Bruno de Finetti forerunner of modern finance

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Abstract. In this paper we discuss the role of de Finetti as a forerunner of some of the more relevant concepts and tools of the modern theory of finance. It is shown that de Finetti gave some ground breaking contributions in such fields as arbitrage free pricing, mean variance efficiency, expected utility and risk aversion. We think it is not only a matter of historical remarks: indeed some of his ideas reveal to be fruitful even nowadays so that going on studying de Finetti's papers may be a good investment for those interested in quantitative finance and economics of uncertainty.

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1 Introduction

It is rather natural in a colloquium concerning economics of uncertainty at Trieste to talk about the related work of Bruno de Finetti and his scholars. Indeed de Finetti spent an important part of his scientific life in Trieste, developing in that period some of his more important contributions to the modern theory of finance and laying the foundations, within the faculty of Economics, of the school of quantitative finance, whose relevance does not need to be underlined here.

That is why we choose to dedicate this paper to a review of de Finetti's early contributions to the modern theory of finance as well as to the new results that have been obtained by some of his direct or indirect scholars following and developing his ground breaking ideas.

Let us start from a premise: as already underlined elsewhere ([30], [31]), de Finetti is universally known as one of the great mathematicians of the XX century, the founder of the theory of subjective probability and a refined scholar of actuarial sciences, but a few people abroad and surprisingly also in Italy have been fully aware of the outstanding relevance of the contributions he gave to the foundations of economics under uncertainty and of the modern theory of finance. Then we intend to offer here a short critical discussion of such contributions and of its connections with recent advances in such fields of research (sometimes due to de Finetti's direct or indirect scholars). On the other side, we do not aim to offer an exhaustive treatment of de Finetti's contributions in economic subjects; rather we concentrate only on some specific points of overwhelming importance.

The plan of the paper is as follows: section 2 introduces a scheme showing connections between inspiration sources, fundamental papers and main topics of economics of uncertainty to be treated in the paper; section 3 discusses the pervasive role of economic thinking in the subjective probability approach and the connections with the Arrow-Debreu theory of complete markets for contingent claims; section 4 describes the Pareto influence and the gambler's ruin theorems as fundamental backgrounds of the paper "Il problema dei pieni". Section 5 describes connections between de Finetti's and Markowitz treatment of the mean-variance paradigm. Section 6 is devoted to a discussion of the new insights obtained through the application of de Finetti's tool of the advantage functions to the portfolio selection problem. Section 7 recalls early de Finetti's treatment of simple correlation structures at the light of recent results. Section 8 and 9 review the suggestion of the zero expected utility paradigm to solve multiperiod reinsurance problems along with his connection with the original introduction of a theory of risk aversion. Section 10 is devoted to a quick recall of early de Finetti's suggestions on the way of a general treatment of reward-risk analysis. Conclusions follow in section 11.

2 A synthetic network of connections

Let us start with a scheme (see Fig. 1) in which a synthesis of the network showing connections between the inspiration sources (Gobbi, Pareto and Generali Insurance), the fundamental papers ([10], [16], [17]) and the main key concepts (arbitrage free pricing, reward-risk (mean-variance, mean-ruin probability and mean-VaR) efficiency, expected utility, CARA and CRRA risk aversion) characterizing de Finetti's scientific production in the field of the economics of uncertainty.

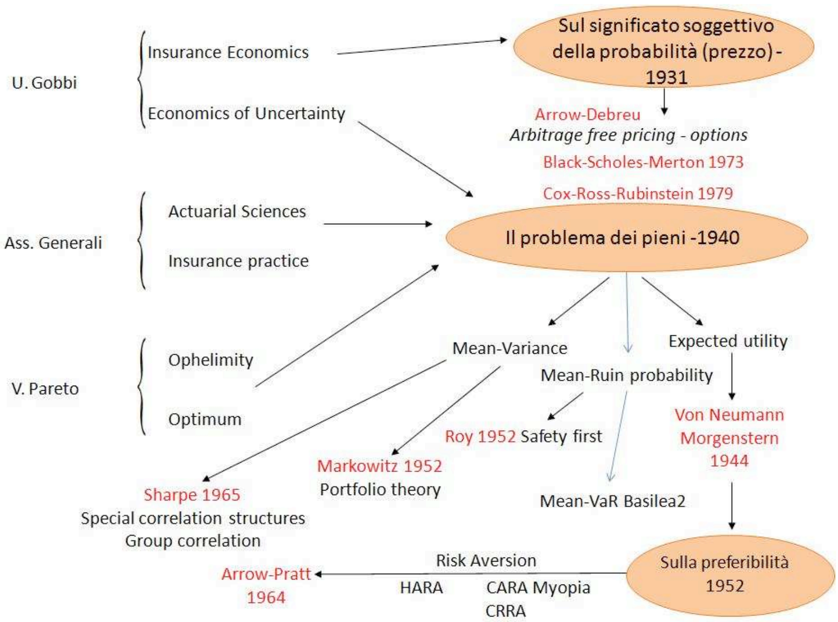

Fig. 1.

3 Sources of inspiration

3.1 U. Gobbi and the subjective theory of probability

While he was a young student of Mathematics at Politecnico of Milan, de Finetti decided to attend a free course in Insurance Economics (actually a course in the Economics of Uncertainty) given by U. Gobbi: an unusual decision for a student of such a degree. This course left an enduring mark in his mind opening large horizons to explore connections between mathematics and economics, which would prove fruitful along his entire scientific life.

Yet the influence of the economic way of thinking was immediate and very strong. It played a decisive role in the definition of the probability of an event as the price of an asset (a bet in de Finetti's terminology) with random gross return: precisely, 1 if the event occurs and 0 otherwise. In turn this definition is the pillar of de Finetti's innovative subjective approach to probability [10], in which the price is given by an agent in a subjective but not arbitrary way as, at that price, the evaluator should be ready to take both the long position of buyer (better) and the short one of seller (bank) in the investment (in the bet).

This idea generates the whole building of the subjective approach to the probability theory, which is entirely based on economic grounds. Indeed de Finetti showed that all fundamental theorems of the probability theory may be derived as consequences of a proper coherency condition on probability assessments (i.e. on prices) regarding logically connected events (combinations of assets). Yet coherency relies entirely upon an economic reasoning. In de Finetti's words: "a person is coherent in evaluating the probabilities of some events if, for any combinations of bets a counterpart makes on whichever set of events among those considered, it is not possible that the gain of the counterpart is in any case positive" (rectius is in any case non negative but positive in some cases).

From a more formal point of view, de Finetti takes as starting point the elements (events) $C_1, C_2, \ldots, C_n$ of a partition of the sure event (Ω). Elementary assets are bets on single events paying, at the end of the period, one unit of money iff the event occurs and nothing otherwise. Under a convenient hypothesis of zero risk free interest rate, the subjective probability of each event, for a specific agent and under a given (agent specific) information set, is the initial price of the asset (i.e. the price of the bet). Probabilities of compound events (those whose value depends logically on the set of elementary events) follow from a condition named by de Finetti economic coherency. According to this condition, the prices must be given in such a way to exclude the building of portfolios of long and short positions (better and bank) with positive initial cash flow and surely non negative final one or with null initial cash flow and non negative (but with at least a positive possibility) final one.

The last sentences make clear that coherency was exactly the counterpart of what is nowadays known in current economic terminology as no arbitrage. And it is straightforward to understand that coherency of prices in the above sense implies that probabilities (that is the prices) must be: a) non negative; b) additive; c) normalized to one; or, said another way, that the classical properties of non negativity, additivity and normalization to one which characterize any theory of probability arise logically from the condition that probabilities behave as prices obeying the coherency (i.e. the arbitrage free) condition).

More generally, expectations of random variables may be seen as prices of bundles of bets on the elementary events. In force of the linearity, such a price is just the combination of the prices of the single bets.

It is astonishing to see that the link connecting (through coherency) probabilities and prices is exactly symmetrical to the one connecting prices and probabilities (through the no arbitrage condition) in the extension to a world of uncertainty of the formal theory of competitive equilibrium provided sometimes later by Arrow (1964), (1971) and Debreu (1959).

In their single period version of their theory of competitive equilibrium under uncertainty Arrow-Debreu too keep as starting point a partition of the sure event Ω in states of the world (states of nature) $C_1, C_2, \ldots, C_n$, incompatible and exhaustive. In such a world (and under the additional assumption of zero risk free rate of interest) pure assets or Arrow-Debreu assets pay at the end of the period one unit of account if and only if a given state of nature happens and nothing elsewhere. Let us denote by $\pi_h = \pi(A_h)$ the initial price of the pure asset corresponding to the state C_h. It is always possible to build portfolios X made by x_h units of any pure asset. The initial price of such a portfolio under an obvious linearity principle, will be given by $\sum_h x_h \pi_h$.

Treating prices as probabilities, it turns out that the price $\Pi(X)$ equals the expectation (under the π distribution) $E(X) = \sum_h x_h \pi_h$ of its final value.

Now, the prices may be seen as probabilities iff they are non negative and summing up to 1. It is easy to check that both conditions are satisfied only under no arbitrage opportunities. For example, with $\pi_h < 0$ building a portfolio made only by long positions on the $h-$th asset would give a positive cash flow at time 0 followed by a cash flow surely non negative at the end of the period. In turn, a portfolio made by long positions in exactly one unit of any pure assets pays at maturity exactly 1. It is then equivalent to one unit of the risk free asset, whose initial price must be (under the zero risk free rate assumption) equal to 1. It is then $\sum_h \pi_h = 1$.

At the end of the story, in a world described by the π distribution, the prices of the pure assets are probabilities and the prices of any portfolio (compound of pure assets) are expectations.

Recently, it has been clearly perceived that such a world is a risk neutral world, that is made by risk neutral agents. Indeed, if the price is an index of preference and if prices are expectations, this implies that the agents preferences are driven solely by expectations, so that they are risk neutral.

The following table resumes the symmetry of de Finetti's and respectively Arrow-Debreu approaches centered on seeing probabilities as counterpart of prices of pure assets and expectations as counterparts of compound assets (i.e. bundles of pure assets).

de Finetti	**Arrow-Debreu**
- Probabilities are prices	- Prices are probabilities
- Properties of probabilities are properties of the prices under no arbitrage	- Properties of prices are properties of probabilities under no arbitrage
- Expectations are prices of portfolios	- Portfolios prices are expectations

It is worth to remark also that more than forty years after the groundbreaking paper by de Finetti the arbitrage free approach would become, through the work of Black-Scholes (1973) and their closed form formulas for option pricing, a pillar of the theory of modern finance and a booster of the exchange and over the counter markets for derivatives.

3.2 Generali Insurance and gambler's ruin, V. Pareto and Pareto efficiency

Two other sources of inspiration played an important role in de Finetti's thought during the thirties.

On one side, in 1931 he was appointed by Generali Insurance, one of the leading (at that time and nowadays) worldwide insurance companies. As head of the research department he found there the opportunity to face with concrete insurance problems, while at the same time keeping in touch with the world of actuarial sciences, whose national and international meetings he regularly attended since the beginning of the decade. Of course in treating actuarial problems he had a big advantage coming from his enormous culture in probability theory coupled with the high ability to apply mathematical tools.

An important example of this synergy between mathematics, probability theory and actuarial sciences is given by de Finetti's treatment of the gambler's ruin problem applied to insurance companies [15].

Let us shortly recall that in the classical approach going back to the origins of probability calculus (de Moivre), gambler's ruin models consider an infinite sequence of fair games (with zero expectation conditionally to any past path of the game) played by two agents. The main result of the theory was that the probability of asymptotic ruin of each player was the ratio between the opponent's initial wealth and the global initial wealth of both players. Hence in the asymmetric case (with only one strong player endowed with unlimited wealth), the sure ruin of the other weak player (as the ratio tends to one). Tailoring this theory to the ruin probability of an insurance company, de Finetti treated it as a weak gambler with finite wealth facing an asymmetric game versus a community of insured people seen as a unique player with infinite wealth.

But differently from the classical problem, the company's ruin is not sure because the sequence of games is no more fair. Indeed the safety loadings induce positive expectation, so as the game becomes advantageous for the weak player.

In this modified scenario de Finetti obtains the following result: let us denote by G_h the company's random gain from its h year portfolio and by β, a coefficient satisfying for any integer h the condition $E(exp(-\beta G_h)) = 1$; then a company with initial wealth W_0 which follows a strategy to insure a sequence of single periods independent portfolios whose random gains are characterized by the common coefficient β, has asymptotic ruin probability $p = exp(-W_0\beta)$. The reciprocal of β is named by de Finetti risk level of each year portfolio. This was a milestone path breaking result in the branch of actuarial sciences known as collective risk theory. Indeed it launched a bridge between the classical, Scandinavian school, approach (see Lundberg (1909), Cramer (1930)) and a modern preference based approach.

On the other side, in the same decade de Finetti payed a lot of attention to the work and thinking of W. Pareto (1909) in order to understand and analyze the mathematical foundations of economic theory.

There is no need to recall here that Pareto was a many sided scientist, well known as one of the leading members of the so called Lausanne school of Economics, also labelled the Mathematical school, due to its stress on mathematical tools.

Let us say here that de Finetti was rather critical on some aspects of Pareto theory; for example, he denied the possibility to extend to an uncertain world the coincidence between competitive equilibrium and Pareto optimality. In some sense, this is surprising because, as correctly perceived by Arrow-Debreu, the key for the extension comes from the introduction of contingent claims as typical goods to be traded and priced in such a world. And in turn this was just at the core of de Finetti's subjective probability approach.

Anyway, at the end of his meditations on the foundations of economic theory, de Finetti was convinced that any approach to pure and applied economics should be based only on the powerful pillars of two Paretian concepts: ophelimity (reflecting an ordinal system of preferences of economic agents) and optimum (set of the allocations which, under a plurality of evaluation criteria, may be changed only worsening the situation at least with respect to one criterion).

In line with his applicative guidelines, he worked on an analytic characterization of the optimum set and in 1937 wrote two milestone papers concerning the issue ([13], [14]). By the end of the decade, he could safely be considered one of the leading experts of the theory of optimum, both on the technical mathematical side as well as on applications to economic theory.

4 "Il problema dei pieni" and reward-risk efficiency

These human events, cultural propensities and technical results were the background of an extraordinary paper: "Il problema dei pieni" [16], written by de Finetti at the end of the thirties, surely one of the most relevant writings in the theory of modern finance. Indeed, as recently recognized by leading scholars of modern finance (see Markowitz (2006) and Rubinstein (2006a)), this paper contains the core of the mean-variance (or rather of the reward-risk) approach to financial decisions under uncertainty and the seed of the theory of expected utility in economic decisions, that is two fundamental paradigms of the economic science of the XX century.

Indeed, as we shall see in the next section, de Finetti and Markowitz formalised two at first sight quite different problems in a very similar way. In detail, de Finetti's problem is as follows: an insurance company is faced with n risks (policies), The net profit of these risks is represented by a vector of random variables with expected value $\mathbf{m} := m_i > 0; i = 1, \ldots, n$ and a non singular covariance matrix $C := \sigma_{ij} : i, j = 1, \ldots, n$. The company has to choose a proportional reinsurance or retention strategy specified by a retention vector $\mathbf{x}$. The retention strategy is feasible if $0 \leq x_i \leq 1$ for all i. A retention x induces a random profit with expected value $E = \mathbf{x}^T\mathbf{m}$ and variance $V = \mathbf{x}^T C\mathbf{x}$. A retention x is by definition mean-variance efficient or Pareto optimum if for no feasible retention y we have both $\mathbf{x}^T\mathbf{m} \leq \mathbf{y}^T\mathbf{m}$ and $\mathbf{x}^T C\mathbf{x} \geq \mathbf{y}^T C\mathbf{y}$, with at least one inequality strict. Let X^* be the set of optimal retentions.

De Finetti looked at the set of feasible retentions as represented by the points of the n dimensional unit cube. The set X^* is a path in this cube. It connects the natural starting point, the vertex $\mathbf{1}$ of full retention (with largest expectation), to the opposite vertex $\mathbf{0}$ of full reinsurance (zero retention and hence minimum null variance).

The core of de Finetti's approach is that (under proportional reinsurance) each additional reinsurance has a twofold effect. It lowers the risk of the retained portfolio, but at the same time lowers its profitability. While in a first attempt to face the problem, de Finetti chose the ruin probability as the proper measure of risk of the company (we will come back to this point in more detail in section 10), he suddenly switched to the variance (a quadratic function of the retention quotas); in turn, the expectation (a linear function) of the retained portfolio represented a proper profitability index. Then, coherently with his economic ideas, this made possible to look at the problem like a typical (two criteria, mean-variance indeed) optimum problem, contrarily to the approach prevailing then in actuarial circles, exclusively concerned with the control of risk. As already said, this is to be seen as the original proposal to apply the mean-variance approach to face portfolio problems under uncertainty. And this is by no means only a methodological innovation.

Looking for a system to solve concrete reinsurance problems and making recourse, as usual for him, to brilliant geometrical constructions, he offered a procedure to obtain the optimum set, in the n dimensional space of retention quotas, as a sequence of line segments, joining the vertex $\mathbf{1}$ of the unit hypercube corresponding to full retention of all policies with the vertex $\mathbf{0}$ of total reinsurance.

On this "optimum reinsurance path" the corner points are the points corresponding to the entrance in active reinsurance of another new policy, joining some other already partially reinsured.

It is interesting to give a sketch of de Finetti reasoning because it offers an idea of his ability to capture in a very simple way the very essence of a problem so as to be able to obtain a friendly solution. Rather than dealing with two goal variables (variance and expectation), it is convenient to view the expectation as a numeraire and to work with a sort of normalized variance. More precisely, de Finetti introduced the so called advantage functions:

$$F_i(\mathbf{x}) = \frac{1}{2}\,\frac{\frac{\partial V}{\partial x_i}}{\frac{\partial E}{\partial x_i}} := \sum_{j=1}^{n} \frac{\sigma_{ij}}{m_i}\, x_j \qquad i = 1, \ldots, n \tag{1}$$

which intuitively capture the advantage coming at x from a small (additional or initial) reinsurance of the $i-$th risk. The connection between the efficient path and the advantage functions is then straightforward: move in such a way to provide additional or initial reinsurance only to the set of those risks giving the largest benefit (that with the largest value of the advantage function). If this set is a singleton the direction of the optimum path is obvious, otherwise the direction should be the one preserving the equality of the advantage functions among all the best performers.

Given the form of the advantage functions, it was easily seen that this implied a movement on a segment of the cube characterized by the set of equations $F_i(\mathbf{x}) = \lambda$ for all the current best performers. Here λ plays the role of the benefit parameter. And we should go on this segment until the advantage function of another, previously non efficient, risk matches the current value of the benefit parameter, thus becoming a member of the efficient set. Accordingly at this point the direction of the efficient path is changed as it is defined by a new set of equations, with the addition of the equation of the newcomer risk. Through a repeated sequential application of this matching logic, de Finetti was able to define the whole efficient set, offering closed form formulas for the no correlation case and giving a largely informal sketch of the sequential procedure in case of correlated risks. This procedure reveals to be an early counterpart of the celebrated critical line algorithm proposed by Markowitz in his subsequent treatment of the portfolio selection problem.

5 de Finetti versus Markowitz

In the following decade Markowitz published his milestone papers ([23], [24], [25]) on mean-variance portfolio selection, which brought him in 1990 the Nobel prize in Economics and the reputation to be the founder of modern finance; meanwhile de Finetti's paper fell into oblivion. Only recently thanks to the work of de Finetti's scholars[1], leading economists began to recognize the importance of de Finetti's paper. It came in the words of M. Rubinstein in [40] "it has recently came to the attention of economists in the English speaking world that among de Finetti's papers is a treasurer trove of results in economics and finance written well before the works of the scholars that are traditionally credited with these ideas......de Finetti's 1940 paper anticipating

[1] Rubinstein (2006b) quotes verbal signaling by C. Albanese, L. Barone and F. Corielli and the paper by F. Pressacco (1986).

much of mean variance portfolio's theory later developed by H. Markowitz", and of Markowitz itself in [26]: "it has come to my attention that in the context of choosing optimum reinsurance levels, de Finetti essentially proposed mean variance portfolio analysis using correlated risks."

Additionally Markowitz underlined that de Finetti has worked out a special case of the so called global optimality conditions in quadratic programming, which Karush-Kuhn-Tucker (Karush (1939), Kuhn-Tucker (1952)) developed at the beginning of the fifties thus paving the way for the critical line algorithm. Concerning this point, we wish to underline de Finetti's ability to face and solve his problem without having at his disposal the powerful technology of KKT conditions. On the other side, we will see hereafter that there is an nice connection between the friendly intuition of the advantage function and the advanced technology of optimality conditions.

Let us consider the proportional reinsurance problem in its formal version:

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{x}^\top C\,\mathbf{x} \\ & \mathbf{x}^\top \mathbf{m} \geq E \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{1} \end{aligned} \tag{2}$$

And let us write the Lagrangian of that problem:

$$L(\mathbf{x}, \lambda, \mathbf{u}, \mathbf{v}) = \frac{1}{2} \cdot \mathbf{x}^T C \cdot \mathbf{x} + \lambda \cdot (E - \mathbf{m}^T \cdot \mathbf{x}) + \mathbf{u} \cdot (\mathbf{x} - \mathbf{1}) - \mathbf{v} \cdot \mathbf{x} \tag{3}$$

Making recourse to the KKT conditions, it turns out that x^* is optimal iff there exists a triplet $(\lambda^*, \mathbf{u}^*, \mathbf{v}^*)$ with $\mathbf{u}^*, \mathbf{v}^*$ non negative vectors such that:

1. $\mathbf{x}^*$ minimizes $L(\mathbf{x}, \lambda, \mathbf{u}^*, \mathbf{v}^*)$
2. $\mathbf{x}^*$ is feasible
3. either $x_j^* = 0$ or $v_j^* = 0$ (or both) and either $x_j^* = 1$ or $u_j^* = 0$ (or both)

It is rather surprising to discover that optimality conditions may be expressed in the simplest and expressive way just through the advantage functions. Indeed, the optimality conditions turn out to be: $\mathbf{x}^*$ is mean-variance efficient iff there exists $\lambda \geq 0$ such that:

$$\begin{aligned} \text{i)}\ & F_i(\mathbf{x}^*) = \lambda \text{ if } 0 < x_i^* < 1 \\ \text{ii)}\ & F_i(\mathbf{x}^*) \geq \lambda \text{ if } x_i^* = 0 \\ \text{iii)}\ & F_i(\mathbf{x}^*) \leq \lambda \text{ if } x_i^* = 1 \end{aligned}$$

To capture the intuitive meaning of the condition, look at the advantage function $F_i(\mathbf{x})$ as the (pseudo) marginal utility at $\mathbf{x}$ of buying reinsurance of the i-th risk and at λ as the shadow price of any (marginal in quota terms) reinsurance. After that, the optimality conditions mean that, given the shadow price, reinsurance of a risk is bought if the marginal utility is larger than the price and up to the point where the (diminishing) marginal utility just matches the price, or obviously if zero retention has been reached this way.

6 The application of de Finetti's ideas in portfolios selection

As may be perceived by a glance to the following formal version of Markowitz portfolio selection problem:

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{x}^\top C\,\mathbf{x} \\ & \mathbf{x}^\top \mathbf{m} \geq E \\ & 1^\top \mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{4}$$

this is quite similar to the reinsurance problem treated by de Finetti. The difference regards the constraints: besides being restricted to the condition $0 \leq x_i \leq 1$, we must add the one $\sum_i x_i = 1$. Let us recall once more here that Markowitz solved the problem making recourse to the KKT optimality conditions and that he developed for the solution a sequential algorithm known as critical line algorithm. It may be thought as a path in the hyper plane of the feasible portfolios: such a path is piecewise linear joining the point of largest expectation with the one of minimum absolute variance. The corner points of the path correspond to points where the composition of the assets entering in the efficient portfolio is changing. There are still two types of corners: those in which a new asset enters in the efficient portfolio and those in which an asset stops to be part of the efficient portfolio (see Fig. 2). The role of the corners will be cleared later.

Now it is exciting to find that, despite the fact that de Finetti did not treat at all the asset portfolio selection problem, his ideas are in direct close relation with the mean-variance optimum portfolio. Indeed, still Pressacco-Serafini (2009) showed that, through a proper reformulation of the advantage functions it is possible to build a procedure mimicking the one suggested by de Finetti for the reinsurance case to obtain in a natural and straightforward way something analogous to the critical line algorithm and a simple and meaningful characterization of the optimum mean-variance portfolio (not only in the classical problem but also in case of additional upper and lower bound collective constraints).

The advantage function is precisely given by:

$$F_{ij}(\mathbf{x}) := \frac{1}{2}\,\frac{\frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial x_j}}{\frac{\partial E}{\partial x_i} - \frac{\partial E}{\partial x_j}} = \sum_{h=1}^{n} \frac{\sigma_{ih} - \sigma_{jh}}{m_i - m_j}\, x_h \tag{5}$$

The idea behind this new version of the advantage function is that small movements of single assets are no more feasible; hence, we consider as basic feasible movement those coming from small bilateral trading between pairs i, j of assets. The corresponding advantage function $F_{ij}(\mathbf{x})$ is designed to capture the consequences (still measured by the ratio decrease of variance over decrease of expectation) coming from such basic movements. It turns out that the easiest way to express the optimality conditions is reached also in this case through the advantage functions. Indeed, given the Lagrangian of problem (4):

$$L(\mathbf{x}, \lambda, \mathbf{u}, \mathbf{v}) = \frac{1}{2}\,\mathbf{x}^T\, C\,\mathbf{x} + \lambda\,(E - \mathbf{m}^T\,\mathbf{x}) + \mathbf{u}\,(1 - \mathbf{1}^T\,\mathbf{x}) - \mathbf{v}\,\mathbf{x} \tag{6}$$

and denoting by k a reference asset such that $x_k > 0$ and partitioning the other assets into three sets as follows:

$$I_k^* := \{h \neq k : x_h > 0\}, \quad I_0^k := \{h < k : x_h = 0\}, \quad I_k^0 := \{h > k : x_h = 0\}$$

the following result holds, at least if $I_k^* \neq \emptyset$ (note that I_k^* is empty in the extreme case $x_k^* = 1; x_h^* = 0$ for $h \neq k$)[2]:
Optimality conditions: let k such that $x_k^* > 0$. Then $\mathbf{x}^* \geq 0$ is optimal iff $\mathbf{1}^T\,\mathbf{x} = 1$ and there exists $\lambda \geq 0$ such that:

i) $F_{kh}(\mathbf{x}^*) = \lambda$ for $h \in I_k^*$
ii) $F_{kh}(\mathbf{x}^*) \geq \lambda$ for $h \in I_0^k$
iii) $F_{kh}(\mathbf{x}^*) \leq \lambda$ for $h \in I_k^0$

Then the optimality conditions require simply that all basic bilateral trading between each pair of active assets share the same advantage at the level λ, while all basic trading between an active i and a non active j have a lower advantage level[3]. There are, once more, two corner types: matching corners in which a matching asset becomes efficient and enters in the efficient portfolio; breaking corners when along a segment of the optimum path one of the active assets reaches his boundary value 0 before a matching occurs. At a breaking corner,

[2] For details about the case $I_k^* = \emptyset$, see [33], pp. 265.).
[3] For $h < k$ and $x_h = 0$ in the optimality conditions ii) describe a toll rather than a premium.

the breaking asset leaves the efficient set and remain frozen maybe temporally at level 0 (see Fig. 2).

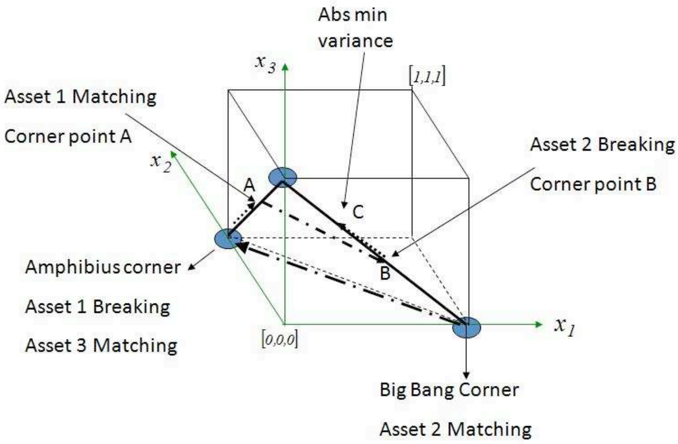

Fig. 2. Triangle of feasible portfolios lying on the plane defined by points (1,0,0);(0,1,0);(0,0,1). Path of efficient portfolios defined by dotted lines. Big Bang: initial point (1,0,0) of efficient path with largest expectation, asset 2 (matching) entering in efficient portfolios. Amphibious corner: point (0,1,0), asset 1 (breaking) leaving, asset 3 (matching) entering in efficient portfolios. Corner point A: asset 1 (matching) entering in efficient portfolios. Corner point B: asset 2 (breaking) leaving the efficient portfolios. Point C: terminal point of the efficient path with minimum absolute variance.

7 Special correlation structures

It is well known that either for reasons of computational speed or in order to get a better understanding of financial markets, many important authors introduced models exploiting specific simple correlation structures. In particular, the so called diagonal model of W. Sharpe (1964) had an overwhelming relevance in this field. Let us note that also on this point de Finetti was a forerunner. In [16], indeed, inspired by the same motivations, he suggested to face the reinsurance problem by utilizing a special correlation structure, named group correlation by him, offering also a quick hint to some key properties of the efficient set (see [16], p. 27-30).

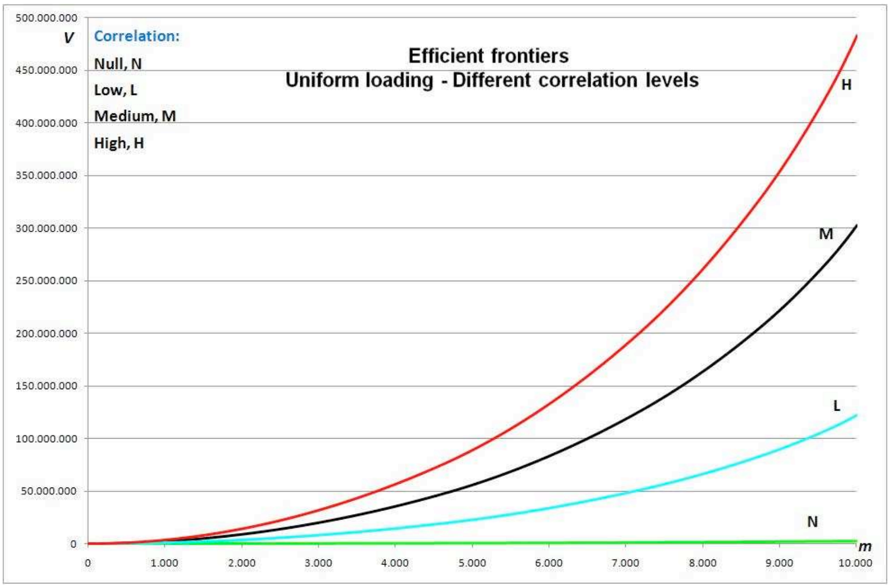

Fig. 3. Influence of correlation.

Also this contribution has been ignored for a long time with the partial exception of the paper by Gigante (1990). We signal that in some recent papers treating the group correlation problem, we obtained ([34], [35]) closed form formulae of efficient retentions, both in the retention space and in the mean-variance one.

This way relevant information may be derived about the influence on the efficient retentions of various correlation levels (see Figure 3) reprinted by Pressacco-Ziani (2010, Fig. 1, pp. 7).

8 de Finetti and the expected utility approach

Up to now we have discussed results obtained by de Finetti in the first part of his 1940 paper concerning the search for the efficient retentions in a single period problem. In the second part of the paper de Finetti faced the problem to select a single "best" from the optimum set and to reach the goal he moved to a multiperiod horizon, indeed an asymptotic one, aiming to choose a retention strategy consistent with a given acceptable (asymptotic) ruin probability.

On the basis of the gambler's ruin background, his proposal was simply to fix the retentions at a level corresponding to the risk coefficient β granting the desired level of ruin probability. We do not enter in technical details here, simply signalling that for example in the toy model of normal no correlated risks in each year it is $x_i = \min\left(\frac{2m_i}{\beta V_i}; 1\right)$ (see [30], footnote 11, pp. 22). According to this approach the issue is mathematically clear but rather obscure as to its true economic meaning. Only after a careful reflection, it could be realized that organizing a sequence of portfolios characterized by a common coefficient β, is for the company equivalent to accept a sequence of indifferent games under exponential utility with risk aversion coefficient β , that is formally with utility of wealth $u(x) = 1-e^{-\beta x}$. Thus it could be said that the second part of the 1940 paper is to be considered as an unconscious anticipation of the application of the expected utility paradigm and in particular the founder of the vast actuarial literature concerning the so called zero utility principle.

9 de Finetti and the theory of risk aversion

But at that time de Finetti was not aware of the importance of his suggestions. He clearly perceived it only some years later, after reading the fundamental work (Von Neumann- Morgenstern (1944)), where a neo Bernoullian theory of measurable (up to linear transforms) utility, concerning preferences among random variables, was coherently exposed[4].

Hence, recognizing the connections between his early intuitions and the new paradigm, de Finetti was able to define some key concepts of the expected utility theory in another groundbreaking paper ([17], 1952). In detail and with the aim of defining proper measures of risk aversion, associated to a given cardinal utility function $u(x)$, he introduced three new tools linking expected utility and risk aversion. Precisely: the absolute risk aversion function $\alpha(x) = -\frac{u''(x)}{u'(x)}$, invariant to linear transforms of u; the probability premium defined as the difference between winning and losing probability which renders a bet of amount h indifferent; the risk premium defined as the sure loss indifferent to a fair bet of amount h.

Then he gave a sketch of the proof that both above premiums are (at least for "small" values of h) directly proportional to the value at the starting wealth of the risk aversion function ([17], pp. 700). Indeed, he showed that the probability premium is $(1/2)h\alpha$ while the risk premium is $(1/2)h^2\alpha$.

In addition, he recognized the exponential utility $u(x) = 1 - exp(-\alpha x)$ as the one associated with an attitude of constant (for any initial wealth x) risk aversion at the level α. And linked such attitude to the asymptotic theory of risk, with the explicit assertion that "the classical criterion of the riskiness level

[4] As well known, the Von Neumann-Morgenstern theory may be considered a rigorous version of an old approach suggested more than two centuries early by Bernoulli and more recently by Ramsey (1931).

(applied in the second part of de Finetti (1940)) is coincident with the utility criterion under constant risk aversion". Note that the criterion is to be intended here in the zero utility sense rather than in the optimizing one.

Finally, he asserted that it would be $u(x) = ln(x)$ for $\alpha(x) = 1/x$ and $u(x) = -x^{1-c}$ for $\alpha(x) = c/x$;in this way also highlighting utility functions characterized by hyperbolic absolute risk aversion. Thus he should be considered also a forerunner of CARA and HARA utility functions.

All these are tools and concepts are of major relevance in the foundations of economics of uncertainty, and universally credited to papers by Arrow (1971) and Pratt (1964), written well after de Finetti's paper.

We signal also that the claim of this primacy came from Italian de Finetti's scholars in the eighties (see Daboni-Pressacco (1987), de Ferra-Pressacco (1986)), and found international imprimatur once more by Rubinstein (2006a): "in 1952 anticipating K. Arrow and J. Pratt by over a decade, he formulated the notion of absolute risk aversion, used it in connection with risk premia for small bets and discussed the special case of constant risk aversion".

10 de Finetti and risk-reward analysis

Even more unknown is de Finetti work as a forerunner of reward-risk analysis (with the application of alternative risk measures) in finance , which he introduced in his 1940 paper too. Indeed, in his early introduction of the reinsurance problem, de Finetti considered at first as the proper target the mean-ruin probability efficiency.

More precisely, in a single period setting, given a company with an initial free capital W and defining as ruin the event that the single period result X of the operations (net retained premiums minus retained liabilities) is lesser than $-W$, De Finetti's looked at the problem of choosing the $\mathbf{x}$ vector of retentions just as that to define the mean-ruin probability efficient set (suggesting thus that the expected profit should be the proper reward measure, while the ruin probability should be the proper risk measure).

Then he argued that under normality of X with parameters (m, s):

$$p(W + X \leq 0) = p(X \leq -W) = p\left(\frac{X - m}{s} \leq -\frac{W + m}{s}\right) = N(-t)$$

with $t = \frac{W+m}{s}$ and $N(-t)$ the value of the distribution function of the standard normal at $-t$.

In order to launch a bridge between the mean-ruin probability and the mean-variance approach, he considered in the mean-variance plane, the equation of the parabola $s^2 = \frac{(W+m)^2}{t^2}$ giving this way a geometrical representation of all the portfolios sharing the same ruin probability at the level $N(-t)$.

Exploiting geometric considerations, he then deduced the perfect equivalence between mean-variance efficiency and mean-ruin probability efficiency for his proportional reinsurance problem. The he paved the way to show that the set of mean-variance efficient retentions was also the set of efficient retentions in mean-free capital (or in modern language mean-VaR) for any fixed level of feasible ruin probability. In some sense it could be said that mean-VaR solutions naturally arise as the dual of the mean-ruin probability ones.

In de Finetti's opinion this was the key to connect mean-variance efficiency, expected utility and ruin probability in order to find optimal reinsurance strategies in a dynamic discrete framework (see Fig. 4).

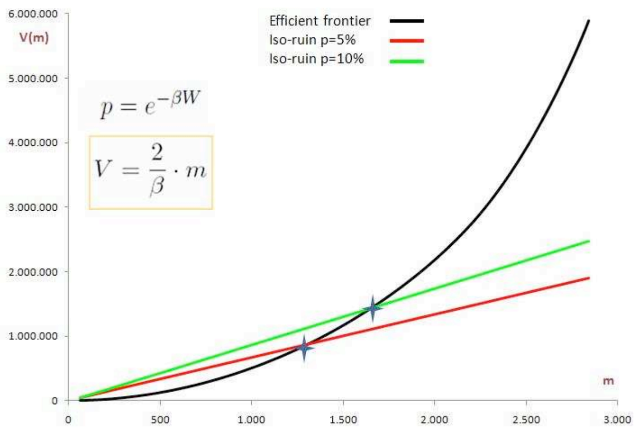

Fig. 4.

We signal here that the bridge between ruin probability and variance is quite similar to that used by Roy (1952) in his safety first approach to the portfolio selection problem. In turn, modern reward-risk approaches based on VaR, C-VaR and other sophisticated risk measures (Artzner et al. (1999), Gaivoronsky-Pflug (2005), Rockafeller-Uryasev (2000)) gained a great popularity and have a big impact on theory and practice of finance.

11 Conclusions

In this paper we discussed the yet not well known role of de Finetti as a forerunner of some of the more relevant concepts and tools of the modern theory of finance. In detail, we recalled the role of the arbitrage free approach in the theory of subjective probability, the early suggestions toward a theory of reward-risk efficiency and in particular the mean-variance approach in the theory of proportional reinsurance. Finally the application of the expected utility paradigm and of an early intuition of a risk aversion theory to discrete multi-period optimization. Some comments on the connected recent works of some of his scholars is also offered.

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