Complex Hyperbolic Cone Structures on the Configuration Spaces

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SUMMARY. - *The space of marked n distinct points on the complex projective line up to projective transformations will be called a configuration space. There are two families of complex hyperbolic structures on the configuration space constructed by Deligne-Mostow and by Thurston. We review that these families are the same, and then exhibit the families for $n = 4, 5$ in constrast with the deformation theory of real hyperbolic cone 3-manifolds.*

1. Introduction

The space of marked n distinct points on the complex projective line $\mathbf{CP}^1$ up to projective transformations will be called a configuration space in this paper and we denote it by $\mathcal{Q}$. It admits a structure of a complex manifold of dimension $n - 3$, and has a long history for attracting many mathematicians. We focus in this paper only on results related with complex hyperbolic geometry.

Deligne and Mostow construct a family of equivariant maps of the universal cover of $\mathcal{Q}$ to the $(n - 3)$-dimensional complex projective space with respect to the action of $\pi_1(\mathcal{Q})$ and the projective transformations in [3]. It is parameterized by the exponents of an integral representation of a several variable analogue of the

hypergeometric function. The main focus of their paper is to discuss when the holonomy representation, which is shown to lie in $\mathrm{PU}(1, n-3) \subset \mathrm{PGL}_{n-2}(\mathbf{C})$ is discrete, and to find many complex hyperbolic lattices.

On the other hand, Thurston provides a different construction of complex hyperbolic structures on $\mathcal{Q}$ in [10] based on euclidean cone structures on $\mathbf{CP}^1$, each of which is assigned to a configuration via a generalized Schwarz-Christofell correspondence. It is parameterized by the cone angles. His approach re-discovers complex hyperbolic lattices found by Deligne and Mostow. Strictly speaking, Thurston constructed structures not on $\mathcal{Q}$ but rather on the quotient of $\mathcal{Q}$ by the action of remarking cone points with the same cone angles, and in fact he found more lattices.

Although the discovery of lattices has been emphasized as a common part of their results, they both actually constructed the continuous families of incomplete complex hyperbolic structures on $\mathcal{Q}$ which provide lattices in particular cases. The purpose of this paper is to expository review the fact that their underlying families of complex hyperbolic structures on $\mathcal{Q}$ are the same, and then to exhibit their families as the deformations of complex hyperbolic cone structures on $\mathcal{Q}$ for $n = 4, 5$. The later discussion is motivated by the deformation theory of real hyperbolic cone 3-manifolds developed in [9, 2, 8, 4, 5, 1].

2. Configuration space

A configuration of marked n points on $\mathbf{CP}^1$ is the way to distribute points with markings on $\mathbf{CP}^1$ disjointly. Let $\mathcal{Q}$ be the space of configurations of marked n points up to projective transformations, and call it a configuration space. That is to say, if we let the space of configurations,

$$\mathcal{M} = \overbrace{\mathbf{CP}^1 \times \cdots \times \mathbf{CP}^1}^{n \text{ times}} - \mathcal{D},$$

where $\mathcal{D}$ is the big diagonal set, then

$$\mathcal{Q} = \mathcal{M} / \mathrm{PGL}_2(\mathbf{C}),$$

where $\mathrm{PGL}_2(\mathbf{C})$ acts diagonally. By sending the last three marked points to $\{0, 1, \infty\}$, we can always normalize a configuration so that the first $n-3$ points lie in $\mathbf{C} - \{0, 1\}$. This normalization gives a canonical identification of $\mathcal{M}$ with the product $\mathcal{Q} \times \mathrm{PGL}_2(\mathbf{C})$. By definition, $\mathcal{Q}$ admits a canonical action of the symmetry group of n letters by remarking the points.

EXAMPLE 2.1. *When $n = 4$, $\mathcal{Q}$ is homeomorphic to $\mathbf{CP}^1 - \{0, 1, \infty\}$. The action of the symmetry group of markings, say $\{1, 2, 3, 4\}$, on $\mathcal{Q}$ is not effective, because the action of the Klein permutation group $\{e, (12)(34), (13)(24), (14)(23)\}$ is realized by projective transformations. The quotient group Γ, isomorphic to the dihedral group of order 6, acts effectively on $\mathcal{Q}$. $\mathcal{Q}/\Gamma$ is naturally extends to an orbifold isomorphic to the moduli space $\mathbf{H}/\mathrm{PSL}_2(\mathbf{Z})$ of elliptic curves. Such ineffectiveness of the action of the symmetry group occurs only when $n = 4$.*

EXAMPLE 2.2. *Example 1 of §4 in [3] discusses what $\mathcal{Q}$ looks like when $n = 5$. It can be identified with the complement of seven rational curves in $\mathbf{CP}^1 \times \mathbf{CP}^1$ defined below,*

$$
x = \left\{ \begin{array}{l} 0 \\ 1 \\ \infty, \end{array} \right. \qquad y = \left\{ \begin{array}{l} 0 \\ 1 \\ \infty, \end{array} \right. \qquad x = y,
$$

where $(x, y) \in \mathbf{CP}^1 \times \mathbf{CP}^1$. $(0, 0)$, $(1, 1)$ and (∞, ∞) are the points where three curves meet, see Figure 1. To get a more symmetric representative with respect to the action of the symmetry group of five letters, we may blow up these three points. Then $\mathcal{Q}$ is homeomorphic to the complement of ten -1 rational curves in $(\mathbf{CP}^1 \times \mathbf{CP}^1) \# 3\overline{\mathbf{CP}^2} \approx \mathbf{CP}^2 \# 4\overline{\mathbf{CP}^2}$.

The complex hyperbolic structure on $\mathcal{Q}$ by Deligne-Mostow to be discussed depends on the weight which will be described by a vector of real numbers,

$$\mu = (\mu_1, \mu_2, \cdots, \mu_n) \quad \text{such that}$$
$$0 < \mu_j < 1 \quad \text{and} \quad \sum_j \mu_j = 2. \qquad (1)$$
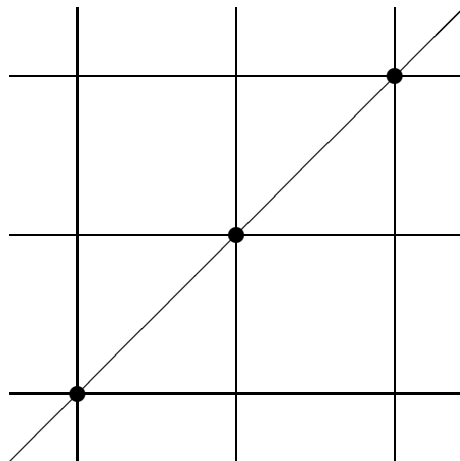
Figure 1: $\mathcal{Q}$ for $n = 5$

This appears soon as exponents of some multi-valued 1-form. It is related with an angle vector

$\theta = (\theta_1, \theta_2, \cdots, \theta_n)$ such that
$$0 < \theta_j < 2\pi \quad \text{and} \quad \sum_j (2\pi - \theta_j) = 4\pi$$

in Thurston's complex hyperbolization subject to the identity,

$$\theta_j = 2\pi(1 - \mu_j).$$

The weight μ can be regarded as a curvature vector from Thurston's viewpoint.

To construct structures in both methods, the common root is an integrand of an integral representation of a several variable analogue of the hypergeometric function

$$\omega_m = \prod (z - m_j)^{-\mu_j} dz \qquad (2)$$

assigned to each configuration

$$m = (m_1, m_2, \cdots, m_n) \in \mathcal{M}.$$

If one of m_j's is ∞, we should appropriately understand the representation (2) as carefully explained in [3]. We will see their constructions more precisely in the next two sections.

3. Deligne-Mostow's construction

Let $\mathbf{P}_m$ be the complement of the point set $\{m_1, m_2, \cdots, m_n\}$ in $\mathbf{CP}^1$, namely

$$\mathbf{P}_m = \mathbf{CP}^1 - \{m_1, \cdots, m_n\}.$$

The construction by Deligne and Mostow in [3] starts with choosing a flat complex line bundle L_m on $\mathbf{P}_m$ with holonomy so that the image of a tiny circle surrounding the point marked by m_j is the rotation of $2\pi\mu_j$. In other words, the holonomy around m_j acts on the fiber as a complex multiplication by $e^{2\pi i \mu_j}$. L_m admits a hermitian structure, and we choose one, though the structure is not unique since $\mathrm{Aut}\, L_m$ is isomorphic to $\mathbf{C}^*$. The monodromy of ω_m around m_j is the inverse of that of a horizontal section of L_m. Hence any section of $\Omega^1(L_m)$ can be written as a tensor product of ω_m, a non zero multi-valued section of L_m and a holomorphic function on $\mathbf{P}_m$.

Then consider de Rham cohomology of $\mathbf{P}_m$ with coefficients in L_m. Since L_m is nontrivial by definition of μ, the zero-th cohomology vanishes. Thus by Euler characteristic argument, the first cohomology group is an $(n-2)$-dimensional complex vector space. The hermitian structure we put on L_m defines a hermitian structure on $H^1(\mathbf{P}_m; L_m)$.

Since each μ_j lies between 0 and 1, or the rotation angles lie between 0 and 2π, Proposition 2.6.1 in [3] identifies the cohomology group in question with that with compact support by the induced homomorphism of the inclusion. Namely

$$H^1(\mathbf{P}_m; L_m) \cong H_c^1(\mathbf{P}_m; L_m)$$

Poincaré duality pairing in this setting defines a perfect pairing

$$\psi_0 : H_c^1(\mathbf{P}_m; L_m) \times H_c^1(\mathbf{P}_m; \overline{L_m}) \to H_c^2(\mathbf{P}_m; \mathbf{C}) \cong \mathbf{C}$$

by sending $\omega_1 \in H_c^1(\mathbf{P}_m; L_m)$ and $\omega_2 \in H_c^1(\mathbf{P}_m; \overline{L_m})$ to

$$\psi_0(\omega_1, \omega_2) = \int_{\mathbf{P}_m} \omega_1 \wedge \omega_2,$$

where $\overline{L_m}$ is the complex conjugate to L_m. This now gives a hermitian form

$$\psi(\omega, \eta) = \frac{-1}{2\pi i} \psi_0(\omega, \overline{\eta})$$

on $H_c^1(\mathbf{P}_m; L_m)$. Corollary 2.21 in [3] shows that the hermitian form ψ is nondegenerate and has signature $(1, n-3)$ by the Hodge theory. Moreover ω_m represents a non zero class which lies in the positive part with respect to ψ in $H_c^1(\mathbf{P}_m; L_m)$.

Let U be a contractible neighborhood of m in $\mathcal{M}$. Then L_m extends uniquely to a flat line bundle L_U on $\cup_{m \in U} \mathbf{P}_m$. Regarding L_U as a sheaf of horizontal sections, and taking a higher direct image of L_U of the projection $\pi : \cup_{m \in U} \mathbf{P}_m \to U$, we obtain a sheaf $R^1 \pi_* L_U$ on $U \subset \mathcal{M}$ whose stalk at m is identified with a vector space $H_c^1(\mathbf{P}_m; L_m)$. Hence $R^1 \pi_* L_U$ can be viewed also as a flat vector bundle. Now the flat projective space bundle $P R^1 \pi_* L_U$ is independent of the choice of L_U up to unique isomorphism, and hence for variable U, they glue into a flat projective space bundle on the whole $\mathcal{M}$. We denote this flat projective space bundle by $B(\mu)$ where the fiber is the projective space of the first cohomology group H_c^1.

Lemma 3.5 in [3] shows that the assignment of $[\omega_m]$ to each $m \in \mathcal{M}$ defines a holomorphic section

$$\omega_\mu : \mathcal{M} \to B(\mu)$$

which is equivariant with respect to the action of $\mathrm{PGL}_2(\mathbf{C})$. Hence restricting ω_μ to $\mathcal{Q}$, we get a section on $\mathcal{Q}$

$$\omega_\mu|_{\mathcal{Q}} : \mathcal{Q} \to B(\mu)|_{\mathcal{Q}}.$$

Let $p : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ be the universal covering. Then the pull back $p^* B(\mu)|_{\mathcal{Q}}$ admits the product structure $\widetilde{\mathcal{Q}} \times B(\mu)|_0$ induced by the flat structure, where 0 denotes a fixed base configuration lying in

$\mathcal{Q} \subset \mathcal{M}$. Hence composing the pull back of ω_μ and the projection $: \widetilde{\mathcal{Q}} \times B(\mu)|_0 \to B(\mu)|_0$, we get a map

$$\widetilde{\omega_\mu} : \widetilde{\mathcal{Q}} \to B(\mu)|_0 = \mathbf{CP}^{n-3}.$$

Proposition 3.9 in [3] establishes that $\widetilde{\omega_\mu}$ is locally biholomorphic. Moreover (3.10) in [3] shows that the image of $\widetilde{\omega_\mu}$ is contained in the complex ball $\mathbf{B} \subset B(\mu)|_0$, where $\mathbf{B}$ is the quotient of positive part of ψ by $\mathbf{C}^*$ action. Thus the action of $\pi_1(\mathcal{Q})$ on $\mathbf{CP}^{n-3}$ is contained in $\mathrm{PU}(1, n-3)$ and $\widetilde{\omega_\mu}$ is equivariant with respect to the action of $\pi_1(\mathcal{Q})$

To end the construction, notice that ψ induces a Bergman metric on $\mathbf{B}$ which we call a complex hyperbolic metric. Pull back this metric on $\widetilde{\mathcal{Q}}$ by $\widetilde{\omega_\mu}$. Since the holonomy representation of $\pi_1(\mathcal{Q})$ preserves the metric, the metric on $\widetilde{\mathcal{Q}}$ is preserved by the action of the covering transformations. Hence it descends to a complex hyperbolic structure on $\mathcal{Q}$. The structure depends continuously on μ, and hence we obtained a family of complex hyperbolic structures on $\mathcal{Q}$ parameterized by the weight μ. This summarizes the construction by Deligne and Mostow.

Fixing μ, we thus obtained a complex hyperbolic structure on $\mathcal{Q}$. Let us denote by $\mathcal{MD}(\mu)$ the completion of a complex hyperbolic manifold so constructed.

4. Thurston's construction

The method of complex hyperbolization by Deligne and Mostow involves the complex Lorentz space supported on the first cohomology group of $\mathbf{P}_m$ with a twisted coefficient L_m together with a hermitian form ψ derived from Poincaré duality pairing. Thurston gave a completely different aspect of these machineries. Here we describe how he translated these ideas to his own.

Fixing a base point $*$ in $\mathbf{P}_m$, Thurston regards the integral of ω_m along a path from $*$ to z in $\mathbf{P}_m$,

$$h(z) = \int_*^z \omega_m = \int_*^z \prod (t - m_j)^{-\mu_j} dt$$

as a developing map of some euclidean structure on $\mathbf{P}_m$ which extends to an euclidean cone structure on $\mathbf{CP}^1$ with prescribed cone

data, and relate the family of euclidean cone spheres obtained by varying m with a complex hyperbolic structure on $\mathcal{Q}$.

The reason why the euclidean cone structure appears comes from the fact that the pre Schwarzian of a multi-valued map h has the form

$$\frac{h''}{h'} = \sum_j \frac{-\mu_j}{z - m_j},$$

and is single-valued. This fact implies that the change of the analytic continuation around singular point m_j is a post composition of a map which is necessarily affine. Moreover direct computation shows that the map must preserve an euclidean metric.

Proposition 6.1 in [10] shows a method to assign to each configuration an euclidean cone sphere as follows. Fix an euclidean metric on $\mathbf{C}$. For each configuration $m \in \mathcal{M}$, we choose a representative such that non of m_j's is ∞. Computation shows that the pre Schwarzian is holomorphic at ∞ since $\sum_j \mu_j = 2$. Thus h defines a $\pi_1(\mathbf{P}_m)$-equivariant map of the universal cover of $\mathbf{P}_m$ to $\mathbf{C}$. The image of the holonomy representation is contained in the group of euclidean isometries. By pulling back the euclidean metric of $\mathbf{C}$ on the universal cover of $\mathbf{P}_m$, and pushing down to $\mathbf{P}_m$, we get an euclidean metric there. The metric is not complete, and the completion yields a cone point of cone angle $2\pi(1 - \mu_j)$, or curvature μ_j, at each punctured point. We denote such an euclidean cone sphere by Δ_m.

This correspondence is not quite one to one since there are several choices we made. However, it turns out to be one to one if we regard it as a correspondence between the set of projective classes of configurations with weight μ and the set of similarity classes of euclidean cone spheres with prescribed curvature μ. In fact, the converse is obtained by remembering only a conformal structure on $\mathbf{P}_m$ induced from an euclidean structure and extend it to the unique conformal structure on $\mathbf{CP}^1$.

The method of complex hyperbolization by Thurston is to give a local coordinate around Δ_m. To do this, choose a geodesic triangulation T of Δ_m such that vertices consists of cone points. Such a triangulation certainly exists by Proposition 2.1 in [10]. Fixing a triangulation T, we consider the set E of oriented edges of the universal cover of $\Delta_m - \{$ cone points $\}$. Assigning to each edge in E

the difference of the images of the end point and the terminal point by h, we get a map $z_m : E \to \mathbf{C}$. The map z_m satisfies the following cocycle conditions with twisted coefficients in L_m,

1. $z_m(e_1) + z_m(e_2) + z_m(e_3) = 0$,   when e_1, e_2, e_3 surround a triangle,

2. $z_m(\gamma e) = H(\gamma) z_m(e)$,   where $H(\gamma)$ is a rotation part of the holonomy of γ

This is well defined up to $\mathbf{C}^*$ action. Note that the rotation part H depends only on the curvature μ and not on the location of cone points m.

The set of euclidean cone spheres close to Δ_m up to similarity can be parameterized locally by cocycles such as

$$Z = \{ z : E \to \mathbf{C} \mid z(e_1) + z(e_2) + z(e_3) = 0, \ z(\gamma e) = H(\gamma) z(e) \}$$

up to $\mathbf{C}^*$ action. Proposition 2.2 in [10] shows that Z is a complex vector space of dimension $n - 2$, and each cocycle can be determined by choosing the values of $n - 2$ edges $e_1, e_2, \cdots, e_{n-2}$ which form a tree in E and also in Δ_m.

LEMMA 4.1. *The assignment of $\omega_m \in B(\mu)|_0$ to $z_m \in PZ$ provides a local bijection, where PZ is a projective space of Z.*

Proof. Fix a configuration m_0. Then z_m near z_{m_0} is parameterized by the value of appropriate $n - 2$ edges $e_1, \cdots, e_{n-2}$ up to $\mathbf{C}^*$ action and hence $(z(e_1), \cdots, z(e_{n-2}))$ provides its virtual coordinate in $\mathbf{C}^{n-2}$. Easy calculation shows

$$z_m(e_j) = \int_{h(e_j)} dw = \int_{e_j} h^* \, dw = \int_{e_j} h' \, dz = \int_{e_j} \omega_m.$$

On the other hand, identifying Δ_m with a conformal extension of $\mathbf{P}_m$ to $\mathbf{CP}^1$, and listing the evaluation of ω_m along the edges $e_1, \cdots, e_{n-2}$ in the last term, we get a period integral, which induces a virtual coordinate of ω_m in $B(\mu)|_0$. $\square$

Assigning the area of Δ_m to each cocycle z_m, we get a hermitian form Area on Z,

$$\text{Area} : Z \to \mathbf{R} \subset \mathbf{C}.$$

Proposition 2.3 in [10] shows that Area turns out to be a hermitian form of signature $(1, n-3)$, and hence induces a complex hyperbolic metric on the ball in PZ.

Each cocycle under the triangulation gives a virtual local chart up to $\mathbf{C}^*$ action. The coordinate change is attained by changing triangulations. However Area is invariant under the coordinate change up to $\mathbf{C}^*$ action. Hence the system of coordinate charts so constructed defines a complex hyperbolic structure on $\mathcal{Q}$. We denote its completion by $\mathcal{T}(\mu)$.

LEMMA 4.2. Area *equals $\pi\psi$ by the correspondence in Lemma 4.1.*

Proof. It is enough to verify the identity for a geodesic triangle Δ on Δ_m. The area of Δ is equal by definition to

$$
\begin{aligned}
\frac{-1}{2i} \int_{h(\Delta)} dw \wedge d\overline{w} &= \frac{-1}{2i} \int_{\Delta} h^*(dw \wedge d\overline{w}) \\
&= \frac{-1}{2i} \int_{\Delta} |h'(z)|^2 dz \wedge d\overline{z} \\
&= \frac{-1}{2i} \int_{\Delta} \omega_m \wedge \overline{\omega_m}.
\end{aligned}
$$

$\square$

THEOREM 4.3. $\mathcal{DM}(\mu)$ *is canonically isometric to* $\mathcal{T}(\mu)$.

Proof. Fix the weight or curvature μ. Then since the local charts of Deligne-Mostow and Thurston for $\mathcal{Q}$ are equivalent, and the metrics they put are the same, they are isometric. So are their completions.

$\square$

REMARK 4.4. *As mentioned in the introduction, Thurston constructed a complex hyperbolic structure not on $\mathcal{Q}$ but on the quotient of $\mathcal{Q}$ by the action of remarking the cone points with the same cone angles. Hence very precisely speaking, $\mathcal{T}(\mu)$ agrees with his only when cone angles all are mutually distinct.*

5. Deformations

Both constructions provide a family of incomplete complex hyperbolic structures on $\mathcal{Q}$. Deligne-Mostow discussed the compactification in relation with Mumford's geometric invariant theory [7]. In particular, topological stratification of the completion has been clarified. For example, the role of stable and semistable points is extensively studied in §§6-7 in [3]. On the other hand, Thurston discussed the completion from geometric viewpoints by introducing complex hyperbolic cone structures. For example, he showed

PROPOSITION 5.1 (PROPOSITION 2.5 IN [10]). *The cone angle around the complex codimension one singularity arisen as collisions of two points with curvature μ_j, μ_i such that $\mu_j + \mu_i \leq 1$ is $2\pi(1 - \mu_j - \mu_i)$.*

The family provides the deformations of complex hyperbolic cone structures on fairly stable underlying topological space. We will look at them from deformation theoretic viewpoint in this section.

By virtue of Theorem 4.3, we denote both $\mathcal{DM}(\mu)$ and $\mathcal{T}(\mu)$ by $\mathcal{Q}(\mu)$.

Start with the classical case when $n = 4$. Recall that $\mathcal{Q}$ is homeomorphic to $\mathbf{CP}^1 - \{0, 1, \infty\}$. $\mathcal{Q}(1/2, 1/2, 1/2, 1/2)$ is isometric to a hyperbolic surface homeomorphic to a three punctured sphere. When the weight varies to $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, then $\mathcal{Q}(\mu)$ becomes a hyperbolic cone sphere. Since the total sum of μ_j's equals 2, at most three pairs of μ_j's have the sum $\mu_j + \mu_i$ less than 1. Such a pair provides a cone singularity of cone angle $= 2\pi(1 - \mu_j - \mu_i)$. If there are less than three such pairs, then there are pairs whose sum equals 1. Such a pair provides a cusp. The total number of cusps and cone points must be three.

THEOREM 5.2. *Any real hyperbolic cone sphere with 3 cone points (including cusps) whose cone angles all are less than 2π occurs as $\mathcal{Q}(\mu)$ for some $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$.*

Proof. The isometry classes of hyperbolic cone spheres with three cone points are classified by the cone angles. Hence it is sufficient to

solve an equation, for example,

$$\begin{cases} A = 2\pi(1 - \mu_2 - \mu_3), \\ B = 2\pi(1 - \mu_3 - \mu_1), \\ C = 2\pi(1 - \mu_1 - \mu_2), \end{cases}$$

for given nonnegative constants A, B, C such that $0 \le A + B + C < 2\pi$, and to let $\mu_4 = 4\pi - 2\pi(\mu_1 + \mu_2 + \mu_3)$. □

EXAMPLE 5.3. *Different weights still can give isometric cone spheres in this case. For instance,* $\mathcal{Q}(1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 - \varepsilon, 1/2 + 3\varepsilon)$ *and* $\mathcal{Q}(1/2 - 3\varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon, 1/2 + \varepsilon)$ *both give a hyperbolic cone sphere with three cone points of cone angle* $= 2\varepsilon$*. These weights cannot be transformed by any permutation of markings.*

When $n = 5$, the situation is a bit complicated. Recall that $\mathcal{Q}$ is homeomorphic to the complement of the union of ten -1 rational curves in $X = (\mathbf{CP}^1 \times \mathbf{CP}^1) \# 3\overline{\mathbf{CP}^2} \approx \mathbf{CP}^2 \# 4\overline{\mathbf{CP}^2}$ as in Example 2.2. We denote the union of these curves by $\mathcal{L}$. The pair $(X, \mathcal{L})$ will be a basic underlying topological space of complex hyperbolic manifolds we discuss. There is a natural way to index each irreducible component of $\mathcal{L}$ by $\mathcal{L}_{ji}$ where j, i are integers such that $1 \le j < i \le 5$. The index has the property that $\mathcal{L}_{ji}$ does intersect with $\mathcal{L}_{kl}$ iff $\{j, i\} \cap \{k, l\} = \emptyset$. In fact, $\mathcal{L}_{ji}$ can be identified with the set of degenerate configurations by the collision of the points marked by m_j and m_i under some weight μ.

EXAMPLE 5.4. $\mathcal{Q}(2/5, 2/5, 2/5, 2/5, 2/5)$ *is a compact complex hyperbolic cone manifold, where the singular set is located exactly as* $\mathcal{L} = \cup_{ji}\mathcal{L}_{ji}$*. The cone angles around* $\mathcal{L}_{ji}$ *all are* $2\pi/5$ *and hence it is an orbifold. The intersection of* $\mathcal{L}_{ji}$ *and* $\mathcal{L}_{kl}$ *if any corresponds to the simultaneous collision of two pair of points. When* $\mu = (\mu_1, \cdots, \mu_5)$ *varies satisfying*

$$\mu_j + \mu_i < 1 \quad for \ all \ i \ne j, \tag{3}$$

then the underlying topology of $\mathcal{Q}(\mu)$ *is stable and the pair with the singular set is homeomorphic to* $(X, \mathcal{L})$*. The cone angle around* $\mathcal{L}_{ji}$ *is equal to* $2\pi(1 - \mu_j - \mu_i) \, (< 2\pi)$ *by Proposition 5.1 and it is easy to*

see that the parameter space of μ under the condition (3) injects into the space of marked complex hyperbolic cone structures on $(X, \mathcal{L})$ by looking at cone angles appeared in $\mathcal{Q}(\mu)$.

To see the limiting case and beyond when $n = 5$, we briefly review what happens in real dimension 3. There are essentially two types of corresponding deformations in real hyperbolic cone 3-manifolds, which are cusp openings.

One is provided by throwing a geodesic cone singularity away to ∞ and opening a cusp, which was discussed originally in [9] as a part of the hyperbolic Dehn filling theory, and studied as a deformation of cone manifolds in [5]. This is due to the existence of codimension two euclidean line. In this case, the continuous deformations beyond the limit may be regarded as cone manifolds with different topology. Some particular discussions of such deformations related with the configuration space can be found in [6].

The other example is discussed in Example 7.2 in [5]. It is provided by collapsing a totally geodesic hyperbolic cone sphere in a real hyperbolic cone 3-manifold to a splitting euclidean cone sphere. This is due to the existence of geodesic hypersurfaces. In this case, the continuous deformations beyond the limit may be regarded as one having a vertex singularity where the cone axis which were stuck through the cone sphere meet.

The complex hyperbolic geometry of $\dim_{\mathbf{C}} \geq 2$ does not admit neither real geodesic hypersurfaces, nor real codimension two euclidean surfaces. Hence it is not conceivable to expect a direct analogue of a cusp opening deformations in the real case. However one sees below that we certainly have cusp opening deformations when the condition (3) breaks down. It can be understood as a mixed type of two cases in real dimension 3.

EXAMPLE 5.5. *When the weight approaches $(1/2, 1/2, 1/3, 1/3, 1/3)$, then the cone angle around $\mathcal{L}_{12}$ becomes zero and $\mathcal{L}_{12}$ itself escapes away to the cusp. $\mathcal{L}_{12}$ in $\mathcal{Q}(2/5, 2/5, 2/5, 2/5, 2/5)$ is metrically a hyperbolic cone sphere with three cone points of cone angles $2\pi/5$. Since the first Chern class of the normal bundle of $\mathcal{L}_{12}$ is -1, the boundary of an equidistant neighborhood of $\mathcal{L}_{12}$ supports $\widetilde{\mathrm{SL}_2(\mathbf{R})}$ geometry. According to the deformation, $\mathcal{L}_{12}$ approaches ∞ where its*

rescaling limit is an euclidean cone sphere with three cone points of cone angle $2\pi/3$, and the section at the cusp supports nilgeometry.

When the weight goes beyond the point, then the cusp comes into the actual point which can be interpreted as the intersection of $\mathcal{L}_{34}, \mathcal{L}_{35}$ and $\mathcal{L}_{45}$. The boundary of an equidistant neighborhood of the point enjoys a spherical geometry. The global topology change from $\mathcal{Q}(2/5, 2/5, 2/5, 2/5, 2/5)$ to $\mathcal{Q}(\mu)$ can be described by collapsing down a -1 rational curve to a point, which is nothing but a blowing down.

THEOREM 5.6. *Any topology change within a family $\mathcal{Q}(\mu_1, \cdots, \mu_5)$ under the condition (1) is attained by a sequence of blowing up and down along $\mathcal{L}_{ij}$'s.*

Proof. Since the total sum of the μ_j's equals 2, possible values of μ_j's such that some pair has the sum equal to 1 are limited. Either one pair does, two pairs with a common value do or three values equal $1/2$. In particular, at most three irreducible components of $\mathcal{L}$, forced to be disjoint, are involved with cusp opening or closing. Hence the claim follows easily from this naive observation with Proposition 5.1.   □

LEMMA 5.7. *Suppose that $1 \leq j < i \leq 5$ both are different from $1 \leq k < l \leq 5$. When $\mu_j + \mu_i > 1$, then $\mathcal{L}_{kl}$ is a hyperbolic cone sphere with cone singularity at blown down $\mathcal{L}_{ji}$ with cone angle $= -2\pi(1 - \mu_j - \mu_i)$.*

Proof. We may assume without loss of generality that $j = 1, i = 2, k = 3, l = 4$. Then $\mathcal{L}_{34}$ can be identified with $\mathcal{Q}(\mu_1, \mu_2, \mu_3 + \mu_4, \mu_5)$ by definition. It has three cone points coming from the intersection with $\mathcal{L}_{15}$, $\mathcal{L}_{25}$, and $\mathcal{L}_{35}$ and $\mathcal{L}_{45}$ simultaneously which is appeared by blowing down $\mathcal{L}_{12}$. Hence the cone angle around the last cone point is calculated as

$$2\pi(1 - ((\mu_3 + \mu_4) + \mu_5)) = -2\pi(1 - \mu_1 - \mu_2)\theta$$

by Proposition 5.1.   □

The following effectiveness of deformations should be compared with Example 5.3.

THEOREM 5.8. *Suppose $n = 5$ and the weight satisfies the condition (1). If $\mathcal{Q}(\mu)$ is isometric to $\mathcal{Q}(\lambda)$, then there is a permutation σ of five letters such that $\sigma(\mu) = \lambda$.*

Proof. Given the weight μ, we get ten numerical invariants $2\pi(1 - \mu_j - \mu_i)$ by running $1 \leq j < i \leq 5$, which describe cone angles appeared in $\mathcal{Q}(\mu)$. If these ten numerical invariants are the same for $\mathcal{Q}(\mu)$ and $\mathcal{Q}(\lambda)$, then it is quite easy to check that the sets of components of μ and λ must be the same. $\qquad\square$

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