Rend. Istit. Mat. Univ. Trieste
Vol. XXXIX, 457–476 (2007)

An Algorithm for Reconstructing a Convex Polygon from its Covariogram

Carlo Benassi and Giuliana D'Ercole (*)

Summary. - *The covariogram of a compact convex set $K \subset \mathbb{R}^n$ is the function that at each point $x \in \mathbb{R}^n$ associates the volume of $K \cap (K+x)$. The covariogram determines, among all convex bodies, any planar convex polygon. In this paper we present an algorithm for reconstructing an arbitrary convex polygon from its covariogram.*

1. Introduction

Let K be a convex body in $\mathbb{R}^n$, that is a compact, convex subset of $\mathbb{R}^n$ with non empty interior. By $\mathbb{S}^{n-1}$ we denote the set of all vectors $u \in \mathbb{R}^n$ whose Euclidean norm is 1. The n-dimensional Lebesgue measure in $\mathbb{R}^n$ is denoted by λ_n. If $x \in \mathbb{R}^n$, then $K+x$ denotes the translate of K by x, i.e.

$$K + x = \{k + x \in \mathbb{R}^n : k \in K\}.$$

The *covariogram* of a convex body $K \subset \mathbb{R}^n$ is the function g_K

(*) Authors' addresses: Carlo Benassi, Dipartimento di Matematica Pura ed Applicata, Universit di Modena e Reggio Emilia, 41100 Modena (MO), Italy; E-mail: benassi.carlo@unimore.it
Giuliana D'Ercole, Dipartimento di Matematica Pura ed Applicata, Universit di Modena e Reggio Emilia, 41100 Modena (MO), Italy; E-mail: dercole.giuliana@unimore.it
Keywords: Convex Polygon, Covariogram, Reconstructive Procedure.
AMS Subject Classification: 52A10.

defined for $x \in \mathbb{R}^n$ by

$$g_K(x) = \lambda_n \left(K \cap (K + x)\right).$$

Some properties of the covariogram are immediate. The support of g_K coincides with the *difference set* DK of K (see, for example, [13]):

$$DK = K - K = \{k_1 - k_2 \in \mathbb{R}^n : k_1, k_2 \in K\}.$$

Moreover g_K is unchanged by a translation or a reflection (in a point) of K. Matheron introduced in [9] the covariogram and posed in [8] the following question.

Covariogram problem *Does the covariogram determine a convex body, among all convex bodies, up to translations and reflections?*

Matheron [9] observed that, for $u \in \mathbb{S}^{n-1}$ and for $r > 0$, the derivatives $dg_K(ru)/dr$ give the distribution of the lengths of the chords of K parallel to u. The covariogram problem is thus equivalent to the problem of determining a convex body from these data. This information is available in stereology, statistical shape recognition and image analysis, when properties of an unknown body have to be inferred from chord length measurements; see, for example, [7], [12], [14].

The covariogram problem appears in several other contexts. Adler and Pyke [1] asked whether the distribution of the difference $X - Y$ of two independent random variables X and Y uniformly distributed over K determines K up to translations and reflections. It is easy to check that

$$g_K = 1_K * 1_{-K} \tag{1}$$

and this convolution, up to a multiplicative factor, is just the probability density of $X - Y$. Hence this problem is equivalent to the covariogram problem.

The covariogram problem is also a particular case of the *phase retrieval problem* in Fourier analysis. This problem involves determining a function f from the modulus of its Fourier transform $\hat{f}$; see [11]. Taking Fourier transforms in (1) and using the relation $\widehat{1_{-K}} = \overline{\widehat{1_K}}$, we obtain

$$\widehat{g_K} = \widehat{1_K}\widehat{1_{-K}} = |\widehat{1_K}|^2.$$

Thus, the phase retrieval problem reduces to the covariogram problem when f is the characteristic function of a convex body.

The covariogram also arises in X-ray crystallography. Here the covariogram appears in the cut and projection scheme for quasicrystals; see [4]. In particular, to obtain the atomic structure of a quasi-crystal, it is necessary to determine a lattice N and a convex set W, called window. The diffraction image provides N and the covariogram of W.

The first contribution to the question posed by Matheron was made by Nagel [10], who gave a positive answer when K is a planar convex polygon. After a long period without any new result and stimulated by a renewed interest in geometric tomography, Bianchi, Segala and Volčič [6] confirmed that if K is a planar convex body whose boundary is C^2_+ (that is, C^2 with strictly positive curvature), then K is determined, among all convex bodies, by its covariogram. Bianchi [5] and Averkov and Bianchi [3] gave other two partial answers to the covariogram problem in the plane. Only recently Averkov and Bianchi [2] have settled Matheron's conjecture for arbitrary planar convex bodies.

The present paper is devoted to present an algorithm which reconstructs any convex polygon from its covariogram. Till now the only result in this direction is due to Schmitt [12] who describes a procedure to reconstruct convex polygons under the assumption that the polygon does not have any pair of parallel edges. It will be clear that our algorithm, if applied under the assumption made by Schmitt, is greatly simplified. If the polygon has no pairs of parallel edges, then out of the many cases that the algorithm deals with, only one has to be considered.

2. Determination of edges

Let $K \subset \mathbb{R}^2$ be a convex polygon and denote by $F(K,u)$ the face of K with outward normal u, that is the subset of the boundary of K made by points with unit outward normal u. Clearly $F(K,u)$ is either an edge or a vertex of K. Let now $F(K,u)$ be an edge of K. If $F(K,-u)$ is a vertex, then $F(K,u)$ will be called a *single edge*; if $F(K,-u)$ is an edge and $\lambda_1(F(K,u)) = \lambda_1(F(K,-u))$, we will refer to $(F(K,u), F(K,-u))$ as to a *pair of parallel edges of equal length*; if $F(K,-u)$ is an edge and $\lambda_1(F(K,u)) \neq \lambda_1(F(K,-u))$, we will refer to $(F(K,u), F(K,-u))$ as to a *pair of parallel edges of different length.*

In order to determine the lengths of the edges of K, let us consider the edges of the polygon DK. As DK is origin symmetric, its edges are pairwise parallel and with equal length. Let $v_0, ..., v_{r-1}, -v_0, ..., -v_{r-1} \in \mathbb{S}^1$ be the unit outward normals to the edges of DK, in counterclockwise order and let m_i, $i = 0, ..., r-1$, be the midpoint of the edge $F(DK, v_i)$. In what follows the symbol ∂^- denotes the left derivative.

Proposition 2.1. *Each edge of K is parallel to one of the edges of DK and vice versa. The vector difference of the midpoint of $F(K, v_i)$ and $F(K, -v_i)$ equals the midpoint of $F(DK, v_i)$. Moreover*

(i) if $\frac{\partial^- g_K}{\partial v_i}(m_i) = l_i$, with $l_i \neq 0$, then both $F(K, v_i)$ and $F(K, -v_i)$ are edges of K; the shorter edge has length l_i and the longer one has length $\lambda_1(F(DK, v_i)) - l_i$;

(ii) if $\frac{\partial^- g_K}{\partial v_i}(m_i) = \frac{\lambda_1(F(DK,v_i))}{2}$, then both $F(K, v_i)$ and $F(K, -v_i)$ are edges of K, of length $\frac{\lambda_1(F(DK,v_i))}{2}$;

(iii) if $\frac{\partial^- g_K}{\partial v_i}(m_i) = 0$, then exactly one between $F(K, v_i)$ and $F(K, -v_i)$ is an edge of K, of length $\lambda_1(F(DK, v_i))$.

Proof. The first and the second claim follow from standard results on the relation between the edges of K and of DK; see Theorem 1.7.5 (c) in [13].

Suppose both $F(K, v_i)$ and $F(K, -v_i)$ are edges of K, the shorter one of length l_i, with $l_i \neq 0$, and compute $\frac{\partial^- g_K}{\partial v_i}(m_i)$. Then

$$\begin{aligned} \frac{\partial^- g_K}{\partial v_i}(m_i) &= \lim_{h \to 0^-} \frac{\lambda_2(K \cap (K + m_i + hv_i)) - \lambda_2(K \cap (K + m_i))}{h} \\ &= \lim_{h \to 0^-} \frac{l_i h + O(h^2)}{h} = l_i, \end{aligned} \tag{2}$$

because in this case $K \cap (K + m_i + hv_i)$ is the union of triangles with edges of length proportional to h and of a rectangle of height h and base l_i, whereas $K \cap (K + m_i)$ is a segment (see Fig. 1).

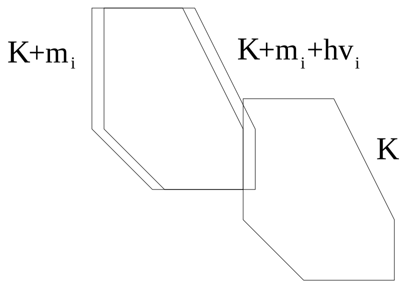

Figure 1: The intersections of a body K, having a pair of parallel edges of different length, with its translated $K+m_i$ and $K+m_i+hv_i$.

As $\lambda_1(F(DK,v_i))$ is the sum of $\lambda_1(F(K,v_i))$ and $\lambda_1(F(K,-v_i))$, then the longer edge has length $\lambda_1(F(DK,v_i)) - l_i$.

Suppose now both $F(K,v_i)$ and $F(K,-v_i)$ are edges of K, of equal length $\frac{\lambda_1(F(DK,v_i))}{2}$. Then

$$\begin{aligned}\frac{\partial^- g_K}{\partial v_i}(m_i) &= \lim_{h\to 0^-} \frac{\lambda_2(K\cap(K+m_i+hv_i)) - \lambda_2(K\cap(K+m_i))}{h} \\ &= \lim_{h\to 0^-} \frac{\lambda_1(F(DK,v_i))h + O(h^2)}{2h} \\ &= \frac{\lambda_1(F(DK,v_i))}{2}, \end{aligned} \tag{3}$$

because in this case $K\cap(K+m_i+hv_i)$ is the union of triangles with edges of length proportional to h and of a rectangle of height h and base $\frac{\lambda_1(F(DK,v_i))}{2}$, whereas $K\cap(K+m_i)$ is a segment.

Finally, suppose exactly one between $F(K,v_i)$ and $F(K,-v_i)$ is an edge of K. Then

$$\begin{aligned}\frac{\partial^- g_K}{\partial v_i}(m_i) &= \lim_{h\to 0^-} \frac{\lambda_2(K\cap(K+m_i+hv_i)) - \lambda_2(K\cap(K+m_i))}{h} \\ &= \lim_{h\to 0} \frac{O(h^2)}{h} = 0, \end{aligned} \tag{4}$$

because $K\cap(K+m_i+hv_i)$ is a triangle with edges of length proportional to h, whereas $K\cap(K+m_i)$ is a point.

The above three cases complete all the possibilities and the comparison of $\frac{\partial^- g_K}{\partial v_i}(m_i)$ with $\lambda_1(F(DK,v_i))$ implies the choice. In each case these two values determine the lengths of the corresponding edges of K. □

In this way we can determine all the lengths and the directions of the edges of the polygon K. However we do not know on which side of K each edge is. That is, given an edge F of K orthogonal to a given direction v_i, we do not know wether the outward normal to K in F is v_i or $-v_i$. As a first step to overcome this difficulty let us remark that the proof of Lemma 3.2 in [5] shows the following results. Given a single edge $F(K, v_i)$, we can determine the directions $v_{i,p}$ and $v_{i,f}$ of the edges of K which are adjacent to the vertex $F(K, -v_i)$. More precisely, that proof shows that there are some level sets of g_K which contain line segments parallel to $F(K, v_i)$. Moreover, the endpoints of these segments are aligned on lines parallel to $v_{i,p}$ and $v_{i,f}$. A similar result holds also in the presence of a pair of parallel edges $(F(K, v_i), F(K, -v_i))$ of different length. If, say, $\lambda_1(F(K, v_i)) < \lambda_1(F(K, -v_i))$, then we can determine the directions $v_{i,p}$ and $v_{i,f}$ of the edges adjacent to $F(K, v_i)$. In fact the proof of Lemma 3.2 in [5] shows that some level sets of g_K contain segments parallel to $F(K, v_i)$. The endpoints of these segments are aligned on lines parallel to $v_{i,p}$ and $v_{i,f}$.

Let us order in a table all the information obtained till now. Fill the first line with the unit vectors v_i, $i = 0, ..., r-1$ and the second line with the vectors $-v_i$, $i = 0, ..., r-1$. Dispose in the third and fourth line the lengths $\lambda_1(F(K, \pm v_i))$: if in correspondence to v_i there is a single edge of K, then write its length in the third line of the i-th column and 0 in the fourth line of the i-th column. If in correspondence to v_i there are two edges of equal length, then write this length both in the third and fourth line of the i-th column. If in correspondence to v_i there are two edges of different length, then write in the third line of the i-th column the value of the shorter edge and in the fourth line of the i-th column the value of the longer edge. Finally, if in correspondence to v_i there is a single edge, then dispose in the fifth line of the i-th column of the table the directions $v_{i,p}$ and $v_{i,f}$ of the edges adjacent to the opposite vertex; if in correspondence to v_i there are two edges of different length, then dispose in the fifth line of the i-th column the directions $v_{i,p}$ and $v_{i,f}$ of the edges adjacent to the shorter edge; if in correspondence to v_i there are two edges of equal length, then let the fifth line of the i-th column be empty.

v_0	v_1	...	v_{r-1}
$-v_0$	$-v_1$	...	$-v_{r-1}$
l_i	$\lambda_1(F(DK, v_i))$	...	$\frac{\lambda_1(F(DK,v_i))}{2}$
$\lambda_1(F(DK, v_i)) - l_i$	0	...	$\frac{\lambda_1(F(DK,v_i))}{2}$
$v_{0,p}, v_{0,f}$	$v_{1,p}, v_{1,f}$	...	- -

3. Reconstruction of K

The reconstruction of the boundary of K develops along two different "antipodal" sequences, $S_1 := (S_1^i)_{i=0,...,r-1}$ and $S_2 := (S_2^i)_{i=0,...,r-1}$, of consecutive edges. For each $i = 0, ..., r-1$, at the end of the i-th step of the reconstruction, S_1^i and S_2^i will coincide with the portion of the boundary of K made of all the edges with outward normal v_h and $-v_h$ respectively, $0 \leqslant h \leqslant i$.

The $(i+1)$-th step consists in positioning all edges with outward normals $\pm v_{i+1}$. Clearly, we have only to decide which edge, between the edges of K orthogonal to v_{i+1}, has to be added to S_1^i and which one has to be added to S_2^i. Once this decision is made, each edge is added at the corresponding sequence so that the added edge follows, in counterclockwise order, the other edges of the sequence. For each positioned edge, we call *first vertex* its endpoint which precedes the edge in the sequence, and *second vertex* its endpoint which follows the edge in the sequence.

In what follows, if in correspondence to $\pm v_i$ there is a single edge, F_i^- will denote the single edge and F_i^+ will denote the opposite vertex. If in correspondence to $\pm v_i$ there is a pair of parallel edges of different length, we denote by F_i^- the shorter edge and by F_i^+ the longer one. Moreover, if in correspondence to $\pm v_i$ there is a pair of parallel edges of equal length, we denote by F_i^- and F_i^+ these two edges.

In the reconstruction of K we denote by $\widehat{uv}$ one of the two angles formed by the vectors u and v. We will see that it is not necessary to specify the choice.

Let us start with the 0-th step. Distinguish the following cases:

1. *In K there is no pair of parallel edges.*

   Locate F_0^- so that it is orthogonal to v_0 and dispose the opposite vertex F_0^+ so that the difference of the midpoint of F_0^- and F_0^+ coincides with the midpoint of $F(DK, v_0)$. This choice is motivated by Proposition 2.1. In this case S_1^0 is the edge F_0^- and S_2^0 is the vertex F_0^+.

2. *In K there are pairs of parallel edges of equal length and there are not pairs of parallel edges of different length.*

   Rename the edges $F(DK, \pm v_i)$, $i = 0, ..., r-1$, of DK so that $\pm v_0$ are the outward normals of a pair of parallel edges of equal length. Dispose F_0^- and F_0^+ so that they are orthogonal to v_0 and the difference of the midpoints of F_0^- and F_0^+ coincides with the midpoint of $F(DK, v_0)$. In this case S_1^0 is the edge F_0^- and S_2^0 is the edge F_0^+.

3. *In K there are pairs of parallel edges of different length.*

   Rename the edges $F(DK, \pm v_i)$, $i = 0, ..., r-1$, of DK so that $\pm v_0$ are the outward normals of a pair of parallel edges of different length. Dispose F_0^- and F_0^+ so that they are orthogonal to v_0 and the difference of the midpoints of F_0^- and F_0^+ coincides with the midpoint of $F(DK, v_0)$. In this case S_1^0 is the edge F_0^- and S_2^0 is the edge F_0^+.

To make the i-th step we have to consider many cases and subcases. Let us first summarize them:

a) The vectors $\pm v_i$ are in correspondence to a single edge F_i^-;

   aa) The vectors $\pm v_{i-1}$ are in correspondence to a single edge F_{i-1}^-;

   ab) The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of equal length (F_{i-1}^-, F_{i-1}^+);

ab1) The sequence S_1^{i-1} is the reflection of S_2^{i-1} with respect to some point;

ab2) The sequences S_1^{i-1} and S_2^{i-1} are not one the reflection of the other (this case splits into two further subcases);

ac) The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of different length (F_{i-1}^-, F_{i-1}^+);

b) The vectors $\pm v_i$ are in correspondence to a pair of parallel edges of equal length (F_i^-, F_i^+);

c) The vectors $\pm v_i$ are in correspondence to a pair of parallel edges of different length (F_i^-, F_i^+);

ca) The vectors $\pm v_{i-1}$ are in correspondence to a single edge F_{i-1}^-;

cb) The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of equal length (F_{i-1}^-, F_{i-1}^+) (this case splits into two further subcases);

cc) The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of different length (F_{i-1}^-, F_{i-1}^+).

Let us now describe in details each case.

a) *The vectors $\pm v_i$ are in correspondence to a single edge F_i^-.*

Consider the vectors $\pm v_{i-1}$.

aa) *The vectors $\pm v_{i-1}$ are in correspondence to a single edge F_{i-1}^-.*

In this case, if one of $v_{i,p}$, $v_{i,f}$ is orthogonal to v_{i-1}, then add F_i^- to the sequence which does not contain F_{i-1}^-; otherwise add F_i^- to the sequence which contains the single edge F_{i-1}^-. Proceed with the $(i+1)$-th step.

ab) *The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of equal length,* (F_{i-1}^-, F_{i-1}^+).

In this case consider the following cases:

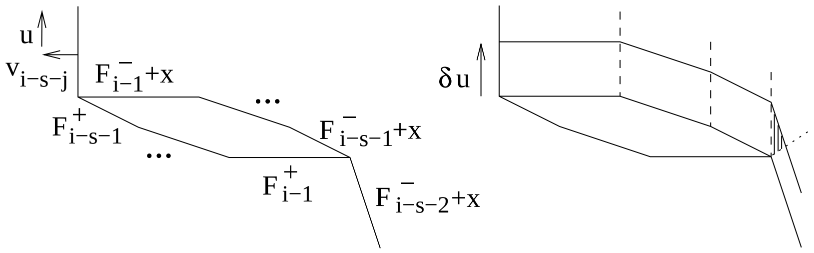

Figure 2: The intersection of K with $K + x$ (left) and of K with $K + x + \delta u$ (right).

ab1) *The sequence S_1^{i-1} is the reflection of S_2^{i-1} with respect to some point.*
This situation occurs when all edges already positioned are pairwise parallel and of equal length. In this case the edge F_i^- can be added indifferently to S_1 or to S_2. Add F_i^- to S_1 and proceed with the $(i+1)$-th step.

ab2) *The sequences S_1^{i-1} and S_2^{i-1} are not one the reflection of the other.*
This means that at a certain moment we have positioned a single edge or a pair of parallel edges of different length. Let s, $s \geqslant 0$, be the smallest index such that F_{i-1}^-, ..., F_{i-s-1}^- and F_{i-1}^+, ..., F_{i-s-1}^+ are consecutive edges of pairs of parallel edges of equal length and $(F_{i-s-2}^-, F_{i-s-2}^+)$ is not a pair of parallel edges of equal length.

ab2s) *If $\pm v_{i-s-2}$ is in correspondence to a single edge* F_{i-s-2}^-, then consider the translation x which maps the second vertex of F_{i-1}^- to the first vertex of F_{i-s-1}^+ (see Fig. 2 on the left). Compute $g_K(x)$.
Afterwards, consider the two edges which are adjacent and precede the edges F_{i-s-1}^- and F_{i-s-1}^+. One of them has outward normal $\pm v_{i-s-2}$, whereas the other one has outward normal $\pm v_{i-s-j}$, $j > 2$. Denote by u the unit vector

orthogonal to v_{i-s-j} with orientation such that $g_K(x+\delta u) > g_K(x)$, where δ is a positive number such that

$$\delta \leq \min_{k=i,\dots,r-1} \left\{ \lambda_1(F^-_{i-s-2}) \frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_k v_{i-s-j}})|}, \right.$$
$$\left. \lambda_1(F^-_{i-s-j}), \lambda_1(F^-_k) \frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_{i-s-2} v_{i-s-j}})|} \right\}.$$

The number

$$g_K(x+\delta u) - g_K(x) +$$
$$- \sum_{k=0}^{s} \lambda_1(F^-_{i-k-1}) |\cos(\widehat{u v_{i-k-1}})| \delta$$

is equal to the area of a triangle (see the shadowed triangle in Fig. 2 on the right).
Here, the maximum value for δ is computed considering separately the three borderline cases for which one of the edges of the aforementioned triangle coincides with a whole edge of K.
About this triangle we know that the length of the edge parallel to u is δ, and we also know the direction of an edge adjacent to the aforementioned edge, since it is orthogonal to v_{i-s-2}. By these data and the area of the triangle, we can determine also the direction of the other edge. If this direction is orthogonal to v_i, then add F^-_i to the sequence which does not contain F^-_{i-s-2}; otherwise add F^-_i to the sequence which contains F^-_{i-s-2}. Proceed with the $(i+1)$-th step.

ab2p) *If* $\pm v_{i-s-2}$ *is in correspondence to a pair of parallel edges of different length,* $(F^-_{i-s-2}, F^+_{i-s-2})$, then consider the translation x which maps the first vertex of F^-_{i-s-2} to the second vertex of the edge with outward normal $\pm v_{i-1}$ located in the sequence which does not contain F^-_{i-s-2} (see Fig. 3 on the left). Compute $g_K(x)$.

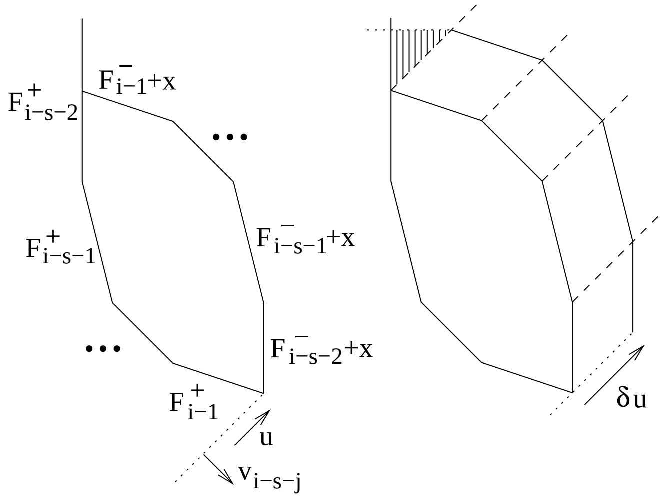

Figure 3: The intersection of K with $K+x$ (left) and of K with $K+x+\delta u$ (right).

Afterwards, let u be the direction of the edge which belongs to the sequence containing F^{-}_{i-s-2} and that precedes F^{-}_{i-s-2} (u is parallel to $v_{i-s-2,p}$ or $v_{i-s-2,f}$). Let j, $j>2$, be such that u is orthogonal to v_{i-s-j}. Let us orient u in such a way that $g_K(x+\delta u) > g_K(x)$, where δ is a positive number such that

$$\begin{aligned}\delta \quad \leq \quad & \min_{k=i,\ldots,r-1}\left\{\lambda_1(F^{-}_{k})\frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_{i-s-2}v_{i-s-j}})|},\right.\\ & [\lambda_1(F^{+}_{i-s-2})-\lambda_1(F^{-}_{i-s-2})]\frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_k v_{i-s-j}})|},\\ & \left.\lambda_1(F^{-}_{i-s-j})\right\}.\end{aligned}$$

The number

$$\begin{aligned} & g_K(x+\delta u) - g_K(x) + \\ & \qquad - \sum_{k=0}^{s+1} \lambda_1(F^{-}_{i-k-1})|\cos(\widehat{u v_{i-k-1}})|\delta \end{aligned}$$

is equal to the area of a triangle (see the shadowed triangle in Fig. 3 on the right).

About this triangle we know that the length of the edge parallel to u is δ, and we also know the direction of an edge adjacent to the aforementioned edge, since it is orthogonal to v_{i-s-2}. By these data and the area of the triangle, we can determine also the direction of the other edge. If this direction is orthogonal to v_i, then add F_i^- to the sequence which contains F_{i-s-2}^-; otherwise add F_i^- to the sequence which contains F_{i-s-2}^+. Proceed with the $(i+1)$-th step.

ac) *The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of different length, (F_{i-1}^-, F_{i-1}^+).*

In this case, if one between $v_{i-1,p}$ and $v_{i-1,f}$ is orthogonal to v_i, then add F_i^- to the sequence which contains F_{i-1}^-; otherwise add F_i^- to the sequence which contains F_{i-1}^+. Proceed with the $(i+1)$-th step.

b) *The vectors $\pm v_i$ are in correspondence to a pair of parallel edges of equal length, (F_i^-, F_i^+).*

In this case add F_i^- to S_1 and add F_i^+ to S_2. Proceed with the $(i+1)$-th step.

c) *The vectors $\pm v_i$ are in correspondence to a pair of parallel edges of different length, (F_i^-, F_i^+).*

Consider $\pm v_{i-1}$.

ca) *The vectors $\pm v_{i-1}$ are in correspondence to a single edge F_{i-1}^-.*

In this case, if one between $v_{i,p}$ and $v_{i,f}$ is orthogonal to v_{i-1}, then add F_i^- to the sequence which contains F_{i-1}^- and add F_i^+ to the other sequence. Otherwise add F_i^+ to the sequence which contains F_{i-1}^- and add F_i^- to the other sequence. Proceed with the $(i+1)$-th step.

cb) *The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of equal length, (F_{i-1}^-, F_{i-1}^+).*

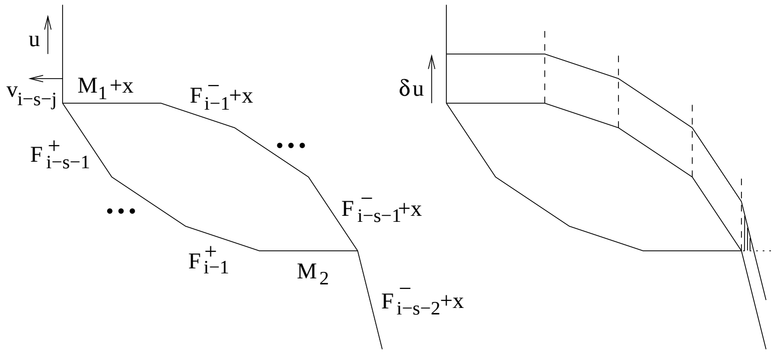

Figure 4: The intersection of K with $K + x$ (left) and of K with $K + x + \delta u$ (right).

In this case we have to decide if F_i^- has to be added to S_1 and F_i^+ to S_2 or vice versa. In any case both S_1^i and S_2^i contain a segment orthogonal to v_i and of length $\lambda_1(F_i^-)$; call these two segments M_1 and M_2. Add M_1 and M_2 to S_1 and S_2, respectively. Let s, $s \geqslant 0$, be the smallest index such that F^-_{i-1}, ..., F^-_{i-s-1} and F^+_{i-1}, ..., F^+_{i-s-1} are consecutive edges of pairs of parallel edges of equal length and $(F^-_{i-s-2}, F^+_{i-s-2})$ is not a pair of parallel edges of equal length.

cbs) *If* $\pm v_{i-s-2}$ *is in correspondence to a single edge* F^-_{i-s-2}, then consider the translation x which maps the second vertex of M_1 to the first vertex of the edge with outward normal $\pm v_{i-s-1}$ located in the sequence which does not contain M_1 (see Fig. 4 on the left). Compute $g_K(x)$.
Afterwards, consider the two edges which are adjacent and precede the edges F^-_{i-s-1} and F^+_{i-s-1}. One of them has outward normal $\pm v_{i-s-2}$, whereas the other one has outward normal $\pm v_{i-s-j}$, $j > 2$. Denote by u the unit vector orthogonal to v_{i-s-j} with orientation such that $g_K(x + \delta u) > g_K(x)$, where δ is a positive

number such that

$$\delta \leq \min_{k=i+1,\dots,r-1} \left\{ \lambda_1(F_k^-) \frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_{i-s-2} v_{i-s-j}})|}, \right.$$
$$\lambda_1(F_{i-s-2}^-) \frac{|\sin(\widehat{v_i v_{i-s-2}})|}{|\sin(\widehat{v_i v_{i-s-j}})|},$$
$$\lambda_1(F_{i-s-2}^-) \frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_i v_{i-s-j}})|},$$
$$[\lambda_1(F_{i-s-j}^-), \lambda_1(F_i^+) - \lambda_1(F_i^-)] \times$$
$$\left. \times \frac{|\sin(\widehat{v_i v_{i-s-2}})|}{|\sin(\widehat{v_{i-s-2} v_{i-s-j}})|} \right\}.$$

The number

$$g_K(x+\delta u) - g_K(x) - \lambda_1(F_i^-)|\cos(\widehat{uv_i})|\delta +$$
$$- \sum_{k=0}^{s} \lambda_1(F_{i-k-1}^-)|\cos(\widehat{uv_{i-k-1}})|\delta$$

is equal to the area of a triangle (see the shadowed triangle in Fig. 4 on the right).
About this triangle we know that the length of the edge parallel to u is δ, and we also know the direction of an edge adjacent to the aforementioned edge, since it is orthogonal to v_{i-s-2}. By these data and the area of the triangle, we can determine also the direction of the other edge. If this direction is orthogonal to v_i, then remove M_1 and M_2 and add F_i^- to the sequence which contains F_{i-s-2}^- and add F_i^+ to the sequence which does not contain F_{i-s-2}^-; otherwise remove M_1 and M_2 and add F_i^- to the sequence which does not contain F_{i-s-2}^- and add F_i^+ to the sequence which contains F_{i-s-2}^+. Proceed with the $(i+1)$-th step.

cbp) *If* $\pm v_{i-s-2}$ *is in correspondence to a pair of parallel edges of different length,* $(F_{i-s-2}^-, F_{i-s-2}^+)$, then consider the translation x which maps the first vertex of F_{i-s-2}^- to the second vertex of the segment M_i, $i = 1, 2$, located in the sequence which does not contain F_{i-s-2}^- (see Fig. 5 on the left). Compute $g_K(x)$.

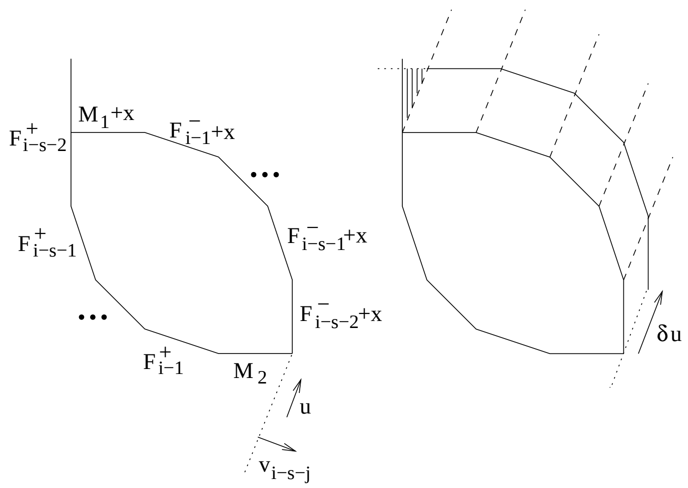

Figure 5: The intersection of K with $K+x$ (left) and of K with $K+x+\delta u$ (right).

Afterwards, let u be the direction of the edge which belongs to the sequence containing F^-_{i-s-2} and that precedes F^-_{i-s-2} (u is parallel to $v_{i-s-2,p}$ or $v_{i-s-2,f}$). Let j, $j>2$, be such that u is orthogonal to v_{i-s-j}. Let us orient u in such a way that $g_K(x+\delta u) > g_K(x)$, where δ is a positive number such that

$$\begin{aligned}\delta \quad \leq \quad \min_{k=i+1,\dots,r-1} \Bigg\{ & \lambda_1(F^-_k)) \frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_{i-s-2} v_{i-s-j}})|}, \\ & \lambda_1(F^-_{i-s-j}), [\lambda_1(F^+_{i-s-2}) - \lambda_1(F^-_{i-s-2})] \times \\ & \times \frac{|\sin(\widehat{v_k v_{i-s-2}})|}{|\sin(\widehat{v_k v_{i-s-j}})|}, \\ & [\lambda_1(F^+_{i-s-2}) - \lambda_1(F^-_{i-s-2})] \frac{|\sin(\widehat{v_i v_{i-s-2}})|}{|\sin(\widehat{v_i v_{i-s-j}})|}, \\ & [\lambda_1(F^+_i) - \lambda_1(F^-_i)] \frac{|\sin(\widehat{v_i v_{i-s-2}})|}{|\sin(\widehat{v_{i-s-2} v_{i-s-j}})|} \Bigg\}.\end{aligned}$$

The number

$$g_K(x+\delta u) - g_K(x) - \lambda_1(F^-_i))|\cos(\widehat{u v_i})|\delta +$$

$$-\sum_{k=0}^{s+1} \lambda_1(F^-_{i-k-1})|\cos(\widehat{uv_{i-k-1}})|\delta$$

is equal to the area of a triangle (see the shadowed triangle in Fig. 5 on the right).

About this triangle we know that the length of the edge parallel to u is δ, and we also know the direction of an edge adjacent to the aforementioned edge, since it is orthogonal to v_{i-s-2}. By these data and the area of the triangle, we can determine also the direction of the other edge. If this direction is orthogonal to v_i, then remove M_1 and M_2 and add F^-_i to the sequence which contains F^+_{i-s-2} and add F^+_i to the sequence which contains F^-_{i-s-2}; otherwise remove M_1 and M_2 and add F^-_i to the sequence which contains F^-_{i-s-2} and add F^+_i to the sequence which contains F^+_{i-s-2}. Proceed with the $(i+1)$-th step.

cc) *The vectors $\pm v_{i-1}$ are in correspondence to a pair of parallel edges of different length, (F^-_{i-1}, F^+_{i-1}).*

In this case we have to decide if F^-_i has to be added to S_1 and F^+_i to S_2 or vice versa. In any case both S^i_1 and S^i_2 contain a segment orthogonal to v_i and of length $\lambda_1(F^-_i)$; call these two segments M_1 and M_2. Add M_1 and M_2 to S_1 and S_2, respectively. Consider the translation x which maps the first vertex of F^-_{i-1} to the second vertex of the segment M_i, $i = 1, 2$, located in the sequence which does not contain F^-_{i-1} (see Fig. 6 on the left). Compute $g_K(x)$. Afterwards, let u be the direction of the edge which belongs to the sequence containing F^-_{i-1} and that precedes F^-_{i-1} (u is parallel to $v_{i-1,p}$ or $v_{i-1,f}$). Let j, $j > 2$, be such that u is orthogonal to v_{i-j}. Let us orient u in such a way that $g_K(x+\delta u) > g_K(x)$, where δ is a positive number such that

$$\begin{aligned}\delta \quad \leq \quad & \min_{k=i+1,\dots,r-1} \left\{ [\lambda_1(F^+_i) - \lambda_1(F^-_i)] \frac{|\sin(\widehat{v_i v_{i-1}})|}{|\sin(\widehat{v_{i-1} v_{i-j}})|}, \right. \\ & \lambda_1(F^-_k)) \frac{|\sin(\widehat{v_k v_{i-1}})|}{|\sin(\widehat{v_{i-1} v_{i-j}})|},\end{aligned}$$

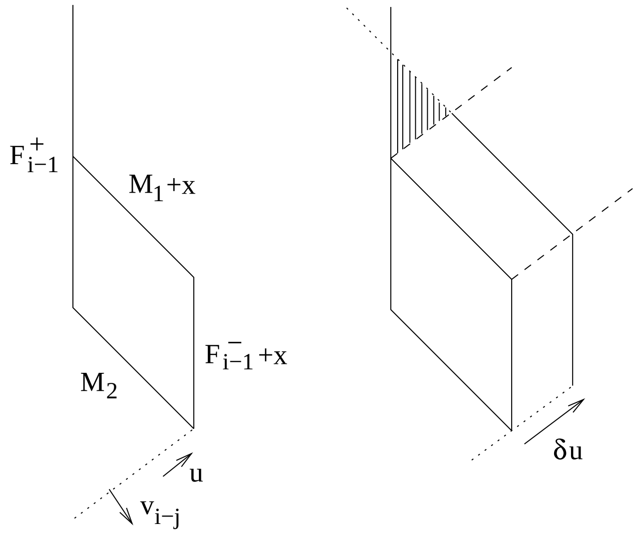

Figure 6: The intersection of K with $K+x$ (left) and of K with $K+x+\delta u$ (right).

$$[\lambda_1(F_{i-1}^+) - \lambda_1(F_{i-1}^-)]\frac{|\sin(\widehat{v_k v_{i-1}})|}{|\sin(\widehat{v_k v_{i-j}})|},$$
$$\lambda_1(F_{i-j}^-), [\lambda_1(F_{i-1}^+) - \lambda_1(F_{i-1}^-)]\frac{|\sin(\widehat{v_i v_{i-1}})|}{|\sin(\widehat{v_i v_{i-j}})|}\Bigg\}.$$

The number

$$g_K(x+\delta u) - g_K(x) - \lambda_1(F_{i-1}^-)|\cos(\widehat{u v_{i-1}})|\delta +$$
$$-\lambda_1(F_i^-)|\cos(\widehat{u v_i})|\delta$$

is equal to the area of a triangle (see the shadowed triangle in Fig. 6 on the right).

About this triangle we know that the length of the edge parallel to u is δ, and we also know the direction of an edge adjacent to the aforementioned edge, since it is orthogonal to v_{i-1}. By these data and the area of the triangle, we can determine also the direction of the other edge. If this direction is orthogonal to v_i, then remove M_1 and M_2 and add F_i^- to the sequence which contains F_{i-1}^+ and add F_i^+

to the sequence which contains F_{i-1}^-; otherwise remove M_1 and M_2 and add F_i^- to the sequence which contains F_{i-1}^- and add F_i^+ to the sequence which contains F_{i-1}^+. Proceed with the $(i+1)$-th step.

After concluding the $(r-1)$-th step, the union of S_1^{r-1} and S_2^{r-1} coincides with the boundary of K and the algorithm stops.

References

[1] R. J. Adler and R. Pyke, *Problem 91-3*, The Inst. of Math. Stat. Bull. **20** (1991), 409.
[2] G. Averkov and G. Bianchi, *Problem 91-3*, in preparation.
[3] G. Averkov and G. Bianchi, *Retrieving convex bodies from restricted covariogram functions*, preprint.
[4] M. Baake and U. Grimm, *Homometric model sets and window covariograms*, Zeitschrift für Kristallographie **222**, no. 2.
[5] G. Bianchi, *Matheron's conjecture for the covariogram problem*, J. London Math. **71** (2005), 203–220.
[6] G. Bianchi, F. Segala, and A. Volčič, *The solution of the covariogram problem for plane C_+^2 bodies*, J. Diff. Geom. **60** (2002), 177–198.
[7] A. Cabo and A. Baddeley, *Estimation of mean particle volume using the set covariance function*, Adv. Appl. Prob. (SGSA) **35** (2003), 27–46.
[8] G. Matheron, *Le covariogramme gomtrique des compacts convexes de $\mathbb{R}^2$*, Tech. Report 2, Centre de Gostatistique, Ecole des Mines de Paris.
[9] G. Matheron, *Random sets and integral geometry*, Wiley, New York, U.S.A. (1975).
[10] W. Nagel, *Orientation-dependent chord lenght distributions characterize convex polygons*, J. Appl. Prob. **30** (1993), 730–736.
[11] J. L. C. Sanz, *Mathematical considerations for the problem of Fourier transform phase retrieval from magnitude*, SIAM J. Appl. Math. (SGSA) **45** (1985), 651–664.
[12] M. Schmitt, *On two inverse problems in mathematical morphology*, Mathematical Morphology in Image Processing (Dougherty, ed.), Dekker, New York, U.S.A. (1993), pp. 151–169.
[13] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, U.K. (1993).

[14] J. Serra, *Image analysis and mathematical morphology*, Academic Press, London, U.K. (1984).

Received January 10, 2008.