# Big Cohen-Macaulay test ideals in equal characteristic zero via ultraproducts

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Let $(R, \mathfrak{m})$ be a Noetherian local ring.

**Definition 1**

$R$-algebra $B$ is said to be a *(balanced) big Cohen-Macaulay algebra* (or simply *BCM* algebra) if every system of parameters of $R$ is a regular sequence on $B$.

We explain BCM regularity introduced by [Ma, Schwede 21].

**Setting 1**

Let $(R, \mathfrak{m})$ be a normal local domain of dimension d.

1. We fix an embedding $R \subseteq \omega_R \subseteq \operatorname{Frac} R$. Hence we also fix an effective canonical divisor $K_R$.
2. $\Delta \geqslant 0$ is a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier.
3. Since $K_R + \Delta$ is effective and $\mathbb{Q}$-Cartier, there exists $n \in \mathbb{N}_{>0}$ and $f \in R$ such that $n(K_R + \Delta) = \operatorname{div}(f)$.

$R[f^{1/n}]^{\mathrm{N}} \subseteq R^+$ denotes the normalization of $R[f^{1/n}]$.

## Definition 2 ([Ma, Schwede 21])

With notation as in Setting 1, if $B$ is a BCM $R$-algebra and an $R[f^{1/n}]^{\mathbb{N}}$-algebra, then we define $0^{B,K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)} := \operatorname{Ker}\psi$, where $\psi$ is the homomorphism determined by the below commutative diagram:

$$\begin{tikzcd} H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(B) \arrow[r, "\cdot f^{1/n}"] \arrow[d] & H^d_{\mathfrak{m}}(B) . \\ H^d_{\mathfrak{m}}(\omega_R) \arrow[r] \arrow[rru, bend right, "\psi"'] & H^d_{\mathfrak{m}}(B \otimes_R \omega_R) \arrow[ru] & \end{tikzcd}$$

**Definition 3 ([Ma, Schwede 21])**

Moreover, if $R$ is complete, then we define

$$\tau_B(R, \Delta) := \operatorname{Ann}_R 0^{B, K_R + \Delta}_{H^d_{\mathfrak{m}}(\omega_R)}.$$

We call $\tau_B(R, \Delta)$ *the BCM test ideal of* $(R, \Delta)$ *w.r.t.* $B$. We call $(R, \Delta)$ is $\text{BCM}_B$-regular if $\tau_B(R, \Delta) = R$.

## Proposition 1 ([Ma, Schwede 21])

*Let*

1. $(R, \mathfrak{m})$ *a Noetherian complete normal local domain of characteristic* $p > 0$
2. $\Delta \geqslant 0$ *a* $\mathbb{Q}$*-Weil divisor on* $\operatorname{Spec} R$ *such that* $K_R + \Delta$ *is* $\mathbb{Q}$*-Cartier*
3. $B$ *a BCM* $R^+$*-algebra*

*Then*

$$\tau_B(R, \Delta) = \tau(R, \Delta).$$

*In particular,* $R$ *is strongly* $F$*-regular if and only if* $R$ *is* $\mathrm{BCM}_B$*-regular.*

We fix an infinite set $W$. We use $P(W)$ to denote the power set of $W$.

### Definition 4

A nonempty subset $\mathcal{F} \subseteq P(W)$ is called a *filter* if the following two conditions hold.

1. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq W$, then $B \in \mathcal{F}$.

### Definition 5

Let $\mathcal{F}$ be a filter on $W$.

1. $\mathcal{F}$ is called an *ultrafilter* if for all $A \in P(W)$, we have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.
2. An ultrafilter $\mathcal{F}$ is called *principal* if there exists a finite subset $A \subseteq W$ such that $A \in \mathcal{F}$.

## Proposition 2

*Every infinite set has non-principal ultrafilters.*

## Definition 6

Let

1. $A_w$ a family of set indexed by $W$
2. $\mathcal{F}$ a non-principal ultrafilter on $W$

The *ultraproduct of* $A_w$ is defined by

$$\underset{w}{\text{ulim}}\, A_w = A_\infty := \prod_{w} A_w / \sim,$$

where $(a_w) \sim (b_w)$ if and only if $\{w \in W | a_w = b_w\} \in \mathcal{F}$.

Following [Schoutens 03], we explain approximations and non-standard hulls.
Let

1. $R$ be a local ring essentially of finite type over $\mathbb{C}$.
2. $\mathcal{P}$ be the set of prime numbers
3. $\mathcal{F}$ a non-principal ultrafilter on $\mathcal{P}$.

Then we can construct an approximation $R_p$ and the non-standard hull $R_\infty$ of $R$.
They have the following properties.

1. $R_p$ local rings essentially of finite type over $\overline{\mathbb{F}_p}$
2. $R_\infty = \operatorname{ulim}_p R_p$
3. $R \to R_\infty$ faithfully flat

### Definition 7

Let

1. $\mathcal{F}$ a non-principal ultrafilter on $\mathcal{P}$
2. $\varphi$ a property

Then we say $\varphi(p)$ *holds for almost all* $p$ if $\{p \in \mathcal{P} | \varphi(p) \text{ holds}\} \in \mathcal{F}$.

Let $R$ be a local ring essentially of finite type over $\mathbb{C}$.

### Example 1

1. $\dim R_p = d$ for almost all $p$ if and only if $\dim R = d$.
2. $R_p$ is Cohen-Macaulay (resp. Gorenstein, regular) for almost all $p$ if and only if $R$ is Cohen-Macaulay (resp. Gorenstein, regular).

Let $R$ be a local domain essentially of finite type over $\mathbb{C}$.

**Definition 8 ([Schoutens 04])**

We define the *canonical BCM algebra* $\mathcal{B}(R)$ by

$$\mathcal{B}(R) := \operatorname{ulim}_{p} R_p^+,$$

where $R_p$ is an approximation of $R$. (Schoutens calls this the *quasi-hull*.)

**Proposition 3 ([Schoutens 04])**

*$\mathcal{B}(R)$ is a BCM $R$-algebra and an $R^+$-algebra.*

Our main result is stated as follows:

**Theorem 1 (Y 22)**

*Let*

1. *$R$ a normal local domain essentially of finite type over $\mathbb{C}$*
2. *$\Delta \geqslant 0$ a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier*
3. *$\mathcal{B}(R)$ is the canonical BCM algebra*
4. *$\widehat{R}$ and $\widehat{\mathcal{B}(R)}$ are the $\mathfrak{m}$-adic completions of $R$ and $\mathcal{B}(R)$*
5. *$\widehat{\Delta}$ the flat pullback of $\Delta$ by $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$*

*Then we have*

$$\tau_{\widehat{\mathcal{B}(R)}}(\widehat{R}, \widehat{\Delta}) = \mathcal{J}(\widehat{R}, \widehat{\Delta}),$$

*where $\mathcal{J}(\widehat{R}, \widehat{\Delta})$ is the multiplier ideal of $(\widehat{R}, \widehat{\Delta})$.*

As an application of the preceding theorem, we proved the following.

**Theorem 2 (Y 22)**

*Let*

1. $R \hookrightarrow S$ *is a pure local* $\mathbb{C}$*-algebra homomorphism between normal local domains essentially of finite type over* $\mathbb{C}$
2. $R$ *is* $\mathbb{Q}$*-Gorenstein*
3. $\Delta_S \geqslant 0$ *a* $\mathbb{Q}$*-Weil divisor such that* $K_S + \Delta_S$ *is* $\mathbb{Q}$*-Cartier*
4. $\mathfrak{a} \subseteq R$ *a nonzero ideal*
5. $t \in \mathbb{Q}_{>0}$

*Then we have*

$$\mathcal{J}(S, \Delta_S, (\mathfrak{a}S)^t) \cap R \subseteq \mathcal{J}(R, \mathfrak{a}^t).$$

Let $R$ be a normal local domain essentially of finite type over $\mathbb{C}$.

### Question 1

*Let*

1. $\Delta \geqslant 0$ *a* $\mathbb{Q}$-*Weil divisor on* $\operatorname{Spec} R$ *such that* $K_R + \Delta$ *is* $\mathbb{Q}$-*Cartier*
2. $B$ *a BCM* $R^+$-*algebra*

*Then does the following hold?*

$$\mathcal{J}(R, \Delta) \subseteq \tau_{\widehat{B}}(\widehat{R}, \widehat{\Delta})$$

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