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On Polynomial Approximation of Entire Functions with Index-Pair (p, q)

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Summary. - *In this paper we have studied interpolation errors for functions in* $\mathbf{C}(E)$*, the normed algebra of analytic functions on a compact set E. The lower* (p,q)*-order and generalized lower* (p,q)*-type have been characterized in terms of these approximation errors. Finally, we have derived necessary conditions for* $f \in \mathbf{C}(E)$ *to be extended to an entire function of perfectly regular* (p,q)*-growth with respect to a proximate order.*

1. Introduction

Let E be a compact set in complex plane and $\xi^{(n)} = (\xi_{n0}, \xi_{n1}, \ldots, \xi_{nn})$ be a system of $n+1$ points of the set E such that

$$V(\xi^{(n)}) = \prod_{0 \le j < k \le n} |\xi_{nj} - \xi_{nk}|,$$

$$\Delta^{(j)}(\xi^{(n)}) = \prod_{\substack{k=0 \\ k \neq j}}^{n} |\xi_{nj} - \xi_{nk}|, \quad j = 0, 1, \ldots, n.$$

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Again, let $\eta^{(n)} = (\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn})$ be a system of $n+1$ points in E such that

$$V_n \equiv V(\eta^{(n)}) = \sup_{\xi^{(n)} \subset E} V(\xi^{(n)}),$$

$$\Delta^0(\eta^{(n)}) \leq \Delta^{(j)}(\eta^{(n)}), \quad j = 0, 1, \ldots, n.$$

Such a system always exists and is called the n-th *extremal system* of E. The polynomials

$$L^{(j)}(z, \eta^{(n)}) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \left(\frac{z - \eta_{nk}}{\eta_{nj} - \eta_{nk}} \right), \quad j = 0, 1, \ldots, n$$

are called the Lagrange extremal polynomials and the limit $d \equiv d(E) = \lim_{n \to \infty} V_n^{2/n(n+1)}$ is called the *transfinite diameter* of E.

Let $\mathbf{C}(E)$ denote the algebra of analytic function on the set E. Let us define the approximation errors as follows:

$$\mu_{n,1}(f) \equiv \mu_{n,1}(f; E) = \inf_{g \in \pi_n} \|f - g\|,$$

where $\|\cdot\|$ is the sup norm and π_n denotes the set of all polynomials of degree $\leq n$. For the Lagrange interpolating polynomial

$$L_n(z) = \sum_{j=0}^{n} L^{(j)}(z, \eta^{(n)}) f(\eta_{nj}), \quad n \in \mathbb{N}$$

we also define

$$\mu_{n,2}(f) \equiv \mu_{n,2}(f; E) = \|L_n - L_{n-1}\|, \quad n \geq 2,$$

$$\mu_{n,3}(f) \equiv \mu_{n,3}(f; E) = \|L_n - f\|, \quad n \geq 0.$$

Reddy [10] connected classical order and type with polynomial approximation error of an entire function which is an extension of a continuous function on $[-1; 1]$. Juneja [2] extended these results for lower order and Mass [8] studied for the lower type. Contemporarily, Rice [11] and Winiarski [15] studied order and type for different approximation errors of a continuous function on the arbitrary domains. These results fail to compare the approximation errors of those entire functions which have same order but their types are infinity. To include this important class of functions we utilize the concept of proximate order (see [9]) and moreover, their result are extended to (p,q)-scale introduced by Juneja *et al.* ([3], [4]). First we recall t he (p,q)-scale, $p \geq q \geq 1$. For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, set $M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)|$, $M(r)$ is called the maximum modulus of $f(z)$. Let us define

$$P_\chi(L) = \begin{cases} L & p > q \\ \chi + L & p = q = 2 \\ \max(1, L) & p = q \geq 3 \\ \infty & p = q = \infty \end{cases}$$

$$\gamma = \begin{cases} (\rho-1)^\rho/\rho^\rho & (p,q) = (2,2) \\ 1/e\rho & (p,q) = (2,1) \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

DEFINITION 1.1. *An entire function $f(z)$ is said to be of (p,q)-order ρ and lower (p,q)-order λ if it is of index-pair (p,q) such that*

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M(r)}{\log^{[q]} r} = \begin{matrix} \rho \\ \lambda \end{matrix}$$

and the function $f(z)$ having (p,q)-order $\rho (b < \rho(p,q) < \infty)$ is said to be of (p,q)-type T and lower (p,q)-type t if

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r)}{\log^{[q-1]} r^\rho} = \begin{matrix} T \\ t \end{matrix}$$

where $b = 1$ if $p = q$ and $b = 0$ if $p > q$.

Recently, Nadan *et al.* [9] has extended the idea of proximate order to entire functions of (p,q) growth. A positive function $\rho(r)$ is said to be a proximate order if

1. $\rho(r) \to r$ as $r \to \infty, b < \rho < \infty$,
2. $\Lambda_{[q]}(r)\rho\prime(r) \to 0$ as $r \to \infty$,

where $\Lambda_{[q]}(r) = \prod_{k=0}^{q} \log^{[k]}(r)$ and $\rho\prime(r)$ denotes the derivative or $\rho(r)$. It is known that $(\ln^{[q]} r)^{\rho(r)-A}$ is a monotonically increasing function of r for $r > r_0$, where $A = 1$ if $(p,q) = (2,2)$ and $A = 0$ otherwise. Hence we can define the function $\phi(x)$ for $x > x_0$ to be the unique solution of the equation,

$$x = (\ln^{[q]} r)^{\rho(r)-A} \Leftrightarrow \phi(x) = \ln^{[q]} r \quad \text{for} \quad r > r_0. \tag{2}$$

DEFINITION 1.2. *A positive function* $\rho(r)$ *defined on* $[r_0, \infty)$, *where* $r_0 > \exp^{[q-1]} 1$, *is said to be a proximate order of an entire function with index-pair* (p,q) *if*

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M(r)}{\log^{[q-1]} r^{\rho(r)}} = \begin{matrix} T^* \\ t^* \end{matrix} .$$

If the quantity t^* is different from zero and infinite then $\rho(r)$ is said to be the proximate order of a given entire function $f(z)$ and t^* as its generalized lower (p,q)-type. Clearly, proximate order and corresponding generalized lower (p,q)-type of an entire function are not uniquely determined [1].

DEFINITION 1.3. *An entire function with index pair* (p,q) *is said to be of regular* (p,q)*-growth if* $b < \lambda = \rho < \infty$, *and further, it is of perfectly regular* (p,q)*-growth with respaect to a proximate order* $\rho(r)$ *if* $0 < t^* = T^* < \infty$.

Let E_r be the curve $E_r = \{z \in \mathbb{C} : |\psi(z)|d = r\}$, where $\psi(z)$ is holomorphic and maps the unbounded component of the complement of E on $|\psi(z)| > 1$ such that $\psi(\infty) = \infty$ and $\psi\prime(\infty) > 0$. Also, we set $\bar{M}(r) = \sup_{z\in E_r} |f(z)|$, for $r > 1$.

2. Auxiliary Results

Let us now prove some auxiliary results to be used in the sequel:

LEMMA 2.1. *If* $f(z)$ *is an entire function of* (p,q)*-order* ρ *and lower* (p,q)*-order* λ *then*

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \bar{M}(r)}{\log^{[q]} r} = \begin{matrix} \rho \\ \lambda \end{matrix}$$

and, for $\rho(b < \rho(p,q) < \infty)$, T^* *and* t^* *are given by*

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \bar{M}(r)}{\log^{[q-1]} r^{\rho}} = \begin{matrix} T^* \\ t^* \end{matrix}$$

For a proof we refer to [7].

Lemma 2.2. *If a function f is defined and bounded an a compact set E, then*

$$\mu_{n,1}(f) \leq \|f - L_n\| \leq (n+2)\mu_{n,1}(f),$$
$$\|L_n - L_{n-1}\| \leq 2(n+2)\mu_{n-1,1}(f), \quad n = 2, 3, \ldots$$

The proof is illustrated in Winiarski [15].

Proposition 2.3. *Let $f \in \mathbf{C}(E)$. Then f can be extended to an entire function if and only if*

$$\mu_{n,i}^{1/n}(f) \to 0 \quad as \quad n \to \infty, \quad i = 1, 2, 3.$$

This is a direct consequence of Lemma (2.1), Eq. (4.5) of Winiarski [15] and an inequality due to Walsh ([14], p.77).

Proposition 2.4. *For every $f \in \mathbf{C}(E)$ and $\mu_{n,i}(f)$, $i = 1, 2, 3$, there exist an entire function $g_i(z) = \sum_{n=0}^{\infty} \mu_{n,i}(f) z^{n+1}$ such that*

$$\bar{M}(r) \leq a_0 + 2g_i(r/d)$$

where d is the transfinite diameter of E.

Proof. Define the function

$$\bar{f}(z) = \pi_0 + \sum_{n=0}^{\infty} (\pi_{n+1}(z) - \pi_n(z)). \tag{3}$$

Obviously, $\bar{f}(z) = f(z)$ for all $z \in E$. We prove that $\bar{f}(z) = f(z)$ in the whole complex plane. For this is enough to show that this series converges uniformly on every compact subset of the complex plane, since

$$\begin{aligned} |\pi_{n+1}(z) - \pi_n(z)| &\leq \|\pi_{n+1} - \pi_n\| \quad z \in E \\ &\leq \mu_{n+1,1}(f) + \mu_{n,1}(f) \\ &\leq 2\mu_{n,1}(f), \end{aligned}$$

and using Walsh inequality [14], we have

$$|\pi_{n+1}(z) - \pi_n(z)| \leq 2\mu_{n,1}(f) \left(\frac{r}{d}\right)^{n+1}, \quad z \in E_r.$$

Thus,

$$\begin{aligned}|\bar{f}(z)| &= |\pi_0| + \sum_{n=0}^{\infty} |\pi_{n+1}(z) - \pi_n(z)| \\ &\leq a_0 + 2\sum_{n=0}^{\infty} \mu_{n,1}(f) \left(\frac{r}{d}\right)^{n+1}, \quad z \in E_r. \qquad (4)\end{aligned}$$

The last series converges for every r, and therefore the series on the right of (3) converges uniformly on every compact subset of $\mathbb{C}$ and so $\bar{f}(z) = f(z)$. Construct the function

$$g_i(z) = \sum_{n=0}^{\infty} \mu_{n,1}(f) z^{n+1}.$$

Since $\lim_{n\to\infty} \mu_{n,i}^{1/n}(f) = 0$ by (2.3), it follows that each $g_i(z)$ is entire and further, (4) implies the desired inequality. □

3. Main results

THEOREM 3.1. *If $f \in \mathbf{C}(E)$ can be extended to an entire function with index-pair (p, q), lower (p, q)-order $\lambda(b < \lambda < \infty)$ and generalized lower (p, q)-type t^*, then for every $\mu_{n,i}(f)$, there exists an entire functions $g_i(z) = \sum_{n=0}^{\infty} \mu_{n,i}(f) z^{n+1}$ such that*

$$\lambda(f) = \lambda(g_i), \quad t^*(f) = \beta t^*(g_i), \qquad (5)$$

where $\beta = d^{-\rho}$ for $q = 1$, otherwise $\beta = 1$ and $i = 1, 2, 3$.

Proof. In view of Propositions (2.3) and (2.4),$\bar{f}(z) = f(z)$ in $\mathbb{C}$, and for each $\mu_{n,i}(f), g_i(z) = \sum_{n=0}^{\infty} \mu_{n,i}(f) z^{n+1}$ is an entire function. Winiarski [15] has proved that for every $\epsilon > 0$,

$$\mu_{n,3} \leq k\bar{M}(r)\left(\frac{de^{\epsilon}}{r}\right)^n, \qquad (6)$$

where k is a constant and $d > 0$ has its usual meaning. Using (6) in the expansion of $g_i(z)$ with $i = 3$ it is inferred that

$$\begin{aligned} g_3\left(\frac{r}{de^{2\epsilon}}\right) &= \sum_{n=0}^{\infty} \mu_{n,3}(f)\left(\frac{r}{de^{2\epsilon}}\right)^{n+1} \\ &\leq \frac{kr\bar{M}(r)}{de^{2\epsilon}} \sum_{n=0}^{\infty} \frac{1}{e^{n\epsilon}} = \frac{kr\bar{M}(r)}{de^{2\epsilon}(e^{\epsilon}-1)}, \end{aligned}$$

or

$$\ln g_3\left(\frac{r}{de^{2\epsilon}}\right) \leq O(1) + \log \bar{M}(r) + \ln r.$$

This inequality with Lemma (2.1) for $q = 1$ gives

$$\lambda(g_3) \leq \lambda(f), \quad t^*(g_3) \leq e^{2\epsilon\rho} d^{\rho} t^*(f),$$

and, for $q > 1$,

$$\lambda(g_3) \leq \lambda(f), \quad t^*(g_3) \leq t^*(f).$$

Since $\epsilon > 0$ is arbitrary, the inequalities are combined for all (p, q) to yeld

$$\lambda(g_3) \leq \lambda(f), \quad \beta t^*(g_3) \leq t^*(f). \tag{7}$$

Further, using the inequality $\bar{M}(r) \leq a_0 + 2g_i(r/d)$, note that for $q = 1$,

$$\lambda(f) \leq \lambda(g_i), \quad t^*(f) \leq d^{-\rho} t^*(g_i),$$

and for $q > 1$,

$$\lambda(f) \leq \lambda(g_i), \quad t^*(f) \leq t^*(g_i). \tag{8}$$

Combining these inequalities with (7), we have (5). Further, application of Lemma (2.2) makes this result valid for $i = 1$ and $i = 2$ also. □

THEOREM 3.2. *Let $f(z) \in \mathbf{C}(E)$. Then $f(z)$ can be extended to an entire function of lower (p, q)-order $\lambda(b < \lambda(p, q) < \infty)$ if and only if, for $(p, q) \neq (2, 2)$,*

$$\lambda = \max_{\{n_k\}} [P_{\chi}(\ell)], \quad \lambda = \max_{\{n_k\}} [P_{\chi}(\ell^*)], \tag{9}$$

where

$$\chi = \liminf_{k\to\infty} \frac{\ln n_{k-1}}{\ln n_k}, \quad \ell = \liminf_{k\to\infty} \frac{\ln^{[p-1]} n_{k-1}}{\ln^{[q]} \mu_{n_{k,i}}^{-1/n_k}},$$

$$\ell^* = \liminf_{k\to\infty} \frac{\ln^{[p-1]} n_{k-1}}{\ln^{[q-1]}\left(\frac{1}{n_k - n_{k-1}} \ln \frac{\mu_{n_{k-1},i}}{\mu_{n_k,i}}\right)}.$$

Also (9) *holds for* $(p,q) = (2,2)$ *provided* n_k *be the sequences of principal indices satisfying* $\ln n_{k-1} \approx \ln n_k$ *as* $k \to \infty$.

Proof. Propositions (2.3) and (2.4) reveal that $f \in \mathbf{C}(E)$ can be extended to an entire function if and only if $g_i(z)$ is an entire function. Moreover, by (5), $f(z)$ and $g_i(z)$ have the same lower (p,q)-order. Applying Theorem 2 by Juneja *et al.* [3] to the function $g_i(z) = \sum_{n=0}^{\infty} \mu_{n,i}(f) z^{n+1}$, Theorem (3.2) follows at once. □

Remark. For $E = [-1,1]$, $i = 3$ and (p,q), (9) includes a theorem by Singh [13] and a result by Massa [8]. Also, for $(p,q) = (2,2)$, (9) includes Theorem 5 by Reddy [10]. Moreover, (9) gives Theorems 1 and 2 by Juneja [2] for entire functions of Sato growth [12].

Theorem 3.3. *let* $f \in \mathbf{C}(E)$. *Then* $f(z)$ *can be extended to an entire function fo* (p,q)*-order* $\rho(b < \rho(p,q) < \infty)$ *and generalized lower* (p,q)*-type* $t^*(0 < t^*(p,q) < \infty)$ *if and only if*

$$t^* = \beta \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \left(\frac{\phi(\ln^{[p-2]} m_{k-1})}{\ln^{[q-1]} \mu_{m_k,i}^{-1/m_k}} \right)^{\rho} \right\}, \quad p \ge 3 \tag{10}$$

and further, if the sequence of pirncipal indices $\{n_k\}$ *satisfies* $n_{k-1} \simeq n_k$ *as* $k \to \infty$*, the for*

$$t^* = \gamma\beta \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \left(\frac{\phi(m_{k-1})}{\ln^{[A]} \mu_{m_k,i}^{-1/m_k}} \right)^{\rho - A} \right\}, \tag{11}$$

where maximum is taken over all increasing sequences of positive integers and β, γ *and* A *have been defined, respectively in* (5), (1) *and* (2).

Proof. To prove this theorem we apply Theorem 2 by Kasana *et al.* [6] to the function $g_i(z) = \sum_{n=0}^{\infty} \mu_{n,i}(f) z^{n+1}$ and the characterization of t^* in terms of $\mu_{n,i}(f)$ and the relation $t^* = \beta t^*(g_i)$ to conclude (10) and (11). □

COROLLARY 3.4. *Let $f \in \mathbf{C}(E)$. Then $f(z)$ is the restriction of an entire function having (p,q) order $\rho(b < \rho(p,q) < \infty)$ and lower (p,q)-type $t(0 < t(p,q) < \infty)$ if and only if*

$$t^* = \gamma\beta \max_{\{m_k\}} \left\{ \liminf_{k\to\infty} \frac{\ln^{[p-2]} m_{k-1}}{\left(\ln^{[q-1]} \mu_{m_k,i}^{-1/m_k}\right)^{\rho-A}} \right\}.$$

On the domain $E = [-1,1]$ and for approximation error $\mu_{n,3}$ this corollary also includes some results of Reddy [10] when $(p,q) = (2,1)$ or $(p,q) = (2,2)$.

Finally, we study the subsequences $\{n_{k,i}\}$ of n such that for $f \in \mathbf{C}(E)$ it satisfies

$$\mu_{n_{k-1},i} > \mu_{n_k,i}, \quad \mu_{n,i} = \mu_{n_{k-1},i} \quad \text{for} \quad n_{k-1,i} \le n < n_{k,i}. \tag{12}$$

The next theorem shows how this sequence influences the growth of an entire function in reference to its generalized (p,q)-type and generalized lower (p,q)-type. This also depicts the necessary condition for $f \in \mathbf{C}(E)$ which has an extension of perfectly regular (p,q)-growth with respect to a proximate order.

THEOREM 3.5. *Suppose $f \in \mathbf{C}(E)$ can be extended to an entire function having (p,q)-order $\rho(b < \rho(p,q) < \infty)$, generalized (p,q)-type T^* and generalized lower (p,q)-type t^*. Let $\{n_{k,i}\}$ be the sequence defined by (12). Then*

$$t^* \le T^* \liminf_{k\to\infty} \left(\frac{\phi(\ln^{[p-2]} n_{k-1,i})}{\phi(\ln^{[p-2]} n_{k,i})} \right)^{\rho}, \quad p \ge 3.$$

Further, if $\{m_{k,i}\}$ be the sequence of principal indices satisfying $m_{k-1,i} \simeq m_{k,i}$ as $k \to \infty$, then

$$t^* \le T^* \liminf_{k\to\infty} \left(\frac{\phi(n_{k-1,i})}{\phi(n_{k,i})} \right)^{\rho-A}.$$

Proof. Define a function $u_i(z)$ such that

$$\begin{aligned} \mu_i(z) &= \sum_{n=1}^{\infty} [\mu_{n-1,i}(f) - \mu_{n,i}(f)] z^n \\ &= \sum_{n=1}^{\infty} \alpha_{k,i}(f) z^{n_{k,i}}, \end{aligned}$$

where $\alpha_{k,i}(f) = \mu_{n_{k-1},i}(f) - \mu_{n_{k,i}}(f)$. Since the function $g_i(z) = \sum_{n=0}^{\infty} \mu_{n,i}(f) z^{n+1}$ has the same (p,q)-order as that of $f(z)$, it follows that $u_i(z)$ has also the same (p,q)-order. Consequently, by Theorem (3.1) the generalized (p,q)-type and generalized lower (p,q)-type of $u_i(z)$ are given by

$$T^*(f) = \beta T^*(u_i), \quad t^*(f) = \beta t^*(u_i).$$

Thus, using Theorem 1 by Kasana *et al.* [6] it can be shown that

$$T^*(f) = \gamma\beta \limsup_{k\to\infty} \left(\frac{\phi(\ln^{[p-2]} n_{k,i})}{\ln^{[q-1]} \alpha_{n_{k,i}}^{-1/n_{k,i}}} \right)^{\rho-A}$$

Considering the above formula and Theorem (3.3) we observe that for $(p,q) \neq (2,1)$ and $(p,q) \neq (2,2)$,

$$\begin{aligned} t^* &= \beta\gamma \max_{\{k_{m,i}\}} \left\{ \liminf_{m\to\infty} \left(\frac{\phi(\ln^{[p-2]} n_{k_{m-1},i})^{-1/n_{k_{m,i}}}}{\ln^{[q-1]} \alpha_{n_{m,i}}} \right)^{\rho} \right\} \\ &\leq \beta\gamma \max_{\{k_{m,i}\}} \left\{ \limsup_{m\to\infty} \left(\frac{\phi(\ln^{[p-2]} n_{k_{m},i})^{-1/n_{k_{m,i}}}}{\ln^{[q-1]} \alpha_{n_{m,i}}} \right)^{\rho} \right\} \times \\ &\qquad \max_{\{k_{m,i}\}} \left\{ \liminf_{m\to\infty} \left(\frac{\phi(\ln^{[p-2]} n_{k_{m-1},i})^{-1/n_{k_{m,i}}}}{\ln^{[q-1]} \alpha_{n_{m,i}}} \right)^{\rho} \right\} \\ &\leq T^* \liminf_{m\to\infty} \left(\frac{\phi(\ln^{[p-2]} n_{k_{m-1},i})^{-1/n_{k_{m,i}}}}{\ln^{[q-1]} \alpha_{n_{m,i}}} \right)^{\rho} \end{aligned}$$

For $p = 2$ and $q = 1$ or $q = 2$, let $\{m_{k,i}\}$ be the sequence of principal indices that $m_{k-1,i} \simeq m_{k,i}$ as $k \to \infty$, we have

$$t^* \leq T^* \liminf_{k\to\infty} \left(\frac{\phi(n_{k-1,i})}{\phi(n_{k,i})} \right)^{\rho-A}. \qquad \square$$

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