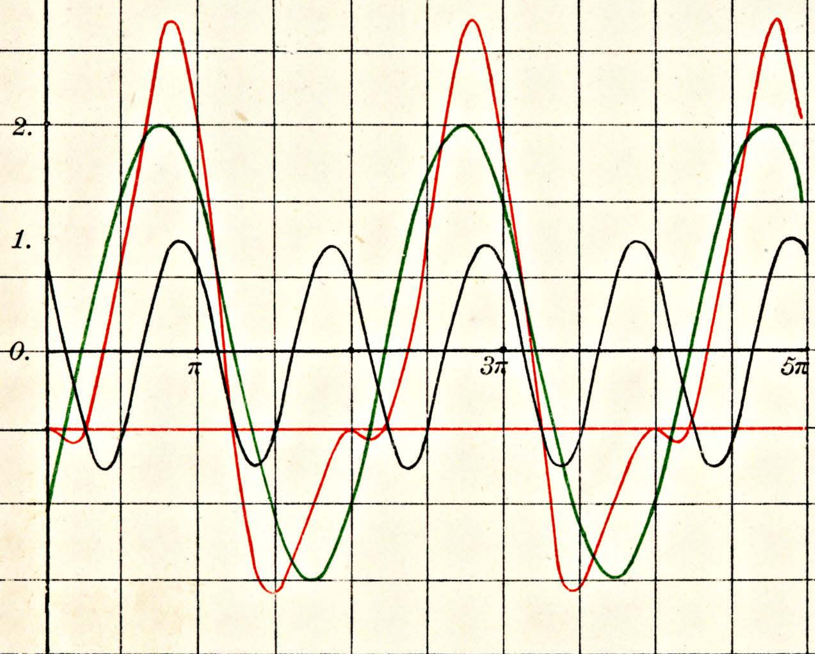


Figure. 11.

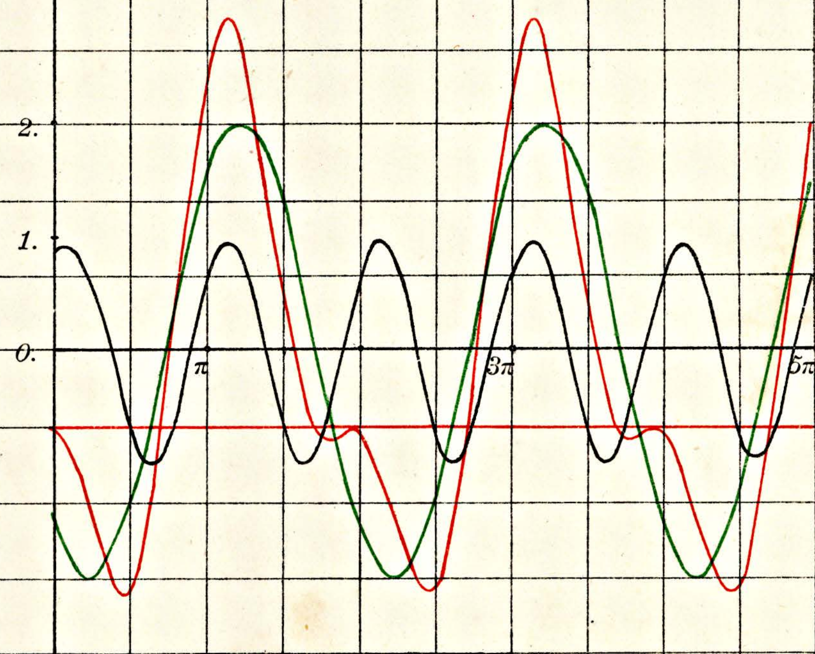
Maximum sudden short-circuit current in the direct field winding of the distortionless alternator with two field windings when both field windings are excited.



$$y_s = \cos\left(2\alpha + \frac{\pi}{4}\right) - 2\cos\left(\alpha + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}$$

Figure. 12.

Maximum sudden short-circuit current in the cross field winding of the distortionles alternator with two field windings when both field windings are excited.



$$z_s = \sin\left(2\alpha + \frac{\pi}{4}\right) - 2\sin\left(\alpha + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}$$