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Some Nonexistence Results for Systems of Nonlinear Partial Differential Inequalities

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Summary. - *We obtain nonexistence results for systems of stationary and evolutional partial differential inequalities that involve p-Laplacian and similar nonlinear operators as well as gradient nonlinearities. Our proofs are based on the nonlinear capacity method.*

1. Introduction

In this paper, we consider problems of existence of positive solutions for systems of quasilinear differential inequalities of different types.

Due both to theoretical reasons and to numerous practical applications (see [13], [14]), necessary conditions for existence of solutions to partial differential equations and inequalities, including quasilinear and higher order ones, got to be treated in quite a number of papers in the last decades. Up to our knowledge, most results in this direction deal with the class of radial solutions, or with those decaying at infinity at a certain rate (see, in particular [12], [15] for single equations and [3], [4] for reaction diffusion systems).

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A more general approach to these problems is due to E. Mitidieri and S. Pohozaev (see their papers [9]–[11] and the monography [8]). Their "nonlinear capacity" method consists in obtaining a priori estimates based on the weak formulation of the problem with a special choice of test functions, usually in the form $\chi(x) = \varphi_R(x)u^\gamma(x)$, where u is the eventual solution, φ_R a standard mollifier, the size of whose support depends on a parameter $R > 0$, and $\gamma \in \mathbb{R}$. The subsequent passage to a limit for $R \to \infty$ (in the case of unbounded domains) or $R \to 0$ (for bounded ones) yields a contradiction to assumed properties of the solution. In evolution problems, the test functions depend upon the temporal variable as well, but the general structure of the argument is similar.

Our method is based on appropriate modifications of this approach. However, for stationary systems with singular terms in bounded domains, we sometimes combine it with the strong maximum principle, similarly to our earlier papers [5] and [6]. We also cover systems with gradient terms, that were not investigated previously to the best of our knowledge.

This paper consists of five sections besides the introduction. In the second one we formulate basic assumptions and introduce test functions for subsequent use. Section 3 deals with systems of stationary differential inequalities with singular nonlinearities, but without gradient ones, which get added in Section 4. In Section 5 we pass on to evolution systems without gradient nonlinearities, and in Section 6, to those that do contain ones.

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2. Basic Definitions

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and let S be a closed (eventually empty) lower dimensional subset of $\overline{\Omega}$. We denote $\Omega \setminus S = \Omega'$. (Evidently, for $S = \emptyset$ or $S \subset \partial\Omega$, one has $\Omega' = \Omega$.) For $\varepsilon > 0$, we shall use the notation $S^\varepsilon = \{x \in \Omega' : \rho(x, S) \leq \varepsilon\}$.

Suppose that $A : \Omega' \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a Carathéodory function, and let

$$L_A u = \operatorname{div}(A(x, u, \nabla u)\nabla u)$$

for any $u \in W^{1,1}_{\text{loc}}(\Omega')$ such that $A(x, u, \nabla u) \in L^1_{\text{loc}}(\Omega')$.

In what follows we shall use the notation $p' = \frac{p}{p-1}$. Our terminology is derived from [1].

DEFINITION 2.1. *The function A and the corresponding elliptic operator L_A are called strongly-p-coercive (S-p-C) if, for some constants $c, C > 0$ and all $(x, u, \eta) \in \Omega' \times \mathbb{R}_+ \times \mathbb{R}^n$, there hold the inequalities*

$$c|\eta|^{p-2} \leq A(x, u, \eta) \leq C|\eta|^{p-2} \tag{1}$$

with some p, $1 < p < n$. The operator $-L_A$ is called strongly-p-anticoercive (S-p-A) in this case.

EXAMPLE 2.2. *The Laplace operator is strongly-2-coercive. In general, the p-Laplace operator with $A = |\eta|^{p-2}$ is strongly-p-coercive.*

To establish a priori estimates of the solution of an inequality, we put into its weak formulation parameter-dependent test functions of the form

$$\chi_\varepsilon(x) = \xi_\varepsilon^\lambda(x) \tag{2}$$

with $\lambda > 0$ large enough (which can be specified further according to the nature of the problem), where ξ_ε (and hence χ_ε itself) belong to $C^1_0(\Omega'; [0, 1])$. In case $S \subset \Omega$, where a model situation may be $S = \{0\}$ and $\rho(x) = |x|$, we shall use functions supported in a thin layer which surrounds S, but does not intersect with it:

$$\xi_\varepsilon(x) = \begin{cases} 0 & (x \in S^\varepsilon \cup (\Omega \setminus S^{4\varepsilon})), \\ \\ 1 & (x \in S^{3\varepsilon} \setminus S^{2\varepsilon}), \end{cases} \tag{3}$$

where $\varepsilon > 0$ is so small that $S^{4\varepsilon} \subset \Omega$.

If $S = \partial\Omega$, we introduce test functions equal to 1 in the most part of the domain Ω, with the exception of a thin layer near the boundary where they vanish:

$$\xi_\varepsilon(x) = \begin{cases} 1 & (x \in \Omega \setminus \partial\Omega^{2\varepsilon}), \\ \\ 0 & (x \in \partial\Omega^\varepsilon). \end{cases} \tag{4}$$

In both cases, we shall additionally assume that

$$|D^\beta \xi_\varepsilon(x)| \leq c\varepsilon^{-|\beta|} \ (x \in \Omega') \tag{5}$$

for each multi-index β that appears in the formulation of the problem or in the proof.

Finally, in case $\Omega = \Omega' = \mathbb{R}^n$ we shall take

$$\chi_R(x) = \xi_R^\lambda(x) \tag{6}$$

with $\lambda > 0$ large enough and $\xi_R \in C_0^1(\mathbb{R}^n; [0,1])$ such that

$$\xi_R(x) = \begin{cases} 1 & (x \in B_R(0)), \\ 0 & (x \in \mathbb{R}^n \setminus B_{2R}(0)) \end{cases} \tag{7}$$

and

$$|D^\beta \xi_R(x)| \leq cR^{-|\beta|} \ (x \in \mathbb{R}^n) \tag{8}$$

for each multi-index β.

Here and in the sequel, c denotes generic positive constants independent of x and u.

3. Stationary Systems

In this section, we consider the problem

$$\begin{cases} \operatorname{div}(A(x,u,\nabla u)\nabla u) \geq fv^{q_1} & \text{in } \Omega', \\ \operatorname{div}(B(x,v,\nabla v)\nabla v) \geq gu^{p_1} & \text{in } \Omega', \\ u, v \geq 0 \quad \text{in } \Omega' \end{cases} \tag{1}$$

assuming that A (or $-A$) is S-p-C and B (or $-B$) is S-q-C with constants $p, q > 1$ respectively.

Let $\gamma \in \mathbb{R}$. We consider the class of solutions

$$X_\gamma(\Omega') = \{(u,v) : \Omega' \to \mathbb{R}_+ \times \mathbb{R}_+, fv^{q_1}u^\gamma, fv^{q_1},$$

$$gu^{p_1}v^\gamma, gu^{p_1}, |\nabla u|^p u^{\gamma-1}, |\nabla v|^q v^{\gamma-1} \in L^1_{loc}(\Omega)\}.$$

We also define intersection sets

$$X_+(\Omega') = \bigcap_{\gamma>0} X_\gamma(\Omega') \text{ and } X_-(\Omega') = \bigcap_{\gamma<0} X_\gamma(\Omega').$$

Definition 3.1. *We shall say that a pair of nonnegative functions* $(u, v) \in W^{1,p}_{\text{loc}}(\Omega') \times W^{1,q}_{\text{loc}}(\Omega')$ *satisfies the system (1) in the weak (distributional) sense if*

$$A(x, u(x), \nabla u(x)),\ B(x, v(x), \nabla v(x)) \in L^1_{\text{loc}}(\Omega'),$$

$$L_A u,\ L_B v \in L^1_{\text{loc}}(\Omega'),$$

and for any nonnegative test function $\varphi \in C^1_0(\Omega')$ *one has*

$$\begin{aligned} -\int_{\Omega'} A(x, u, \nabla u)(\nabla u, \nabla\varphi)\, dx &\geq \int_{\Omega'} f v^{q_1} \varphi\, dx, \\ -\int_{\Omega'} B(x, v, \nabla v)(\nabla v, \nabla\varphi)\, dx &\geq \int_{\Omega'} g u^{p_1} \varphi\, dx, \end{aligned} \tag{2}$$

where all integrals are supposed to exist. If

$$\operatorname*{ess\,inf}_{x\in\Omega'} \max(u(x), v(x)) > 0,$$

then the solution is called strictly positive.

The results of this section are based on

Lemma 3.2. *Let* $A, B : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$. *Suppose that* A *is S-p-C (resp. S-p-A) and* B *is S-q-C (resp. S-q-A) with* $\min(p, q) > 1$ *and*

$$p - 1 < p_1,\ q - 1 < q_1. \tag{3}$$

Then all eventual solutions of (1) satisfy apriori estimates

$$\begin{aligned} &\left(\int_{\Omega'} f v^{q_1} \varphi\, dx\right)^{p_1 q_1 - (p-1)(q-1)} \leq c\mu^{p_1 q_1 - (p-1)(q-1)}(\operatorname{supp}|\nabla\varphi|)\cdot \\ &\cdot \max_{x\in\operatorname{supp}|\nabla\varphi|} \Big(|\nabla\varphi(x)|^{q_1(q(p-1)+pp_1)} f^{-(p-1)(q-1)}(x) g^{-(p-1)q_1}(x)\Big), \\ &\left(\int_{\Omega'} g u^{p_1} \varphi\, dx\right)^{p_1 q_1 - (p-1)(q-1)} \leq c\mu^{p_1 q_1 - (p-1)(q-1)}(\operatorname{supp}|\nabla\varphi|)\cdot \\ &\cdot \max_{x\in\operatorname{supp}|\nabla\varphi|} \Big(|\nabla\varphi(x)|^{p_1(p(q-1)+qq_1)} f^{-(q-1)p_1}(x) g^{-(p-1)(q-1)}(x)\Big), \end{aligned} \tag{4}$$

where φ *is any admissible test function.*

Proof. To prove this lemma, we adapt a method developed in [8]–[9] for quasilinear elliptic systems with $a, b \equiv 1$ in $\mathbb{R}^n$ (see, in particular, Theorem 8.1 of [8]). In case of anti-coercive systems and $S = \partial\Omega$, our result was proven in [5]. Here we assume that both operators are coercive, otherwise we take $\gamma < 0$ and test functions $(u+\delta)^\gamma\varphi$ and $(v+\delta)^\gamma\varphi$ with $\delta \to 0$.

Let $(u, v) \in X_\gamma(\Omega')$ be a solution to (1), and let $\varphi \in C_0^1(\Omega'; \mathbb{R}_+)$ be a standard cut–off function.

By multiplying the first and the second equation of (1) respectively by $u^\gamma\varphi$ and by $v^\gamma\varphi$ and integrating by parts we find

$$\int_{\Omega'} f v^{q_1} u^\gamma \varphi\, dx \le -\int_{\Omega'} A(x, u, \nabla u)(\gamma|\nabla u|^2 u^{\gamma-1}\varphi + (\nabla u, \nabla\varphi)u^\gamma)\, dx,$$

$$\int_{\Omega'} g u^{p_1} v^\gamma \varphi\, dx \le -\int_{\Omega'} B(x, v, \nabla v)(\gamma|\nabla v|^2 v^{\gamma-1}\varphi + (\nabla v, \nabla\varphi)v^\gamma)\, dx.$$

Using inequalities (1), we obtain

$$\int_{\Omega'} f v^{q_1} u^\gamma \varphi\, dx \le -c\left(\gamma\int_{\Omega'} |\nabla u|^p u^{\gamma-1}\varphi\, dx + \int_{\Omega'} |\nabla u|^{p-1}|\nabla\varphi| u^\gamma\, dx\right),$$

$$\int_{\Omega'} g u^{p_1} v^\gamma \varphi\, dx \le -c\left(\gamma\int_{\Omega'} |\nabla v|^q v^{\gamma-1}\varphi\, dx + \int_{\Omega'} |\nabla v|^{q-1}|\nabla\varphi| v^\gamma\, dx\right).$$

Further, by the Young inequality with parameter $\eta > 0$ we have

$$\int_{\Omega'} f v^{q_1} u^\gamma \varphi\, dx + c(|\gamma| - \eta)\int_{\Omega'} |\nabla u|^p u^{\gamma-1}\varphi\, dx \le c_\eta \int_{\Omega'} \frac{|\nabla\varphi|^p}{\varphi^{p-1}} u^{p+\gamma-1}\, dx, \tag{5}$$

$$\int_{\Omega'} g u^{p_1} v^\gamma \varphi\, dx + c(|\gamma| - \eta)\int_{\Omega'} |\nabla v|^q v^{\gamma-1}\varphi\, dx \le d_\eta \int_{\Omega'} \frac{|\nabla\varphi|^q}{\varphi^{q-1}} v^{\gamma+q-1}\, dx, \tag{6}$$

where the constants c_η, $d_\eta > 0$ depend only on the operators and γ, $\eta > 0$.

Next we test (1) with φ and obtain by the Hölder inequality

$$\int_{\Omega'} fv^{q_1}\varphi\,dx \leq \left(\int_{\Omega'} |\nabla u|^p u^{\gamma-1}\varphi\,dx\right)^{\frac{1}{p'}} \left(\int_{\Omega'} \frac{|\nabla\varphi|^p}{\varphi^{p-1}} u^{(1-\gamma)(p-1)}\,dx\right)^{\frac{1}{p}}, \tag{7}$$

$$\int_{\Omega'} gu^{p_1}\varphi\,dx \leq \left(\int_{\Omega'} |\nabla v|^q v^{\gamma-1}\varphi\,dx\right)^{\frac{1}{q'}} \left(\int_{\Omega'} \frac{|\nabla\varphi|^q}{\varphi^{q-1}} v^{(1-\gamma)(q-1)}\,dx\right)^{\frac{1}{q}}. \tag{8}$$

By using (5) and (6), the last estimates imply that

$$\int_{\Omega'} fv^{q_1}\varphi\,dx \leq D_\eta \left(\int_{\Omega'} \frac{|\nabla\varphi|^p}{\varphi^{p-1}} u^{p+\gamma-1}\,dx\right)^{\frac{1}{p'}} \cdot \left(\int_{\Omega'} \frac{|\nabla\varphi|^p}{\varphi^{p-1}} u^{(1-\gamma)(p-1)}\,dx\right)^{\frac{1}{p}}, \tag{9}$$

$$\int_{\Omega'} gu^{p_1}\varphi\,dx \leq E_\eta \left(\int_{\Omega'} \frac{|\nabla\varphi|^q}{\varphi^{q-1}} v^{\gamma+q-1}\,dx\right)^{\frac{1}{q'}} \cdot \left(\int_{\Omega'} \frac{|\nabla\varphi|^q}{\varphi^{q-1}} v^{(1-\gamma)(q-1)}\,dx\right)^{\frac{1}{q}}, \tag{10}$$

where D_η and $E_\eta > 0$ depend only on the operators and γ, $\eta > 0$.

Now we apply the Hölder inequality with parameters a, a' to the first integral on the right–hand side of (9) and we get

$$\left(\int_{\Omega'} \frac{|\nabla\varphi|^p}{\varphi^{p-1}} u^{p+\gamma-1}\,dx\right)^{\frac{1}{p'}} \leq$$

$$\leq \left(\int_{\operatorname{supp}|\nabla\varphi|} gu^{(p+\gamma-1)a}\varphi\,dx\right)^{\frac{1}{p'a}} \left(\int_{\operatorname{supp}|\nabla\varphi|} g^{-\frac{a'}{a}} \frac{|\nabla\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{1}{p'a'}}, \tag{11}$$

where $\dfrac{1}{a}+\dfrac{1}{a'}=1$. By choosing the parameter a so that $(p+\gamma-1)a=p_1$ from (9) and (11) we have

$$\int_{\Omega'} f v^{q_1}\varphi\,dx \le D_\eta\left(\int_{\mathrm{supp}|\nabla\varphi|} g u^{p_1}\varphi\,dx\right)^{\frac{1}{p'a}}\cdot \cdot\left(\int_{\mathrm{supp}|\nabla\varphi|} g^{-\frac{a'}{a}}\frac{|\nabla\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{1}{p'a'}}\cdot \cdot\left(\int_{\mathrm{supp}|\nabla\varphi|} \frac{|\nabla\varphi|^{p}}{\varphi^{p-1}}u^{(1-\gamma)(p-1)}\,dx\right)^{\frac{1}{p}}. \quad (12)$$

By repeating this procedure with parameters $y, y'>1$ with the last integral of (12) we obtain

$$\left(\int_{\Omega'} \frac{|\nabla\varphi|^{p}}{\varphi^{p-1}}u^{(1-\gamma)(p-1)}\,dx\right)^{\frac{1}{p}} \le \le \left(\int\limits_{\mathrm{supp}|\nabla\varphi|} g u^{(1-\gamma)(p-1)y}\varphi\,dx\right)^{\frac{1}{py}}\left(\int\limits_{\mathrm{supp}|\nabla\varphi|} g^{-\frac{y'}{y}}\frac{|\nabla\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{1}{py'}}, \quad (13)$$

where $\dfrac{1}{y}+\dfrac{1}{y'}=1$. By choosing y so that $(1-\gamma)(p-1)y=p_1$ in (13) and combining it with (12) we deduce that

$$\int_{\Omega'} f v^{q_1}\varphi\,dx \le D_\eta\left(\int_{\mathrm{supp}|\nabla\varphi|} g u^{p_1}\varphi\,dx\right)^{\frac{1}{p'a}}\cdot \cdot\left(\int_{\mathrm{supp}|\nabla\varphi|} g u^{p_1}\varphi\,dx\right)^{\frac{1}{py}}\left(\int_{\mathrm{supp}|\nabla\varphi|} g^{-\frac{a'}{a}}\frac{|\nabla\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{1}{p'a'}}\cdot \cdot\left(\int_{\mathrm{supp}|\nabla\varphi|} g^{-\frac{y'}{y}}\frac{|\nabla\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{1}{py'}}$$

that is,

$$\int_{\Omega'} f v^{q_1}\varphi\,dx \le D_\eta \left(\int_{\mathrm{supp}|\nabla\varphi|} g u^{p_1}\varphi\,dx \right)^{\frac{1}{p'a}+\frac{1}{py}} \cdot$$
$$\cdot \left(\int_{\mathrm{supp}|\nabla\varphi|} g^{-\frac{a'}{a}} \frac{|\nabla\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx \right)^{\frac{1}{p'a'}} \cdot \left(\int_{\mathrm{supp}|\nabla\varphi|} g^{-\frac{y'}{y}} \frac{|\nabla\varphi|^{py'}}{\varphi^{py'-1}}\,dx \right)^{\frac{1}{py'}}, \tag{14}$$

where we have chosen the parameters a, y so that

$$\begin{cases} \dfrac{1}{y}+\dfrac{1}{y'}=1, \quad (1-\gamma)(p-1)y=p_1, \\[2ex] \dfrac{1}{a}+\dfrac{1}{a'}=1, \quad (\gamma+p-1)a=p_1. \end{cases} \tag{15}$$

We observe that this choice of a and y is admissible by our assumption (3) provided that $|\gamma|$ is chosen sufficiently small. Introducing new parameters b and z such that

$$\begin{cases} \dfrac{1}{z}+\dfrac{1}{z'}=1, \quad (1-\gamma)(q-1)z=q_1, \\[2ex] \dfrac{1}{b}+\dfrac{1}{b'}=1, \quad (\gamma+q-1)b=q_1 \end{cases} \tag{16}$$

and estimating now the right–hand side of (10), we obtain

$$\int_{\Omega'} g u^{p_1}\varphi\,dx \le E_\eta \left(\int_{\mathrm{supp}|\nabla\varphi|} f v^{q_1}\varphi\,dx \right)^{\frac{1}{q'b}+\frac{1}{qz}} \cdot$$
$$\cdot \left(\int_{\mathrm{supp}|\nabla\varphi|} f^{-\frac{b'}{b}} \frac{|\nabla\varphi|^{qb'}}{\varphi^{qb'-1}}\,dx \right)^{\frac{1}{q'b'}} \cdot \left(\int_{\mathrm{supp}|\nabla\varphi|} f^{-\frac{z'}{z}} \frac{|\nabla\varphi|^{qz'}}{\varphi^{qz'-1}}\,dx \right)^{\frac{1}{qz'}}. \tag{17}$$

Combining (14) and (17), we get

$$\left(\int_{\Omega'} f v^{q_1}\varphi\,dx\right)^{1-m_1m_2} \le D_\eta E_\eta^{m_2}\left(\int_{\operatorname{supp}|\nabla\varphi|} f^{-\frac{b'}{b}}\frac{|\nabla\varphi|^{qb'}}{\varphi^{qb'-1}}\,dx\right)^{\frac{m_2}{q'b'}}\cdot$$
$$\cdot\left(\int_{\operatorname{supp}|\nabla\varphi|} f^{-\frac{z'}{z}}\frac{|\nabla\varphi|^{qz'}}{\varphi^{qz'-1}}\,dx\right)^{\frac{m_2}{qz'}}\cdot\left(\int_{\operatorname{supp}|\nabla\varphi|} g^{-\frac{a'}{a}}\frac{|\nabla\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{1}{p'a'}}\cdot$$
$$\cdot\left(\int_{\operatorname{supp}|\nabla\varphi|} g^{-\frac{y'}{y}}\frac{|\nabla\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{1}{py'}}, \tag{18}$$

$$\left(\int_{\Omega'} g u^{p_1}\varphi\,dx\right)^{1-m_1m_2} \le E_\eta D_\eta^{m_1}\left(\int_{\operatorname{supp}|\nabla\varphi|} g^{-\frac{a'}{a}}(x)\frac{|\nabla\varphi|^{pa'}}{\varphi^{pa'-1}}\,dx\right)^{\frac{m_1}{p'a'}}\cdot$$
$$\left(\int_{\operatorname{supp}|\nabla\varphi|} g^{-\frac{y'}{y}}(x)\frac{|\nabla\varphi|^{py'}}{\varphi^{py'-1}}\,dx\right)^{\frac{m_1}{py'}}\left(\int_{\operatorname{supp}|\nabla\varphi|} f^{-\frac{b'}{b}}(x)\frac{|\nabla\varphi|^{qb'}}{\varphi^{qb'-1}}\,dx\right)^{\frac{1}{q'b'}}\cdot$$
$$\left(\int_{\operatorname{supp}|\nabla\varphi|} f^{-\frac{z'}{z}}(x)\frac{|\nabla\varphi|^{qz'}}{\varphi^{qz'-1}}\,dx\right)^{\frac{1}{qz'}}, \tag{19}$$

and finally

$$\left(\int_{\Omega'} f v^{q_1}\varphi\,dx\right)^{1-m_1m_2} \le D_\eta E_\eta^{m_2}\mu(\operatorname{supp}|\nabla\varphi|)^{\frac{m_2}{q'b'}+\frac{m_2}{qz'}+\frac{1}{p'a'}+\frac{1}{py'}}\cdot$$
$$\cdot\max_{x\in\operatorname{supp}|\nabla\varphi|}\left(|\nabla\varphi(x)|^{m_2q+p} f^{-\frac{m_2}{q'b}-\frac{m_2}{qz}} g^{-\frac{1}{p'a}-\frac{1}{py}}\right), \tag{20}$$

$$\left(\int_{\Omega'} g u^{p_1}\varphi\,dx\right)^{1-m_1m_2} \le E_\eta D_\eta^{m_1}\mu(\text{supp}|\nabla\varphi|)^{\frac{m_1}{p'a'}+\frac{m_1}{py'}+\frac{1}{q'b'}+\frac{1}{qz'}}\cdot$$
$$\cdot \max_{x\in\text{supp}|\nabla\varphi|}\left(|\nabla\varphi(x)|^{m_1p+q} f^{-\frac{1}{q'b}-\frac{1}{qz}} g^{-\frac{m_1}{p'a}-\frac{m_1}{py}}\right), \tag{21}$$

where
$$m_1 := \frac{1}{q'b}+\frac{1}{qz}, \quad m_2 := \frac{1}{p'a}+\frac{1}{py}. \tag{22}$$

An easy computation, by taking into account of (15) and (16), gives the explicit values of m_1 and m_2, that is

$$m_1 = \frac{q-1}{q_1}, \quad m_2 = \frac{p-1}{p_1}. \tag{23}$$

Consequently from assumption (3) it follows that the exponent appearing on the left–hand side of (20)–(21) satisfies

$$1 - m_1m_2 = \frac{p_1q_1-(p-1)(q-1)}{p_1q_1} > 0.$$

Calculating the explicit values of the exponents on the right by (15)–(16), we obtain (4). □

Using Lemma 3.2, we can prove a series of nonexistence results for systems. Here we start with Case 2 (boundary singularity).

Theorem 3.3. *Let the assumptions of Lemma 3.2 hold. If* $f,\ g \in C(\Omega')$ *are non-negative functions such that*

$$f(x) \ge c\rho^{-\alpha}(x),\ \ g(x) \ge d\rho^{-\beta}(x) \tag{24}$$

in Ω *with* $\rho = \text{dist}(x,\partial\Omega)$ *for some* $c, d > 0$ *and* α, β *satisfying the assumption*

$$\max((\alpha-1)(q-1)+(\beta-p_1-q)q_1, (\beta-1)(p-1)+(\alpha-q_1-p)p_1) > 0, \tag{25}$$

then problem (1) with A S-p-C (resp. S-p-A) and B S-q-C (resp. S-q-A) has no solutions $(u, v) \in X_\gamma(\Omega')$ *such that both* $u \not\equiv 0$ *and* $v \not\equiv 0$ *for sufficiently small* $\gamma > 0$ *(resp.* $\gamma < 0$*).*

Proof. Choosing $\varphi = \chi_\varepsilon$ that satisfy (2), (3), and (5), by Lemma 3.2, we get that for any $\varepsilon > 0$ such that $\Omega \setminus \partial\Omega^{2\varepsilon} \neq \emptyset$ one has

$$\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} f v^{q_1}\, dx \le c\varepsilon^{\delta_1+\tau_1} \tag{26}$$

and

$$\int\limits_{\Omega\setminus\partial\Omega^{2\varepsilon}} g u^{p_1}\, dx \le c\varepsilon^{\delta_2+\tau_2}\,, \tag{27}$$

where

$$\begin{aligned} \delta_1 = \delta_v &:= -\frac{(p-1)((p_1+q)q_1+(q-1))}{p_1q_1-(p-1)(q-1)},\\ \delta_2 = \delta_u &:= -\frac{(q-1)((q_1+p)p_1+(p-1))}{p_1q_1-(p-1)(q-1)},\\ \tau_1 = \tau_v &:= \frac{(p-1)(\alpha(q-1)+\beta q_1)}{p_1q_1-(p-1)(q-1)},\\ \tau_2 = \tau_u &:= \frac{(q-1)(\alpha p_1+\beta(p-1))}{p_1q_1-(p-1)(q-1)}. \end{aligned} \tag{28}$$

From assumptions (3) and (25) it follows that either $\delta_1 + \tau_1 \ge 0$, or $\delta_2 + \tau_2 \ge 0$. Letting $\varepsilon \downarrow 0$, we complete the proof . □

In Case 1 ($S \subset \Omega$), as soon as A is S-p-A and B is S-q-A, estimate (4) with test functions φ_ε that satisfy (2), (4) (instead of (3)) and (5) leads to

$$\operatorname*{ess\,inf}_{x\in\Omega'} u(x) \le c\varepsilon^{\frac{(\alpha-p)p_1+(\beta-q)(p-1)}{p_1q_1-(p-1)(q-1)}},$$

$$\operatorname*{ess\,inf}_{x\in\Omega'} v(x) \le c\varepsilon^{\frac{(\alpha-p)(q-1)+(\beta-q)q_1}{p_1q_1-(p-1)(q-1)}}.$$

This results in

Theorem 3.4. *Let (3) hold. Suppose that $S \subset \Omega$ and*

$$\max((\alpha-p)p_1+(\beta-q)(p-1),(\alpha-p)(q-1)+(\beta-q)q_1) > 0. \tag{29}$$

Then problem (1) has no strictly positive solutions in $X_\gamma(\Omega')$.

REMARK 3.5. *If the operators L_A and L_B satisfy the strong maximum principle (see conditions of its validity in [2]), then Theorem 3.4 implies that system (1) has no nonnegative nontrivial solutions at all.*

Finally, consider Case 3, i.e.

$$\begin{cases} \operatorname{div}(A(x,u,\nabla u)\nabla u) \geq f v^{q_1} & \text{in } \mathbb{R}^n\,, \\ \operatorname{div}(B(x,v,\nabla v)\nabla v) \geq g u^{p_1} & \text{in } \mathbb{R}^n\,, \\ u, v \geq 0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{30}$$

In this case, Lemma 3.2 implies

THEOREM 3.6. *Let (3) hold with A S-p-C (resp. S-p-A) and B S-q-C (resp. S-q-A). Suppose that $\Omega = \mathbb{R}^n$ and*

$$\min(\theta_1, \theta_2) \leq 0 \tag{31}$$

with

$$\theta_1 := \tfrac{(\alpha-p)p_1+(\beta-q)(p-1)}{p_1q_1-(p-1)(q-1)} + \tfrac{n-q}{q-1},$$

$$\theta_2 := \tfrac{(\alpha-p)(q-1)+(\beta-q)q_1}{p_1q_1-(p-1)(q-1)} + \tfrac{n-p}{p-1}.$$

Then problem (30) has no positive solution in $X_+(\mathbb{R}^n)$ (resp. $X_-(\mathbb{R}^n)$).

Proof. Putting $\varphi = \chi_R$ that satisfy (6)–(8) into (4), applying Lemma 3.2 and taking $R \to \infty$, we immediately obtain the result if inequality (31) is strict. If, for instance, $\theta_1 \leq \theta_2 = 0$ (the symmetrical case is analogous), then from Lemma 3.2 it follows that $\int_{\mathbb{R}^n} f v^{q_1}\,dx < \infty$ and hence

$$\int_{\operatorname{supp}|\nabla\varphi_R|} f v^{q_1} \varphi_R\,dx \to 0$$

as $R \to \infty$. Combining this with (14) and (18), we obtain $u \equiv v \equiv 0$ in $\mathbb{R}^n$. $\square$

Remark 3.7. *In case $\alpha = \beta = 0$, Theorem 3.6 essentially reduces to Theorem 22.1 from [8], which is known to be sharp (see Remark 22.1 therein). This suggests to consider eventual solutions of the form*

$$u(x) = c(1 + |x|^{s_1})^{\theta_1},\ v(x) = d(1 + |x|^{s_2})^{\theta_2}$$

with a suitable choice of the constants in order to establish positive solvability of problem (30) with

$$A = -\Delta_p,\ B = -\Delta_q,$$
$$f(x) = (1 + |x|^{s_1})^{-\alpha},\ g(x) = (1 + |x|^{s_2})^{-\beta}$$

if the assumptions of the theorem do not hold.

4. Stationary Systems with Gradient Nonlinearities

Consider the system of parabolic inequalities containing gradient terms:

$$\begin{cases} \operatorname{div}(A(x,u,\nabla u)\nabla u)u \geq a_1 v^{q_1}\rho^{-\alpha} - b_1|\nabla u|^{s_1}\rho^{-\alpha_1} \quad (x \in \Omega'), \\ \operatorname{div}(B(x,v,\nabla v)\nabla v)v \geq a_2 u^{p_1}\rho^{-\beta} - b_2|\nabla u|^{s_2}\rho^{-\beta_1} \quad (x \in \Omega'). \end{cases} \tag{1}$$

We can also consider the case $\Omega = \mathbb{R}^n$, $S = \emptyset$ and $\rho(x) = |x|$. We assume that

$$\begin{aligned} &p_1 > p - 1, &&q_1 > q - 1, \\ &0 < s_1 < \frac{pp_1}{p_1 + 1}, &&0 < s_2 < \frac{qq_1}{q_1 + 1}, \end{aligned} \tag{2}$$

and the operator A (resp. B) is S-p-C (resp. S-q-C) with

$$c_1|\eta|^{p-2} \leq A(x,y,\eta) \leq C_1|\eta|^{p-2}, \quad c_2|\eta|^{p-2} \leq B(x,y,\eta) \leq C_2|\eta|^{p-2} \tag{3}$$

for all $(x,y,\eta) \in \Omega' \times \mathbb{R}_+ \times \mathbb{R}^n$.

Suppose that (1) has a nontrivial positive solution in the following sense.

Definition 4.1. *A pair of nonnegative functions* $(u,v)\in(L_{1,\mathrm{loc}}(\Omega'))^2$ *is called a solution of (1) iff for each test function* $\varphi\in C_0^\infty(\Omega';\mathbb{R}_+)$ *there hold inequalities*

$$\begin{aligned} a_1\int_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx &\le \int_{\Omega'}(-A(x,u,\nabla u)(\nabla u,\nabla\varphi)+b_1|\nabla u|^{s_1}\rho^{-\alpha_1}\varphi)\,dx,\\ a_2\int_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx &\le \int_{\Omega'}(-B(x,v,\nabla v)(\nabla v,\nabla\varphi)+b_2|\nabla v|^{s_2}\rho^{-\beta_1}\varphi)\,dx, \end{aligned} \tag{4}$$

where all integrals are supposed to exist.

Now choose a parameter γ so that $\max(1-p,1-q)<\gamma<0$, and let φ be a test function that will be specified later. Testing the first equation against $u^\gamma\varphi$ and the second one against $v^\gamma\varphi$, with the help of inequalities (3) we obtain

$$\begin{aligned} &a_1\int_{\Omega'} v^{q_1}u^\gamma\rho^{-\alpha}\varphi\,dx+c_1|\gamma|\int_{\Omega'}|\nabla u|^p u^{\gamma-1}\varphi\,dx\\ &\le C_1\int_{\Omega'}|\nabla u|^{p-2}(\nabla u,\nabla\varphi)u^\gamma\,dx+b_1\int_{\Omega'}|\nabla u|^{s_1}u^\gamma\rho^{-\alpha_1}\varphi\,dx\\ &\le C_1\int_{\Omega'}|\nabla u|^{p-1}|\nabla\varphi|u^\gamma\,dx+b_1\int_{\Omega'}|\nabla u|^{s_1}u^\gamma\rho^{-\alpha_1}\varphi\,dx,\\ &a_2\int_{\Omega'} u^{p_1}v^\gamma\rho^{-\beta}\varphi\,dx+c_2|\gamma|\int_{\Omega'}|\nabla v|^q v^{\gamma-1}\varphi\,dx\\ &\le C_2\int_{\Omega'}|\nabla v|^{q-2}(\nabla v,\nabla\varphi)v^\gamma\,dx+b_2\int_{\Omega'}|\nabla v|^{s_2}v^\gamma\rho^{-\beta_1}\varphi\,dx\\ &\le C_2\int_{\Omega'}|\nabla v|^{q-1}|\nabla\varphi|v^\gamma\,dx+b_2\int_{\Omega'}|\nabla v|^{s_2}v^\gamma\rho^{-\beta_1}\varphi\,dx, \end{aligned}$$

and, by the Young inequality with appropriate parameters,

$$\begin{aligned} &a_1\int_{\Omega'} v^{q_1}u^\gamma\rho^{-\alpha}\varphi\,dx+\frac{c_1|\gamma|}{2}\int_{\Omega'}|\nabla u|^p u^{\gamma-1}\varphi\,dx\\ &\le c(\gamma)\int_{\Omega'}u^{p+\gamma-1}|\nabla\varphi|^p\varphi^{1-p}\,dx+c(b_1,\gamma)\int_{\Omega'}u^{\gamma+\frac{s_1}{p-s_1}}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi\,dx, \end{aligned} \tag{5}$$

$$a_2\int\limits_{\Omega'} u^{p_1}v^\gamma\rho^{-\beta}\varphi\,dx+\frac{c_2|\gamma|}{2}\int\limits_{\Omega'}|\nabla v|^q u^{\gamma-1}\varphi\,dx$$
$$\le c(\gamma)\int\limits_{\Omega'} u^{q+\gamma-1}|\nabla\varphi|^q\varphi^{1-q}\,dx+c(b_2,\gamma)\int\limits_{\Omega'} v^{\gamma+\frac{s_2}{q-s_2}}\rho^{-\frac{\beta_1 q}{q-s_2}}\varphi\,dx. \tag{6}$$

Now, testing both equations in (1) with φ, we get

$$a_1\int\limits_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx\le\int\limits_{\Omega'}(-A(x,u,\nabla u)(\nabla u,\nabla\varphi)+b_1|\nabla u|^{s_1}\rho^{-\alpha_1}\varphi)\,dx$$
$$\le c_1\int\limits_{\Omega'}|\nabla u|^{p-1}|\nabla\varphi|\,dx+b_1\int\limits_{\Omega'}|\nabla u|^{s_1}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi\,dx \tag{7}$$

and likewise

$$a_2\int\limits_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx\le\int\limits_{\Omega'}(-B(x,v,\nabla v)(\nabla v,\nabla\varphi)+b_2|\nabla v|^{s_2}\rho^{-\beta_1}\varphi)\,dx$$
$$\le c_2\int\limits_{\Omega'}|\nabla v|^{q-1}|\nabla\varphi|\,dx+b_2\int\limits_{\Omega'}|\nabla v|^{s_2}\rho^{-\frac{\beta_1 q}{q-s_2}}\varphi\,dx. \tag{8}$$

By the Young inequality (note that it is applicable since $s_1<p_1$ and $s_2<q_1$ by (2)), we get

$$\int\limits_{\Omega'}|\nabla u|^{p-1}|\nabla\varphi|\,dx\le\int\limits_{\Omega'}(\eta|\nabla u|^p u^{\gamma-1}\varphi+a_\eta u^{(1-\gamma)(p-1)}|\nabla\varphi|^p\varphi^{1-p})\,dx,$$
$$\int\limits_{\Omega'}|\nabla v|^{q-1}|\nabla\varphi|\,dx\le\int\limits_{\Omega'}(\eta|\nabla v|^q v^{\gamma-1}\varphi+b_\eta v^{(1-\gamma)(q-1)}|\nabla\varphi|^q\varphi^{1-q})\,dx,$$
$$\int\limits_{\Omega'}|\nabla u|^{s_1}\varphi\,dx\le\eta\int\limits_{\Omega'}|\nabla u|^p u^{\gamma-1}\varphi\,dx+c_\eta\int\limits_{\Omega'} u^{\frac{(1-\gamma)s_1}{p-s_1}}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi\,dx,$$
$$\int\limits_{\Omega'}|\nabla v|^{s_2}\varphi\,dx\le\eta\int\limits_{\Omega'}|\nabla v|^q v^{\gamma-1}\varphi\,dx+d_\eta\int\limits_{\Omega'} v^{\frac{(1-\gamma)s_2}{q-s_2}}\rho^{-\frac{\beta_1 q}{q-s_2}}\varphi\,dx \tag{9}$$

with some constants $a_\eta, b_\eta, c_\eta, d_\eta > 0$. Combining (5) – (9) leads to

$$\int\limits_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx \le c\Bigg(\int\limits_{\Omega'} u^{p+\gamma-1}|\nabla\varphi|^p\varphi^{1-p}\,dx + \int\limits_{\Omega'} u^{\gamma+\frac{s_1}{p-s_1}}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi\,dx + \int\limits_{\Omega'} u^{(1-\gamma)(p-1)}|\nabla\varphi|^p\varphi^{1-p}\,dx + \int\limits_{\Omega'} u^{\frac{(1-\gamma)s_1}{p-s_1}}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi\,dx\Bigg), \tag{10}$$

$$\int\limits_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx \le c\Bigg(\int\limits_{\Omega'} v^{q+\gamma-1}|\nabla\varphi|^q\varphi^{1-q}\,dx + \int\limits_{\Omega'} v^{\gamma+\frac{s_2}{q-s_2}}\rho^{-\frac{\beta_1 q}{q-s_2}}\varphi\,dx + \int\limits_{\Omega'} v^{(1-\gamma)(q-1)}|\nabla\varphi|^q\varphi^{1-q}\,dx + \int\limits_{\Omega'} v^{\frac{(1-\gamma)s_2}{q-s_2}}\rho^{-\frac{\beta_1 q}{q-s_2}}\varphi\,dx\Bigg), \tag{11}$$

where the constant c depends only on the parameters of the system and on the choice of η.

Now define parameters

$$\begin{aligned} \gamma_1 &= \frac{p_1}{p+\gamma-1}, & \delta_1 &= \frac{q_1}{q+\gamma-1},\\ \gamma_2 &= \frac{p_1(p-s_1)}{\gamma(p-s_1)+s_1}, & \delta_2 &= \frac{q_1(q-s_2)}{\gamma(q-s_2)+s_2},\\ \gamma_3 &= \frac{p_1}{(1-\gamma)(p-1)}, & \delta_3 &= \frac{q_1}{(1-\gamma)(q-1)},\\ \gamma_4 &= \frac{p_1(p-s_1)}{(1-\gamma)s_1}, & \delta_4 &= \frac{q_1(q-s_2)}{(1-\gamma)s_2} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \lambda_1 = \lambda_3 = 0, \quad \lambda_2 = \lambda_4 = -\frac{\alpha_1 p}{p-s_1},\\ \mu_1 = \mu_3 = 0, \quad \mu_2 = \mu_4 = -\frac{\beta_1 q}{q-s_2}. \end{aligned} \tag{13}$$

Note that, for $\gamma < 0$ small enough, we have $\gamma_i > 1$ and $\delta_i > 1$ ($i = 1, \ldots, 4$) due to (2). Thus applying the Young inequality with these parameters to respective terms on the right hand sides of (10) – (11) results in

$$\int\limits_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx \le \int\limits_{\Omega'} (\eta u^{p_1}\rho^{-\beta}\varphi + c(\eta)\sum_{i=1}^{4}\rho^{\gamma_i'\left(\frac{\beta}{\gamma_i}+\lambda_i\right)}|\nabla\varphi|^{p\gamma_i'}\varphi^{1-p\gamma_i'})\,dx, \tag{14}$$

$$\int\limits_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx \le \int\limits_{\Omega'} (\eta v^{q_1}\rho^{-\alpha}\varphi + c(\eta)\sum_{i=1}^{4}\rho^{\delta_i'\left(\frac{\alpha}{\delta_i}+\mu_i\right)}|\nabla\varphi|^{q\delta_i'}\varphi^{1-q\delta_i'})\,dx, \tag{15}$$

γ_i', δ_i' $(i = 1, \dots, 4)$ being conjugate exponents to γ_i, δ_i, with $\eta > 0$ arbitrary small.

Now suppose that $S \subset \Omega$, and choose a family of test functions $\varphi = \chi_\varepsilon$ with χ_ε satisfying (2), (3), and (5). With this choice of φ, substituting (14) with $\eta < 1$ into (15) and vice versa, we obtain the following estimates:

$$\max\left(\int\limits_{S^{4\varepsilon}\setminus S^{\varepsilon}} v^{q_1}\rho^{-\alpha}\varphi_\varepsilon\,dx, \int\limits_{S^{4\varepsilon}\setminus S^{\varepsilon}} u^{p_1}\rho^{-\beta}\varphi_\varepsilon\,dx\right) \le c\varepsilon^{\kappa_0+m}, \tag{16}$$

where

$$\begin{aligned} \kappa_0 &= \min(\kappa_1, \kappa_2), \\ \kappa_1 &= \min_{i=1,\dots,4} \gamma_i'\left(\frac{\beta}{\gamma_i} + \lambda_i - p\right), \\ \kappa_2 &= \min_{i=1,\dots,4} \delta_i'\left(\frac{\alpha}{\delta_i} + \mu_i - q\right). \end{aligned} \tag{17}$$

Restricting integration on the right hand to the set $S^{3\varepsilon}\setminus S^{2\varepsilon}$, where $\varphi_\varepsilon(x) \equiv 1$, yields

$$\max\left(\int\limits_{S^{3\varepsilon}\setminus S^{2\varepsilon}} v^{q_1}\rho^{-\alpha}\,dx, \int\limits_{S^{3\varepsilon}\setminus S^{2\varepsilon}} u^{p_1}\rho^{-\beta}\,dx\right) \le c\varepsilon^{\kappa_0+m}$$

and hence

$$\max\left(\inf_{x\in S^{3\varepsilon}} u(x), \inf_{x\in S^{3\varepsilon}} v(x)\right) \le c\left(\varepsilon^{\frac{\kappa_0+\alpha}{q_1}} + \varepsilon^{\frac{\kappa_0+\beta}{p_1}}\right). \tag{18}$$

Taking $\varepsilon \to 0$, we see that inequality (18) contradicts the strong positivity of the solution if $\kappa_0 > \max(-\alpha, -\beta)$. Thus we come to the following result.

Theorem 4.2. *Let (2) hold. Assume that one has $\kappa_0 > \max(-\alpha, -\beta)$ with κ_0 defined by (12), (13), and (17).*

Then problem (1) with $S \subset \Omega$ has no strictly positive solution.

Remark 4.3. *If A, B additionally satisfy the strong maximum principle, Theorem 4.2 implies nonexistence of any nonnegative notrivial solutions to (1).*

In a similar manner, for $\Omega = \mathbb{R}^n$ and $\rho := |x|$, we can choose $\varphi = \chi_R(x)$ with $\chi_R(x) = 1$ ($|x| < R$), $\mathrm{supp}\varphi \subset B_{2R}(0)$. In this situation, an analogue of (16) with φ_R instead of φ_ε implies

$$\max\left(\int\limits_{B_R(0)} v^{q_1}|x|^{-\alpha}\,dx, \int\limits_{B_R(0)} u^{p_1}|x|^{-\beta}\,dx \right) \le cR^{M_0+n},$$

where

$$\begin{aligned} M_0 &= \max(M_1, M_2), \\ M_1 &= \max_{i=1,\dots,4} \gamma_i' \left(\tfrac{\beta}{\gamma_i} + \lambda_i - p \right), \\ M_2 &= \max_{i=1,\dots,4} \delta_i' \left(\tfrac{\alpha}{\delta_i} + \mu_i - q \right). \end{aligned} \tag{19}$$

Thus, taking $R \to \infty$ leads to

Theorem 4.4. *Let (2) hold and $M_0 > -n$, where M_0 is defined by (12), (13), and (19).*

Then problem (1) with $\Omega = \mathbb{R}^n$ and $\rho := |x|$ has no positive solution.

5. Parabolical Systems: Nonexistence

Now consider a system of first order evolutional inequalities

$$\begin{cases} u_t + \operatorname{div}(A(x,u,\nabla u)\nabla u) \geq a\rho^{-\alpha} v^{q_1} & ((x,t) \in Q = \Omega' \times \mathbb{R}_+), \\ v_t + \operatorname{div}(B(x,v,\nabla v)\nabla v) \geq b\rho^{-\beta} u^{p_1} & ((x,t) \in Q), \\ u(x,0) = u_0(x) & (x \in \Omega'), \\ v(x,0) = v_0(x) & (x \in \Omega'). \end{cases} \tag{1}$$

Here we assume that $u_0, v_0 \in C(\Omega'; \overline{\mathbb{R}_+})$ and one has

$$p_1 > \max(p-1, 1),\ q_1 > \max(q-1, 1). \tag{2}$$

The operators A and B are supposed to satisfy (7).

Definition 5.1. *A pair of nonnegative functions $(u,v) \in (C(\bar{Q}) \cup W^{1,p}_{loc}(Q)) \times (C(\bar{Q}) \cup W^{1,q}_{loc}(Q))$, $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ is called a solution of (1) iff for each test function $\varphi \in C_0^\infty(Q; \overline{\mathbb{R}_+})$ there hold inequalities*

$$\begin{aligned} & a \int_{\mathbb{R}_+} \int_{\Omega'} v^{q_1} \rho^{-\alpha} \varphi \, dx\, dt + \int_{\Omega'} u_0 \varphi(x,0)\, dx \\ & \leq - \int_{\mathbb{R}_+} \int_{\Omega'} \left(A(x,u,\nabla u)(\nabla u, \nabla\varphi) + u \frac{\partial \varphi}{\partial t} \right) dx\, dt, \\ & b \int_{\mathbb{R}_+} \int_{\Omega'} u^{p_1} \rho^{-\beta} \varphi \, dx\, dt + \int_{\Omega'} v_0 \varphi(x,0)\, dx \\ & \leq - \int_{\mathbb{R}_+} \int_{\Omega'} \left(B(x,v,\nabla v)(\nabla v, \nabla\varphi) + v \frac{\partial \varphi}{\partial t} \right) dx\, dt. \end{aligned} \tag{3}$$

The argument that follows is based on those from Section 40, [8], where a similar system is considered with $f(x) = g(x) \equiv 1$ and $\Omega = \mathbb{R}^n$.

Assume that problem (1) has a solution in the sense of Definition 5.1, and let φ be a standard test function. Testing the first inequality in (1) with $u^\gamma\varphi$ and the second one with $v^\gamma\varphi$, similarly to the proof of Theorem 4.2, due to (7), we obtain

$$\begin{aligned}
&\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}(av^{q_1}u^\gamma\rho^{-\alpha}+c_1|\gamma||\nabla u|^p u^{\gamma-1})\varphi\,dx\,dt+\int\limits_{\Omega'}u_0^{\gamma+1}\varphi\,dx\\
&\le\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(C_1|\nabla u|^{p-2}(\nabla u,\nabla\varphi)u^\gamma\,dx\,dt-\frac{1}{\gamma+1}u^{\gamma+1}\frac{\partial\varphi}{\partial t}\right)dx\,dt,\\
&\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}(bu^{p_1}v^\gamma\rho^{-\beta}\varphi+c_2|\gamma||\nabla v|^q v^{\gamma-1}\varphi)\,dx\,dt+\int\limits_{\Omega'}v_0^{\gamma+1}\varphi\,dx\\
&\le\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(C_2|\nabla v|^{q-2}(\nabla v,\nabla\varphi)v^\gamma\,dx\,dt-\frac{1}{\gamma+1}v^{\gamma+1}\frac{\partial\varphi}{\partial t}\right)dx\,dt,
\end{aligned}$$

and, by the Young inequality with appropriate parameters,

$$\begin{aligned}
&\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(av^{q_1}u^\gamma\rho^{-\alpha}+\frac{c_1|\gamma|}{2}|\nabla u|^p u^{\gamma-1}\right)\varphi\,dx\,dt+\int\limits_{\Omega'}u_0^{\gamma+1}\varphi\,dx\\
&\le C_1(\gamma)\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(u^{p+\gamma-1}|\nabla\varphi|^p\varphi^{1-p}+u^{\gamma+1}\left|\frac{\partial\varphi}{\partial t}\right|\right)dx\,dt,\\
&\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(bu^{p_1}v^\gamma\rho^{-\beta}+\frac{c_2|\gamma|}{2}|\nabla v|^q v^{\gamma-1}\varphi\right)dx\,dt+\int\limits_{\Omega'}v_0^{\gamma+1}\varphi\,dx\\
&\le C_2(\gamma)\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(v^{q+\gamma-1}|\nabla\varphi|^q\varphi^{1-q}+v^{\gamma+1}\left|\frac{\partial\varphi}{\partial t}\right|\right)dx\,dt
\end{aligned}\tag{4}$$

with some constants $c_1, c_2, C_1(\gamma), C_2(\gamma) > 0$ for $\gamma < 0$ small enough.

Now let $\theta > 0$ and

$$A_\varepsilon=\left\{(x,t):\rho(x),t^\theta\le R\right\},\ B_\varepsilon=A_\varepsilon\cap\{t=0\}.$$

Recall that $\Omega'\subset\mathbb{R}^n$ and $m := n-\dim S$. We introduce the notation

$$\begin{aligned}
a_1=b_1&=\frac{p_1q_1}{(p-1)(q-1)}, &\quad a_2=b_3&=\frac{p_1q_1}{p-1},\\
a_3=b_2&=\frac{p_1q_1}{q-1}, & a_4=b_4&=p_1q_1,
\end{aligned}\tag{5}$$

as well as

$$\begin{aligned}
\gamma_1 &= \frac{(m+\theta)\Lambda_0 + (p-1)(\beta(q-1)+(\alpha-q)q_1) - pp_1q_1}{q_1},\\
\gamma_2 &= \frac{m(p_1q_1-p+1)+\theta\Lambda_1+(p-1)(\alpha q_1+\beta)-pp_1q_1}{q_1},\\
\gamma_3 &= \frac{m(p_1q_1-q+1)+(\beta-\theta)(q-1)+(\alpha-q)q_1}{q_1},\\
\gamma_4 &= \frac{m(p_1q_1-1)-\theta(q_1+1)+\alpha q_1+\beta}{q_1},
\end{aligned} \tag{6}$$

and symmetrically

$$\begin{aligned}
\delta_1 &= \frac{(m+\theta)\Lambda_0 + (q-1)(\alpha(p-1)+(\beta-p)p_1) - qp_1q_1}{p_1},\\
\delta_2 &= \frac{m(p_1q_1-q+1)+\theta\Lambda_2+(q-1)(\beta p_1+\alpha)-qp_1q_1}{p_1},\\
\delta_3 &= \frac{m(p_1q_1-p+1)+(\alpha-\theta)(p-1)+(\beta-p)p_1}{p_1},\\
\delta_4 &= \frac{m(p_1q_1-1)-\theta(p_1+1)+\beta p_1+\alpha}{p_1}
\end{aligned} \tag{7}$$

with $\Lambda_0 := p_1q_1-(p-1)(q-1)$, $\Lambda_1 := p_1q_1-(q_1+1)(p-1)$ and $\Lambda_2 := p_1q_1-(p_1+1)(q-1)$.

Our next aim will be to prove the estimates

$$\begin{aligned}
a\int_{A_\varepsilon}\int v^{q_1}\rho^{-\alpha}\,dx\,dt + \int_{B_\varepsilon} u_0\,dx &\le c\left\{\varepsilon^{\gamma_1 a_1'}+\varepsilon^{\gamma_2 a_2'}+\varepsilon^{\gamma_3 a_3'}+\varepsilon^{\gamma_4 a_4'}\right\},\\
b\int_{A_\varepsilon}\int u^{p_1}\rho^{-\beta}\,dx\,dt + \int_{B_\varepsilon} v_0\,dx &\le c\left\{\varepsilon^{\delta_1 b_1'}+\varepsilon^{\delta_2 b_2'}+\varepsilon^{\delta_3 b_3'}+\varepsilon^{\delta_4 b_4'}\right\},
\end{aligned} \tag{8}$$

where $a_i', b_i'\ (i=1,\dots,4)$ are conjugate exponents to a_i and b_i defined by (5).

For this purpose, we test (1) against φ and, due to positivity of u_0 and v_0, obtain

$$a\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx\,dt \le -\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(A(x,u,\nabla u)(\nabla u,\nabla\varphi)+u\frac{\partial\varphi}{\partial t}\right)dx\,dt$$

and

$$b\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx\,dt \le -\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(B(x,v,\nabla v)(\nabla v,\nabla\varphi)+v\frac{\partial\varphi}{\partial t}\right)dx\,dt,$$

which by (3) and the Hölder inequality leads to

$$\begin{aligned} a\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx\,dt &\le c\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}|\nabla u|^p u^{\gamma-1}\varphi\,dx\,dt\right)^{1/p'}\cdot\\ &\cdot\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} u^{(1-\gamma)(p-1)}|\nabla\varphi|^p\varphi^{1-p}\,dx\,dt\right)^{1/p}\\ &+\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left|\frac{\partial\varphi}{\partial t}\right|^{p_1'}\varphi^{1-p_1'}\rho^{\frac{\alpha p_1'}{p_1}}\,dx\,dt\right)^{1/p_1'}\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} u^{p_1}\rho^{-\alpha}\varphi\,dx\,dt\right)^{1/p_1} \end{aligned}\tag{9}$$

and

$$\begin{aligned} b\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx\,dt &\le c\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}|\nabla v|^q v^{\gamma-1}\varphi\,dx\,dt\right)^{1/q'}\cdot\\ &\cdot\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} v^{(1-\gamma)(q-1)}|\nabla\varphi|^q\varphi^{1-q}\,dx\,dt\right)^{1/q}\\ &+\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left|\frac{\partial\varphi}{\partial t}\right|^{q_1'}\varphi^{1-q_1'}\rho^{\frac{\beta q_1'}{q_1}}\,dx\,dt\right)^{1/q_1'}\left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} v^{q_1}\rho^{-\beta}\varphi\,dx\,dt\right)^{1/q_1}. \end{aligned}\tag{10}$$

We introduce parameters

$$\begin{aligned} \kappa_1 &= \frac{p_1}{p+\gamma-1},\ y_1 = \frac{p_1}{\gamma+1},\ z_1 = \frac{p_1}{(1-\gamma)(p-1)}, \\ \kappa_2 &= \frac{q_1}{q+\gamma-1},\ y_2 = \frac{q_1}{\gamma+1},\ z_2 = \frac{q_1}{(1-\gamma)(q-1)}. \end{aligned} \tag{11}$$

Denote

$$X = \int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} v^{q_1}\rho^{-\alpha}\,dx\,dt,\ Y = \int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} u^{p_1}\rho^{-\beta}\,dx\,dt.$$

Then from (4), (9), and (10) we get

$$X \le c\left\{Y^{\frac{1}{p'\kappa_1}+\frac{1}{pz_1}}A + Y^{\frac{1}{p'y_1}+\frac{1}{pz_1}}B + Y^{\frac{1}{p_1}}C\right\},$$

$$Y \le c\left\{X^{\frac{1}{q'\kappa_2}+\frac{1}{qz_2}}D + X^{\frac{1}{q'y_2}+\frac{1}{qz_2}}E + Y^{\frac{1}{q_1}}F\right\}$$

where

$$A = \left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} |\nabla\varphi|^{p\kappa_1'}\varphi^{1-p\kappa_1'}\rho^{\frac{\alpha\kappa_1'}{\kappa_1}}\,dx\,dt\right)^{\frac{1}{p'\kappa_1'}} \cdot \left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} |\nabla\varphi|^{pz_1'}\varphi^{1-pz_1'}\rho^{\frac{\alpha z_1'}{z_1}}\,dx\,dt\right)^{\frac{1}{pz_1'}},$$

$$B = \left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} \left|\frac{\partial\varphi}{\partial t}\right|^{y_1'}\varphi^{1-y_1'}\rho^{\frac{\alpha y_1'}{y_1}}\,dx\,dt\right)^{\frac{1}{p'y_1'}} \cdot \left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} |\nabla\varphi|^{pz_1'}\varphi^{1-pz_1'}\rho^{\frac{\alpha z_1'}{z_1}}\,dx\,dt\right)^{\frac{1}{pz_1'}},$$

$$C = \left(\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'} \left|\frac{\partial\varphi}{\partial t}\right|^{p_1'}\varphi^{1-p_1'}\rho^{\frac{\alpha p_1'}{p_1}}\,dx\,dt\right)^{\frac{1}{p_1'}},$$

$$D = \left(\int_{\mathbb{R}_+} \int_{\Omega'} |\nabla \varphi|^{q\kappa_2'} \varphi^{1-q\kappa_2'} \rho^{\frac{\beta\kappa_2'}{\kappa_2}}\, dx\, dt \right)^{\frac{1}{q'\kappa_2'}} \cdot$$

$$\cdot \left(\int_{\mathbb{R}_+} \int_{\Omega'} |\nabla \varphi|^{qz_2'} \varphi^{1-qz_2'} \rho^{\frac{\beta z_2'}{z_2}}\, dx\, dt \right)^{\frac{1}{qz_2'}},$$

$$E = \left(\int_{\mathbb{R}_+} \int_{\Omega'} \left| \frac{\partial \varphi}{\partial t} \right|^{y_2'} \varphi^{1-y_2'} \rho^{\frac{\beta y_2'}{y_2}}\, dx\, dt \right)^{\frac{1}{q'y_2'}} \cdot$$

$$\cdot \left(\int_{\mathbb{R}_+} \int_{\Omega'} |\nabla \varphi|^{qz_2'} \varphi^{1-qz_2'} \rho^{\frac{\beta z_2'}{z_2}}\, dx\, dt \right)^{\frac{1}{qz_2'}},$$

$$F = \left(\int_{\mathbb{R}_+} \int_{\Omega'} \left| \frac{\partial \varphi}{\partial t} \right|^{q_1'} \varphi^{1-q_1'} \rho^{\frac{\beta q_1'}{q_1}}\, dx\, dt \right)^{\frac{1}{q_1'}}.$$

Using the definitions of κ_i, y_i and z_i, $i = 1, 2$, from (11), we obtain

$$X \le c\left(Y^{\frac{p-1}{p_1}} A + Y^{\frac{2(p-1)}{pp_1}} B + Y^{\frac{1}{p_1}} C \right),$$
$$Y \le c\left(X^{\frac{q-1}{q_1}} D + X^{\frac{2(q-1)}{qq_1}} E + X^{\frac{1}{q_1}} F \right),$$

that is,

$$X^{p_1} \le c\left(Y^{p-1} A^{p_1} + Y^{\frac{2(p-1)}{p}} B^{p_1} + Y C^{p_1} \right),$$
$$Y^{q_1} \le c\left(X^{q-1} D^{q_1} + X^{\frac{2(q-1)}{q}} E^{q_1} + X F^{q_1} \right). \tag{12}$$

Further, we choose a family of test functions $\varphi(x,t) = \xi_\varepsilon(x) T_\tau(t)$ with ξ_ε satisfying (2), (3), and (5), and $T_\tau \in C_0^\infty([0,\tau];[0,1])$ such that $T_\tau(0) = 1$, $T_\tau(\tau) = 0$, and

$$\int_0^\tau \frac{|T_\tau'|^{s'}}{T_\tau^{s'-1}}\, dx\, dt \le c\tau^{1-s'} \tag{13}$$

with $c > 0$, $\tau = \varepsilon^\theta$ and $s \in \{p_1, q_1\}$. This leads to

$$\begin{array}{c} A \leq c_A \varepsilon^{L_1},\ B \leq c_B \varepsilon^{L_2},\ C \leq c_C \varepsilon^{L_3}, \\ D \leq c_D \varepsilon^{M_1},\ E \leq c_E \varepsilon^{M_2},\ F \leq c_F \varepsilon^{M_3}, \end{array} \tag{14}$$

where

$$\begin{aligned} L_1 &= \frac{(m+\theta)(p_1 - p + 1) - pp_1 + \alpha(p-1)}{p_1}, \\ L_2 &= \frac{(m+\theta)(pp_1 - 2(p-1)) - \theta p_1 (p-1) - pp_1 + 2\alpha(p-1)}{pp_1}, \\ L_3 &= \frac{m(p_1 - 1) - \theta + \alpha}{p_1}, \\ M_1 &= \frac{(m+\theta)(q_1 - q + 1) - qq_1 + \beta(q-1)}{q_1}, \\ M_2 &= \frac{(m+\theta)(qq_1 - 2(q-1)) - \theta q_1 (q-1) - qq_1 + 2\beta(q-1)}{qq_1}, \\ M_3 &= \frac{m(q_1 - 1) - \theta + \beta}{q_1}. \end{aligned} \tag{15}$$

Note that

$$L_2 = \frac{L_1}{p} + \frac{L_3}{p'},\ M_2 = \frac{M_1}{q} + \frac{M_3}{q'}. \tag{16}$$

Combining (12) and (14) results in

$$\begin{aligned} X^{p_1} &\leq c\left(Y^{p-1}\varepsilon^{L_1 p_1} + Y^{\frac{2(p-1)}{p}}\varepsilon^{L_2 p_1} + Y\varepsilon^{L_3 p_1} \right), \\ Y^{q_1} &\leq c\left(X^{q-1}\varepsilon^{M_1 q_1} + X^{\frac{2(q-1)}{q}}\varepsilon^{M_2 q_1} + X\varepsilon^{M_3 q_1} \right), \end{aligned}$$

and, by (16) and the Young inequality,

$$X^{p_1} \leq c\left(Y^{p-1}\varepsilon^{L_1 p_1} + Y\varepsilon^{L_3 p_1} \right),\ Y^{q_1} \leq c\left(X^{q-1}\varepsilon^{M_1 q_1} + X\varepsilon^{M_3 q_1} \right).$$

Thus

$$\begin{aligned} X^{p_1} \leq c\Bigg(& X^{\frac{(p-1)(q-1)}{q_1}}\varepsilon^{M_1(p-1) + L_1 p_1} + X^{\frac{p-1}{q_1}}\varepsilon^{M_3(p-1) + L_1 p_1} \\ & + X^{\frac{q-1}{q_1}}\varepsilon^{M_1 + L_3 p_1} + X^{\frac{1}{q_1}}\varepsilon^{M_3 + L_3 p_1} \Bigg), \\ Y^{q_1} \leq c\Bigg(& Y^{\frac{(p-1)(q-1)}{p_1}}\varepsilon^{L_1(q-1) + M_1 q_1} \\ & + Y^{\frac{q-1}{p_1}}\varepsilon^{L_3(q-1) + M_1 q_1} + Y^{\frac{p-1}{p_1}}\varepsilon^{L_1 + M_3 q_1} + Y^{\frac{1}{p_1}}\varepsilon^{L_3 + M_3 q_1} \Bigg) \end{aligned}$$

or, again by the Young inequality,

$$\begin{aligned}X &\leq c\Big(\varepsilon^{(M_1(p-1)+L_1p_1)a_1'} + \varepsilon^{(M_3(p-1)+L_1p_1)a_2'} \\ &\quad + \varepsilon^{(M_1+L_3p_1)a_3'} + \varepsilon^{(M_3+L_3p_1)a_4'}\Big), \\ Y &\leq c\Big(\varepsilon^{(L_1(q-1)+M_1q_1)b_1'} + \varepsilon^{(L_3(q-1)+M_1q_1)b_2'} \\ &\quad + \varepsilon^{(L_1+M_3q_1)b_3'} + \varepsilon^{(L_3+M_3q_1)b_4'}\Big),\end{aligned}$$

where a_i', b_i' $(i = 1, \ldots, 4)$ are conjugate exponents to a_i and b_i defined by (5). A direct calculation shows that

$$\begin{array}{ll}\gamma_1 = M_1(p-1) + L_1p_1, & \delta_1 = L_1(q-1) + M_1q_1, \\ \gamma_2 = M_3(p-1) + L_1p_1, & \delta_2 = L_3(q-1) + M_1q_1, \\ \gamma_3 = M_1 + L_3p_1, & \delta_3 = L_1 + M_3q_1, \\ \gamma_4 = M_3 + L_3p_1, & \delta_4 = L_3 + M_3q_1,\end{array} \tag{17}$$

and estimates (8) follow.

Now assume that for some constants $\sigma_1, \sigma_2 \in \mathbb{R}$

$$u_0(x) \geq c\rho^{\sigma_1}(x)\ (x \in \Omega') \text{ with } \min_{i=1,\ldots,4} \gamma_i a_i' > \sigma_1 \tag{18}$$

or, alternatively,

$$v_0(x) \geq c\rho^{\sigma_2}(x)\ (x \in \Omega') \text{ with } \min_{i=1,\ldots,4} \delta_i b_i' > \sigma_2. \tag{19}$$

Then, taking $\varepsilon \to 0$, we immediately obtain a contradiction. Hence, the following result can be formulated.

THEOREM 5.2. *Let (2) and either (18) or, alternatively, (19) hold with parameters defined by (5)–(7). Then system (1) has no positive solutions in the sense of Definition 5.1.*

REMARK 5.3. *The optimal choice of θ in this and the next chapter can be made similarly to [8], Theorem 40.6.*

6. Evolution Systems with Gradient Nonlinearities

Now consider the system of evolutional inequalities containing gradient terms:

$$\begin{cases} u_t + \operatorname{div}(A(x,u,\nabla u)\nabla u)u \geq a_1 v^{q_1}\rho^{-\alpha} - b_1|\nabla u|^{s_1}\rho^{-\alpha_1} & ((x,t)\in Q), \\ v_t + \operatorname{div}(B(x,v,\nabla v)\nabla v)v \geq a_2 u^{p_1}\rho^{-\beta} - b_2|\nabla u|^{s_2}\rho^{-\beta_1} & ((x,t)\in Q), \\ u(x,0) = u_0(x) \geq 0 & (x\in\Omega'), \\ v(x,0) = v_0(x) \geq 0 & (x\in\Omega'). \end{cases} \tag{1}$$

We assume that

$$\begin{aligned} &p_1 > p-1, && q_1 > q-1, \\ &0 < s_1 < \frac{pp_1}{p_1+1}, && 0 < s_2 < \frac{qq_1}{q_1+1}. \end{aligned} \tag{2}$$

For $S \subset \Omega$, we shall also suppose that

$$u_0(x) \geq c\rho^{\delta_1}(x),\ \ v_0(x) \geq c\rho^{\delta_2}(x)\ \ (x\in\Omega') \tag{3}$$

with some $c, \delta_1, \delta_2 > 0$.

Definition 6.1. *A pair of nonnegative functions $(u,v) \in (L_{1,\mathrm{loc}}(Q))^2$ is called a solution of (1) iff for each test function $\varphi \in C_0^\infty(Q;\mathbb{R}_+)$ there hold inequalities*

$$\begin{aligned} &a_1 \int\limits_{R_+}\int\limits_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx\,dt + \int\limits_{\Omega'} u_0\varphi(x,0)\,dx \leq \\ &\leq \int\limits_{R_+}\int\limits_{\Omega'} (-A(x,u,\nabla u)(\nabla u,\nabla\varphi) + b_1|\nabla u|^{s_1}\rho^{-\alpha_1})\varphi\,dx\,dt + \\ &\qquad - \int\limits_{R_+}\int\limits_{\Omega'} u\frac{\partial\varphi}{\partial t}\,dx\,dt \end{aligned}$$

$$
\begin{aligned}
& a_2 \int\limits_{R_+} \int\limits_{\Omega'} u^{p_1} \rho^{-\beta} \varphi \, dx \, dt + \int\limits_{\Omega'} v_0 \varphi(x,0) \, dx \le \\
& \le \int\limits_{R_+} \int\limits_{\Omega'} (-B(x,v,\nabla v)(\nabla v, \nabla \varphi) + b_2 |\nabla v|^{s_2} \rho^{-\beta_1}) \varphi \, dx \, dt + \\
& \qquad - \int\limits_{R_+} \int\limits_{\Omega'} v \frac{\partial \varphi}{\partial t} \, dx \, dt.
\end{aligned} \tag{4}
$$

Suppose that (1) has a nontrivial positive solution. We test the first equation against $u^\gamma \varphi$ and the second one against $v^\gamma \varphi$, where $\max(1-p, 1-q, -1) < \gamma < 0$ (if A is S-p-A and B is S-q-A) or $\gamma > 0$ (if A is S-p-C and B is S-q-C) and φ is a test function that will be specified later. Using our assumptions on A and B (see (3)), we obtain

$$
\begin{aligned}
& \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} (a_1 v^{q_1} u^\gamma \rho^{-\alpha} + c_1 |\gamma| \cdot |\nabla u|^p u^{\gamma-1}) \varphi \, dx \, dt + \int\limits_{\Omega'} u_0^{\gamma+1} \varphi \, dx \\
& \le \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} \left(C_1 |\nabla u|^{p-1} \cdot |\nabla \varphi| + b_1 |\nabla u|^{s_1} \rho^{-\alpha_1} \varphi - \frac{u}{\gamma+1} \frac{\partial \varphi}{\partial t} \right) u^\gamma \, dx \, dt, \\
& \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} (a_2 u^{p_1} v^\gamma \rho^{-\beta} + c_2 |\gamma| \cdot |\nabla v|^q v^{\gamma-1}) \varphi \, dx \, dt + \int\limits_{\Omega'} v_0^{\gamma+1} \varphi \, dx \\
& \le \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} \left(C_2 |\nabla v|^{q-1} \cdot |\nabla \varphi| + b_2 |\nabla v|^{s_2} \rho^{-\beta_1} \varphi - \frac{v}{\gamma+1} \frac{\partial \varphi}{\partial t} \right) v^\gamma \, dx \, dt,
\end{aligned}
$$

and, by the Young inequality,

$$
\begin{aligned}
& \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} \left(a_1 v^{q_1} u^\gamma \rho^{-\alpha} + \frac{c_1 |\gamma|}{2} |\nabla u|^p u^{\gamma-1} \right) \varphi \, dx \, dt + \int\limits_{\Omega'} u_0^{\gamma+1} \varphi \, dx \\
& \le c(\gamma) \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} u^{p+\gamma-1} \varphi^{1-p} \, dx \, dt + c(b_1, \gamma) \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} u^{\gamma + \frac{s_1}{p-s_1}} \rho^{-\frac{\alpha_1 p}{p-s_1}} \varphi \, dx \, dt \\
& + \frac{1}{\gamma+1} \int\limits_{\mathbb{R}_+} \int\limits_{\Omega'} u^{\gamma+1} \left| \frac{\partial \varphi}{\partial t} \right| \, dx \, dt,
\end{aligned} \tag{5}
$$

$$\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(a_2u^{p_1}v^{\gamma}\rho^{-\beta}+\frac{c_2|\gamma|}{2}|\nabla v|^q v^{\gamma-1}\right)\varphi\,dx\,dt+\int\limits_{\Omega'}v_0^{\gamma+1}\varphi\,dx$$
$$\le c(\gamma)\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}v^{q+\gamma-1}\varphi^{1-q}\,dx\,dt+c(b_2,\gamma)\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}v^{\gamma+\frac{s_2}{q-s_2}}\rho^{-\frac{\beta_1 q}{q-s_2}}\varphi\,dx\,dt$$
$$+\frac{1}{\gamma+1}\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}v^{\gamma+1}\left|\frac{\partial\varphi}{\partial t}\right|\,dx\,dt. \tag{6}$$

Now, testing both equations in (1) with φ, we get

$$a_1\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}v^{q_1}\rho^{-\alpha}\varphi\,dx\,dt+\int\limits_{\Omega'}u_0\varphi\,dx$$
$$\le\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(-A(x,u,\nabla u)(\nabla u,\nabla\varphi)+b_1|\nabla u|^{s_1}\rho^{-\alpha_1}\varphi-u\frac{\partial\varphi}{\partial t}\right)\,dx\,dt$$
$$\le\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(C_1|\nabla u|^{p-1}|\nabla\varphi|+b_1|\nabla u|^{s_1}\rho^{-\alpha_1}\varphi+u\left|\frac{\partial\varphi}{\partial t}\right|\right)\,dx\,dt \tag{7}$$

and likewise

$$a_2\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}u^{p_1}\rho^{-\beta}\varphi\,dx\,dt+\int\limits_{\Omega'}v_0\varphi(x,0)\,dx$$
$$\le\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(-B(x,v,\nabla v)(\nabla v,\nabla\varphi)+b_2|\nabla v|^{s_2}\rho^{-\beta_1}\varphi-v\frac{\partial\varphi}{\partial t}\right)\,dx\,dt$$
$$\le\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(C_2|\nabla v|^{q-1}|\nabla\varphi|+b_2|\nabla v|^{s_2}\rho^{-\beta_1}\varphi+v\left|\frac{\partial\varphi}{\partial t}\right|\right)\,dx\,dt. \tag{8}$$

Using the Young inequality again, we get

$$\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}|\nabla u|^{p-1}|\nabla\varphi|\,dx\,dt$$
$$\le\int\limits_{\mathbb{R}_+}\int\limits_{\Omega'}\left(\eta|\nabla u|^p u^{\gamma-1}\varphi+c_\eta u^{(1-\gamma)(p-1)}|\nabla\varphi|^p\varphi^{1-p}\right)\,dx\,dt,$$

$$\int_{\mathbb{R}_+}\int_{\Omega'} |\nabla v|^{q-1}|\nabla\varphi|\,dx\,dt \le \int_{\mathbb{R}_+}\int_{\Omega'} \left(\eta|\nabla v|^q v^{\gamma-1}\varphi + c_\eta v^{(1-\gamma)(q-1)}|\nabla\varphi|^q\varphi^{1-q}\right)\,dx\,dt,$$

$$\int_{\mathbb{R}_+}\int_{\Omega'} |\nabla u|^{s_1}\rho^{-\alpha_1}\varphi\,dx\,dt \le \int_{\mathbb{R}_+}\int_{\Omega'} \left(\eta|\nabla u|^p u^{\gamma-1}\varphi + c_\eta u^{\frac{(1-\gamma)s_1}{p-s_1}}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi\right)\,dx\,dt, \tag{9}$$

$$\int_{\mathbb{R}_+}\int_{\Omega'} |\nabla v|^{s_2}\rho^{-\beta_1}\varphi\,dx\,dt \le \int_{\mathbb{R}_+}\int_{\Omega'} \left(\eta|\nabla v|^q v^{\gamma-1}\varphi + c_\eta v^{\frac{(1-\gamma)s_2}{q-s_2}}\rho^{-\frac{\beta_1 q}{q-s_2}}\varphi\right)\,dx\,dt.$$

Combining (5) – (9) leads to

$$\begin{aligned} &a_1\int_{\mathbb{R}_+}\int_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx\,dt + \int_{\Omega'} u_0^{\gamma+1}\varphi\,dx \\ &\le c\Bigg(\int_{\mathbb{R}_+}\int_{\Omega'} (u^{p+\gamma-1}|\nabla\varphi|^p\varphi^{1-p} + u^{\gamma+\frac{s_1}{p-s_1}}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi)\,dx\,dt \\ &+ \int_{\mathbb{R}_+}\int_{\Omega'} (u^{(1-\gamma)(p-1)}|\nabla\varphi|^p\varphi^{1-p} + u^{\frac{(1-\gamma)s_1}{p-s_1}}\rho^{-\frac{\alpha_1 p}{p-s_1}}\varphi)\,dx\,dt \\ &+ \int_{\mathbb{R}_+}\int_{\Omega'} (u^{\gamma+1}+1)\left|\frac{\partial\varphi}{\partial t}\right|\,dx\,dt\Bigg), \end{aligned} \tag{10}$$

$$
\begin{aligned}
& a_2 \int_{\mathbb{R}_+} \int_{\Omega'} u^{p_1} \rho^{-\beta} \varphi \, dx \, dt + \int_{\Omega'} v_0^{\gamma+1} \varphi \, dx \\
& \leq c \Bigg(\int_{\mathbb{R}_+} \int_{\Omega'} (v^{q+\gamma-1} |\nabla \varphi|^q \varphi^{1-q} + v^{\gamma + \frac{s_2}{q-s_2}} \rho^{-\frac{\beta_1 q}{q-s_2}} \varphi) \, dx \, dt \\
& + \int_{\mathbb{R}_+} \int_{\Omega'} \Big(v^{(1-\gamma)(q-1)} |\nabla \varphi|^q \varphi^{1-q} + v^{\frac{(1-\gamma)s_2}{q-s_2}} \rho^{-\frac{\beta_1 q}{q-s_2}} \varphi) \, dx \, dt \\
& + \int_{\mathbb{R}_+} \int_{\Omega'} (v^{\gamma+1} + 1) \left| \frac{\partial \varphi}{\partial t} \right| dx \, dt \Bigg),
\end{aligned}
\tag{11}
$$

where the constant c depends only on the parameters of the system and on the choice of η.

Now define parameters

$$
\begin{aligned}
& \gamma_1 = \frac{p_1}{p+\gamma-1}, && \delta_1 = \frac{q_1}{q+\gamma-1}, \\
& \gamma_2 = \frac{p_1(p-s_1)}{\gamma(p-s_1)+s_1}, && \delta_2 = \frac{q_1(q-s_2)}{\gamma(q-s_2)+s_2}, \\
& \gamma_3 = \frac{p_1}{(1-\gamma)(p-1)}, && \delta_3 = \frac{q_1}{(1-\gamma)(q-1)}, \\
& \gamma_4 = \frac{p_1(p-s_1)}{(1-\gamma)s_1}, && \delta_4 = \frac{q_1(q-s_2)}{(1-\gamma)s_2}, \\
& \gamma_5 = \frac{p_1}{\gamma+1}, && \delta_5 = \frac{q_1}{\gamma+1}, \\
& \gamma_6 = p_1, && \delta_6 = q_1
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
& \lambda_1 = \lambda_3 = 0, \quad \lambda_2 = \lambda_4 = -\frac{\alpha_1 p}{p-s_1}, \\
& \mu_1 = \mu_3 = 0, \quad \mu_2 = \mu_4 = -\frac{\beta_1 q}{q-s_2}.
\end{aligned}
\tag{13}
$$

Note that, for $\gamma < 0$ small enough, we have $\gamma_i' > 1$ and $\delta_i' > 1$ ($i = 1, \ldots, 6$) due to (2). Thus applying the Young inequality with these parameters to respective terms on the right hand sides of (10) – (11) results in

$$
\begin{aligned}
&a_1 \int_{\mathbb{R}_+}\int_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx\,dt + \int_{\Omega'} u_0^{\gamma+1}\varphi\,dx \\
&\leq \eta \int_{\mathbb{R}_+}\int_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx\,dt \\
&+c(\eta)\int_{\mathbb{R}_+}\int_{\Omega'} \sum_{i=1}^{4} \rho^{\gamma_i'\left(\frac{\beta}{\gamma_i}+\lambda_i\right)} |\nabla\varphi|^{p\gamma_i'}\varphi^{1-p\gamma_i'}\,dx\,dt \\
&+c(\eta)\int_{\mathbb{R}_+}\int_{\Omega'} \sum_{i=5}^{6} \rho^{\frac{\gamma_i'\beta}{\gamma_i}} \left|\frac{\partial\varphi}{\partial t}\right|^{p\gamma_i'} \varphi^{1-p\gamma_i'}\,dx\,dt,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&a_2 \int_{\mathbb{R}_+}\int_{\Omega'} u^{p_1}\rho^{-\beta}\varphi\,dx\,dt + \int_{\Omega'} v_0^{\gamma+1}\varphi\,dx \\
&\leq \eta \int_{\mathbb{R}_+}\int_{\Omega'} v^{q_1}\rho^{-\alpha}\varphi\,dx\,dt \\
&+c(\eta)\int_{\mathbb{R}_+}\int_{\Omega'} \sum_{i=1}^{4} \rho^{\delta_i'\left(\frac{\alpha}{\delta_i}+\mu_i\right)} |\nabla\varphi|^{q\delta_i'}\varphi^{1-q\delta_i'}\,dx\,dt \\
&+c(\eta)\int_{\mathbb{R}_+}\int_{\Omega'} \sum_{i=5}^{6} \rho^{\frac{\delta_i'\alpha}{\delta_i}} \left|\frac{\partial\varphi}{\partial t}\right|^{q\delta_i'} \varphi^{1-q\delta_i'}\,dx\,dt,
\end{aligned}
\tag{15}
$$

γ_i', δ_i' $(i = 1, \ldots, 6)$ being conjugate exponents to γ_i, δ_i, with $\eta > 0$ arbitrary small.

Now suppose that $S \subset \Omega$, and choose a family of test functions $\varphi(x,t) := \varphi_\varepsilon(x,t) = \chi_\varepsilon(x)T_\varepsilon(t)$ with χ_ε satisfying (2), (3), and (5), and $T_\varepsilon(t) = T(\varepsilon^\theta t)$ where $T(t) \in C^\infty(\mathbb{R};[0,1])$, $T(t) = 1$ $(0 \leq t \leq 1/2)$ and $T(t) = 0$ $(t \geq 3/4)$, $\theta > 0$ is fixed, $\varepsilon > 0$ arbitrary. We may assume $|\frac{\partial T}{\partial t}| \leq c$ and thus

$$
\left|\frac{\partial T_\varepsilon(t)}{\partial t}\right| \leq c\varepsilon^{-\theta} \ (t > 0) \tag{16}
$$

with some $c > 0$ independent of t and ε. With this choice of φ, substituting (14) with $\eta < \sqrt{a_1 a_2}$ into (15) and vice versa, and taking into account that the solution is supposed to be nonnegative, we obtain

$$\max\left(\int\limits_{S^{4\varepsilon}\setminus S^{\varepsilon}} u_0\varphi_\varepsilon\,dx, \int\limits_{S^{4\varepsilon}\setminus S^{\varepsilon}} v_0\varphi_\varepsilon\,dx\right) \le c\varepsilon^{\kappa_0+m}, \tag{17}$$

where

$$\begin{aligned}
&\kappa_0 = \min_{i=1,\dots,4}\kappa_i,\\
&\kappa_1 = \min_{i=1,\dots,4}\gamma_i'\left(\tfrac{\beta}{\gamma_i}+\lambda_i-p\right), \quad \kappa_2 = \min_{i=1,\dots,4}\delta_i'\left(\tfrac{\alpha}{\delta_i}+\mu_i-q\right),\\
&\kappa_3 = \min_{i=5,6}\gamma_i'\left(\tfrac{\beta}{\gamma_i}-\theta\right), \qquad\qquad \kappa_4 = \min_{i=5,6}\delta_i'\left(\tfrac{\alpha}{\delta_i}-\theta\right).
\end{aligned} \tag{18}$$

Restricting integration on the right hand to the set $S^{3\varepsilon}\setminus S^{2\varepsilon}$, where $\chi_\varepsilon(x)\equiv 1$, yields

$$\max\left(\int\limits_{S^{3\varepsilon}\setminus S^{2\varepsilon}} u_0(x)\,dx, \int\limits_{S^{3\varepsilon}\setminus S^{2\varepsilon}} v_0(x)\,dx\right) \le c\varepsilon^{\kappa_0+m}$$

and hence

$$\max\left(\inf_{x\in S^{3\varepsilon}} u_0(x), \inf_{x\in S^{3\varepsilon}} v_0(x)\right) \le c\varepsilon^{\kappa_0}. \tag{19}$$

Taking $\varepsilon\to 0$, we see that (19) contradicts (3) if $\kappa_0 > \max(\delta_1,\delta_2)$. Thus we come to the following result.

THEOREM 6.2. *Let κ_0 be defined by (12) and (18). Assume that (2) holds and one has $\kappa_0 > \max(\delta_1,\delta_2)$.*

Then problem (1) has no strictly positive solution.

In a similar manner, for $\Omega' = \mathbb{R}^n$ and $\rho := |x|$, we can choose $\varphi(x,t) := \varphi_R(x,t) = \chi_R(x)T(R^\theta t)$ with $\chi_R(x) = 1$ ($|x| < R$), $\operatorname{supp}\varphi \subset B_{2R}(0)$, and T and θ defined as herefore. In this situation, an analogue of (17) with φ_R instead of φ_ε implies

$$\max\left(\int\limits_0^{R^\theta}\int\limits_{B_R(0)} v^{q_1}|x|^{-\alpha}\,dx\,dt, \int\limits_0^{R^\theta}\int\limits_{B_R(0)} u^{p_1}|x|^{-\beta}\,dx\,dt\right) \le cR^{M_0+n+\theta},$$

where

$$
\begin{aligned}
&M_0 = \max_{i=1,\dots,4} M_i, \\
&M_1 = \max_{i=1,\dots,4} \gamma_i' \left(\tfrac{\beta}{\gamma_i} + \lambda_i - p\right), \quad M_2 = \max_{i=1,\dots,4} \delta_i' \left(\tfrac{\alpha}{\delta_i} + \mu_i - q\right), \\
&M_3 = \max_{i=5,6} \gamma_i' \left(\tfrac{\beta}{\gamma_i} - \theta\right), \qquad\qquad M_4 = \max_{i=5,6} \delta_i' \left(\tfrac{\alpha}{\delta_i} - \theta\right).
\end{aligned}
\tag{20}
$$

Thus, taking $R \to \infty$ leads to

THEOREM 6.3. *Let (2) hold and $M_0 > -n$ for some $\theta > 0$, where M_0 is defined by (12) and (20).*

Then problem (1) with $\Omega' = \mathbb{R}^n$ and $\rho := |x|$ has no positive solution.

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