On T-algebra homomorphisms between rational function semifields of tropical curves

宋 珠愛 [1]

城崎代数幾何学シンポジウム 2022

東京都立大学 客員研究員

October 19, 2022

[1] songjuae@tmu.ac.jp

1. Introduction

1. Background

$\boldsymbol{T} := (\boldsymbol{R} \cup \{-\infty\}, \max, +)$: the tropical semifield

tropical geometry = algebraic geometry/ $\boldsymbol{T}$

algebraic geometry $\overset{\text{tropicalization}}{\longrightarrow}$ tropical geometry

tropical curves = tropicalizations of algebraic curves

1. Question

Fact

Category of nonsingular projective curves
$\overset{\text{equiv.}}{\longleftrightarrow}$ Category of function fields of $\dim = 1$

$\varGamma$: a tropical curve
$\mathrm{Rat}(\varGamma)$: the rational function semifield of $\varGamma$
(1) $\mathrm{Rat}(\varGamma)$: finitely generated as a semifield$/\boldsymbol{T}$? $\dim(\mathrm{Rat}(\varGamma)) = 1$?
(2) What kind of semifields$/\boldsymbol{T}$ determines a tropical curve? How?
(3) Does a $\boldsymbol{T}$-algebra homomorphism $\mathrm{Rat}(\varGamma) \to \mathrm{Rat}(\varGamma')$ induce a morphism $\varGamma' \to \varGamma$?

2. Preliminaries

2. Preliminaries

(abstract) tropical curve = metric graph = the underlying metric space of (G, l)
G : a graph
$l : E(G) \to \boldsymbol{R}_{>0}$
$*$ "graph" = unweighted, undirected, finite, connected nonempty multigraph that may have loops

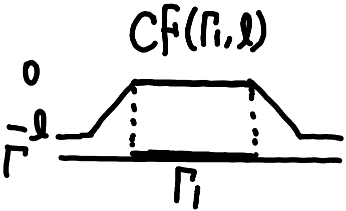
Γ : the tropical curve obtained from (G, l)
(G, l) : a model for Γ
(G, l) : loopless $\overset{\text{def}}{\Longleftrightarrow}$ G : loopless

2. Preliminaries

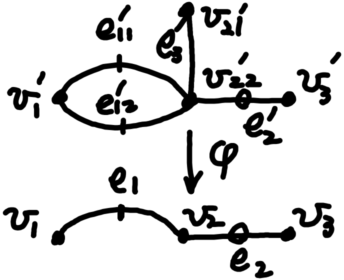
Γ : a tropical curve
$f : \Gamma \to \boldsymbol{R} \cup \{-\infty\}$: a rational function on Γ $\stackrel{\text{def}}{\Longleftrightarrow}$
f : continuous piecewise $\boldsymbol{Z}$-affine or $f \equiv -\infty$
$\mathrm{Rat}(\Gamma) := \{\text{rational functions on } \Gamma\}$
$(\mathrm{Rat}(\Gamma), \max, +)$: a semifield, a $\boldsymbol{T}$-algebra

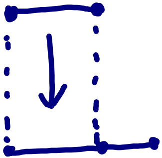
2. Preliminaries

Γ : a tropical curve
$\Gamma_1 \subset \Gamma$: closed, $\#$ connected components $< \infty$
$l \in \boldsymbol{R}_{>0}$
$\mathrm{CF}(\Gamma_1, l)(x) := -\min\{\mathrm{dist}(\Gamma_1, x), l\}\ (x \in \Gamma)$



2. Preliminaries





Γ, Γ' : tropical curves

$\varphi : \Gamma' \to \Gamma$: a continuous map

φ : a morphism[2] $\overset{\text{def}}{\iff}$

$\exists (G, l)$ (resp. (G', l')) : a loopless model for Γ (resp. Γ') s.t.

$\varphi : V(G') \cup E(G') \to V(G) \cup E(G)$:

(1) $\varphi(V(G')) \subset V(G)$,

(2) $e' = x'y' \in E(G'), \varphi(e') \in V(G) \Longrightarrow \varphi(e') = \varphi(x') = \varphi(y')$, and

(3)

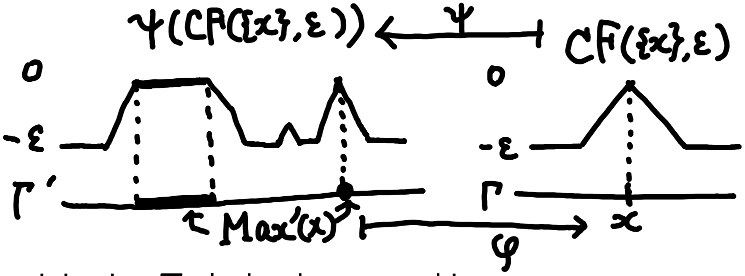
$e' = x'y' \in E(G'), \varphi(e') \in E(G) \Longrightarrow \varphi(e') = \varphi(x')\varphi(y'), l(\varphi(e'))/l'(e') \in \mathbf{Z}_{>0}$.

---

[2] M. Chan, *Tropical hyperelliptic curves*, J. Algebr. Comb. **37**(2):331–359, 2013.

3. Main results

3. Main theorem



Γ, Γ' : tropical curves
$\psi : \mathrm{Rat}(\Gamma) \to \mathrm{Rat}(\Gamma')$: an injective $\boldsymbol{T}$-algebra homomorphism
$\forall x \in \Gamma, \forall \varepsilon > 0, \mathrm{Max}'(x) := \{x' \in \Gamma' \mid \psi(\mathrm{CF}(\{x\}, \varepsilon))(x') = 0\}$

Theorem 1. (S.)

ψ induces a surjective morphism $\varphi : \Gamma' \to \Gamma; \mathrm{Max}'(x) \mapsto x$.
unique

3. Contrary

Proposition 2. (S.)

Γ, Γ' : tropical curves
$\varphi : \Gamma' \to \Gamma$: a surjective morphism
$\Longrightarrow \varphi^* : \mathrm{Rat}(\Gamma) \to \mathrm{Rat}(\Gamma'); f \mapsto f \circ \varphi$ is an injective $\boldsymbol{T}$-algebra homomorphism.

3. Corollary

Corollary 3. (S.)

The following categories $\mathscr{C}, \mathscr{D}$ are isomorphic:

(1) $\mathrm{Ob}(\mathscr{C}) := \{\text{tropical curves}\}$
$\mathrm{Hom}_{\mathscr{C}}(\Gamma, \Gamma') := \{\text{injective } \boldsymbol{T}\text{-algebra homomorphisms } \mathrm{Rat}(\Gamma) \to \mathrm{Rat}(\Gamma')\}$

(2) $\mathrm{Ob}(\mathscr{D}) := \{\text{tropical curves}\}$
$\mathrm{Hom}_{\mathscr{D}}(\Gamma, \Gamma') := \{\text{surjective morphisms } \Gamma \to \Gamma'\}$