A Zariski dense exceptional set in Manin's Conjecture: dimension 2

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Varieties with too many rational points

Definition (thin set)

$f : Y \to X$: a morphism of varieties.
Then f is a **thin map** if it is generically finite onto its image and either

1. f is not dominant, or
2. if f is dominant, then it is not birational.

We say $U \subseteq X(k)$ is a **thin set** if $U = \bigcup f_i(Y_i(k))$ for finite many thin maps $f_i : Y_i \to X$.

For a subset of $X(F)$,

$$\text{non-Zariski-dense set} \subsetneq \text{thin set}$$

Conjecture (Colliot-Thélène)

Let X be a k-unirational variety, then $X(k)$ is not thin.

Invariants from birational geometry

Definition (a-invariant / Fujita invariant)

Let X be a smooth projective variety and let L be a big and nef divisor. Define

$$a(X, L) := \min\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\}.$$

Definition (b-invariant)

Let X be a smooth geometrically integral projective variety over a field F and let L be a big and nef divisor on X. Define

$$b(F, X, L) := \text{codimension of the minimal supported face of } \overline{\mathrm{Eff}}^1(X) \text{ containing } K_X + a(X, L)L.$$

When X is a Fano variety and $L = -K_X$, we have $a(X, L) = 1$ and $b(F, X, L) =$ Picard rank of X.

Manin's Conjecture

X: a geometrically rationally connected and geometrically integral smooth projective variety over a number field F
$\mathcal{L}$: an big and nef line bundle with an adelic metrization X
$H_{\mathcal{L}}$: the height function associated to $\mathcal{L}$
Define

$$N(Q, \mathcal{L}, T) := \# \{P \in Q \mid H_{\mathcal{L}}(P) \leq T\}$$

for any subset $Q \subset X(F)$.

Manin's Conjecture (Batyrev-Manin,1990, Peyre, 2003)

Suppose $X(F)$ is not thin. Then there exists a thin set Z such that

$$N(X(F) \backslash Z, \mathcal{L}, T) \sim c(F, Z, \mathcal{L}) T^{a(X,L)} \log(T)^{b(F,X,L)-1}$$

as $T \to \infty$. The thin set Z is known as the **exceptional set**.

Manin's Conjecture

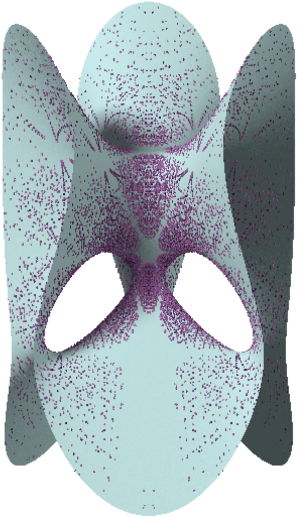

Figure: Rational points of bounded height outside the 27 lines on a smooth cubic surface (from Wikipedia)

Manin's Conjecture

Histories:

- The original version of the conjecture predicted that it is enough to remove a exceptional set Z which is not Zariski dense.
- A counterexample was found: (Batyrev-Tschinkel 1996)
  A hypersurface of bidegree $(1,3)$ in $\mathbb{P}^n \times \mathbb{P}^3$ $(n \geq 1)$.
- The conjecture was refined to assume Z is contained in a thin set. (Peyre 2003)

Some known cases:

1. $\mathbb{P}^n$ over number fields (Schanuel, 1976)
2. toric varieties over number fields, including smooth del Pezzo surface of degree ≥ 6.(Batyrev-Tschinkel 1998)
3. Some del Pezzo surfaces of degree 4.

Exceptional sets in dimension 2

- Batyrev-Tschinkel's counterexamples work for each dimension ≥ 3, but there is no counterexample in dimension 2.
- In all confirmed cases in dimension 2, the exceptional set Z is not Zariski dense.

Question

Does the original Manin's conjecture still hold in dimension 2?

Geometric exceptional sets of Fano varieties

Recently, a conjectural construction of the exceptional set was proposed in [Lehmann-Sengupta-Tanimoto 2019]. We call it the **Geometric exceptional set** Z'. They proved

Theorem ([LST19])

1. Z' is contained in a thin set.
2. If X is a **general** del Pezzo surface, Z' is not Zariski dense.

The only case they can not deal with is when X is a del Pezzo surface of degree 1 and Picard rank 1. Explicitly, the proof appeals to

Proposition

For such X which is general in moduli, the moduli space of rational curves in $|-2K_X|$ is irreducible of genus ≥ 2.

Main theorem

Question ([LST19])

For such X, let M be the curve parametrizing rational curves in $|-2K_X|$. Does every component of M have genus ≥ 2?

If the answer is yes, then we expect the original conjecture still holds in dimension 2. However, we answer this question negatively.

Theorem (G, 2022)

Let

$$S := \{w^2 = z^3 + ax^6 + ay^6\} \subset \mathbb{P}_k(1,1,2,3)$$

over the field $k = \mathbb{Q}(e^{2\pi i/3})$. Then there exists an elliptic family $(\pi : \mathcal{C} \to E,\ \mu : \mathcal{C} \to S)$ of rational curves in $|-2K_S|$, such that

1. When $a = 49$, E is an elliptic curve with positive Mordell–Weil rank.
2. there exists a generically smooth section of π,
3. the Picard rank of S is 1.

Corollaries

Corollary 1

The geometric exceptional set of S is Zariski dense.

Corollary 2

The original version of Manin's Conjecture does not hold for S.

This settles the last dimension in which we don't know the original Manin's Conjecture is true or not.

Key observations

- (well-known) $|-2K_S|$ realizes S as a double cover of a quadric cone, rational curves in $|-2K_S|$ corresponds to bitangent planes of the branch locus B.
- There exists a nontrivial involution $\iota \in \mathrm{Aut}(B)$ fixing the tangent planes.
- The universal family has a defining equation

$$\frac{(x_0^3+y_0^3)^2w^2}{49} = (y_0x^2 - x_0y^2)^2[(2x_0^3y_0+y_0^4)x^2 + (x_0^4+2x_0y_0^3)y^2],$$

  where the only non-square part becomes $(x_0^3 - \zeta_3 y_0^3)^2$ by letting $(x,y) = (\zeta_3 y_0, x_0)$.
- We have in general $\mathrm{Pic}\, S_k \cong (\mathrm{Pic}\, S_K)^{\mathrm{Gal}(K/k)}$, where $K = \mathbb{Q}(\zeta_3, \sqrt[3]{2}, \sqrt[3]{7})$ is the splitting field of S. We compute the Picard rank of S_k explicitly by computing the representation of $\mathrm{Gal}(K/k)$ on $\mathrm{Pic}\, S_K$.

Thanks.