Rend. Istit. Mat. Univ. Trieste
Vol. XXXVIII, 17–20 (2006)

Short Proof of a Cardinal Inequality involving the Weak Extent

D. Basile and A. Bella (*)

Summary. - *We are presenting a short and self contained proof of the cardinal inequality* $|X| \leq we(X)^{psw(X)}$*, by using the Pol-Šapirovskiĭ's technique.*

1. Introduction

One of the most interesting cardinal inequalities obtained in the last years is the formula for a T_1 space X

$$|X| \leq we(X)^{psw(X)} \tag{1}$$

This inequality was proved by Hodel in 1991 [6] as a consequence of a combinatorial theorem of Engelking and Karlowicz [3].

Now, it is well known that soon after the discovery of the most important cardinal inequalities at the end of the sixties, the so-called "closure method", proposed by Pol and Šapirovskiĭ has become a unified approach in the proofs of practically all basic cardinal inequalities, see e.g. [5] or the comments and references in [6].

The purpose of this short note is to give a self contained and direct proof of (1), by using only the Pol-Šapirovskiĭ's approach. The proof presented by Hodel in [6] is at first glance completely different and heavely based on the theorem of Engelking and Karlowicz. On the other hand, a proof of this combinatorial result via a closure

(*) Authors' addresses:
Désirée Basile and Angelo Bella, University of Catania, Italy; E-mail: basile@dmi.unict.it, bella@dmi.unict.it
Keywords: Wead Extent, Extent, Point Separating Weight, Diagonal Degree.

argument has been proposed by Michael in [7]. This, of course, could have suggested an approach via a closure argument even to (1). Indeed, our proof follows the ideas developed in [7]. For notations and undefined notions we refer to [2].

Given a collection of sets $\mathcal{V} \cup \{A\}$, the symbol $\mathcal{V}[A]$ denotes the subcollection of all $V \in \mathcal{V}$ satisfying $V \cap A \neq \emptyset$. When $A = \{x\}$, we simply write $\mathcal{V}[x]$.

The weak extent of a topological space X, denoted by $we(X)$, is the smallest cardinal κ such that for any open cover $\mathcal{U}$ of X there exists a set $A \subseteq X$ such that $|A| \leq \kappa$ and $\mathcal{U}[A]$ is a cover of X.

A cover $\mathcal{V}$ of a set X is separating if $\bigcap \mathcal{V}[x] = \{x\}$ for every $x \in X$.

The point separating weight of a topological space X, denoted by $psw(X)$, is the smallest cardinal κ such that there exists a separating open cover $\mathcal{V}$ of X such that $|\mathcal{V}[x]| \leq \kappa$ for every $x \in X$.

Theorem 1.1. *If X is a T_1 space, then $|X| \leq we(X)^{psw(X)}$.*

Proof. Let $psw(X) = \lambda$ and $we(X) = \kappa$. Let $\mathcal{V}$ be a separating open cover of X such that $|\mathcal{V}[x]| \leq \lambda$ for every $x \in X$. We will construct a family $\{X_\alpha : \alpha < \lambda^+\}$ of subsets of X of cardinality not exceeding κ^λ in the following way. Choose a point $x_0 \in X$ and let $X_0 = \{x_0\}$. Then, assume to have already constructed the sets $\{X_\beta : \beta < \alpha\}$. Let $\mathcal{V}_\alpha = \mathcal{V}[\bigcup\{X_\beta : \beta < \alpha\}]$ and for any $\mathcal{W} \subseteq \mathcal{V}_\alpha$ satisfying $\bigcap \mathcal{W} \neq \emptyset$, choose a non-empty set $S_\mathcal{W} \subseteq \bigcap \mathcal{W}$ in such a way that, for some $\mathcal{W}^* \supseteq \mathcal{W}$, the family $\{\mathcal{V}[x] : x \in S_\mathcal{W}\}$ is maximal with respect to the property that $\mathcal{V}[p] \cap \mathcal{V}[q] = \mathcal{W}^*$ for distinct $p, q \in S_\mathcal{W}$. If $\bigcap \mathcal{W} = \emptyset$, then we put $S_\mathcal{W} = \emptyset$. Notice that $\mathcal{W} \subseteq \mathcal{V}[x]$ for any $x \in S_\mathcal{W}$. Moreover, if there are distinct $p, q \in X$ satisfying $\mathcal{V}[p] \cap \mathcal{V}[q] = \mathcal{W}$, we will choose $S_\mathcal{W}$ in such a way that $\mathcal{W}^* = \mathcal{W}$ and $\{p, q\} \subseteq S_\mathcal{W}$ (this can clearly be done in this case). Next, let $X_\alpha = \bigcup\{S_\mathcal{W} : \mathcal{W} \subseteq \mathcal{V}_\alpha\}$. The evaluation of the cardinality of the set X_α depends on the following observations:
Fact 1: each $S_\mathcal{W}$ is closed discrete. Indeed, suppose that the point $x \in X$ is an accumulation point of $S_\mathcal{W}$. First of all, we cannot have $\mathcal{V}[x] \subseteq \mathcal{W}^*$ as this would imply $S_\mathcal{W} = \{x\}$. So, fix an element $V \in \mathcal{V}[x] \setminus \mathcal{W}^*$. Since x is an accumulation point of $S_\mathcal{W}$, we may take distinct points $p, q \in V \cap S_\mathcal{W}$. But then $V \in \mathcal{V}[p] \cap \mathcal{V}[q] = \mathcal{W}^*$ - a contradiction.

Fact 2: each $S_{\mathcal{W}}$ has cardinality not exceeding $\kappa\lambda$. Assume $|S_{\mathcal{W}}| > 1$ and, for any $x \in S_{\mathcal{W}}$, select an element $V_x \in \mathcal{V}[x] \setminus \mathcal{W}^*$. Observe that the way we choose V_x guarantees that $V_x \cap S_{\mathcal{W}} = \{x\}$ for each $x \in S_{\mathcal{W}}$. Then, let $\mathcal{U} = \{V_x : x \in S_{\mathcal{W}}\} \cup \{X \setminus S_{\mathcal{W}}\}$. By Fact 1, the collection $\mathcal{U}$ is an open cover of X and so there exists a set $A \subseteq X$ such that $|A| \leq \kappa$ and $\mathcal{U}[A]$ is a cover of X. It is obvious that $|\mathcal{U}[A]| \leq psw(X)|A| \leq \lambda\kappa$. Now, the fact that the map $x \mapsto V_x$ is injective does the rest.
Fact 3: the set $S_{\mathcal{W}}$ is not empty only if $|\mathcal{W}| \leq \lambda$ and consequently the cardinality of the family $\{S_{\mathcal{W}} : \mathcal{W} \subseteq \mathcal{V}_\alpha\}$ does not exceed $|\mathcal{V}_\alpha|^\lambda$.

By the inductive assumptions, we have $|X_\beta| \leq \kappa^\lambda$ for each $\beta < \alpha$ and this in turn implies that $|\mathcal{V}_\alpha| \leq \lambda\lambda\kappa^\lambda = \kappa^\lambda$. The latter formula, together with Facts 2 and 3, implies that the set $X_\alpha = \bigcup\{S_{\mathcal{W}} : \mathcal{W} \subseteq \mathcal{V}_\alpha\}$ has actually cardinality not exceeding $\kappa\lambda(\kappa^\lambda)^\lambda = \kappa^\lambda$.

To finish the proof, it is enough to check that $X = \bigcup\{X_\alpha : \alpha < \lambda^+\}$. Assume the contrary, and let $p \in X \setminus \bigcup\{X_\alpha : \alpha < \lambda^+\}$. Let $\mathcal{W} = \mathcal{V}[p] \cap \mathcal{V}[\bigcup\{X_\alpha : \alpha < \lambda^+\}]$. Since $|\mathcal{W}| \leq \lambda$, there exists some $\alpha < \lambda^+$ such that $\mathcal{W} \subseteq \mathcal{V}[\bigcup\{X_\beta : \beta < \alpha\}] = \mathcal{V}_\alpha$. Since $\mathcal{W} \subseteq \mathcal{V}[p]$, it is clear that the set $S_{\mathcal{W}}$ is certainly non-empty. Take any $q \in S_{\mathcal{W}}$ and observe that, by construction, we have $q \in X_\alpha$. Taking into account that $\mathcal{W} \subseteq \mathcal{V}[q]$, we have $\mathcal{W} \subseteq \mathcal{V}[p] \cap \mathcal{V}[q] \subseteq \mathcal{V}[p] \cap \mathcal{V}_{\alpha+1} \subseteq \mathcal{W}$. This implies $\mathcal{V}[p] \cap \mathcal{V}[q] = \mathcal{W}$ and consequently the condition we imposed in the choice of $S_{\mathcal{W}}$ gives $\mathcal{W}^* = \mathcal{W}$. Therefore, for any distinct $q, q' \in S_{\mathcal{W}}$ we have $\mathcal{V}[q] \cap \mathcal{V}[q'] = \mathcal{W}$. But, we have already shown that the formula $\mathcal{V}[p] \cap \mathcal{V}[q] = \mathcal{W}$ holds for any $q \in S_{\mathcal{W}}$ while $p \notin S_{\mathcal{W}}$. This contradicts the maximality of $S_{\mathcal{W}}$ and the proof is then complete. □

In connection with formula(1), it seems appropriate here to mention the question whether the inequality $|X| \leq e(X)^{\Delta(X)}$ holds, for every T_1 space X. This problem appeared ten years ago in [1] and it is still unsolved (we warn the interested reader that the other problems asked in [1] have already been solved). The motivations for the previous question are described in [1]. Here, $e(X)$ and $\Delta(X)$ denote extent and diagonal degree of X.

It is to be remarked that the possible stronger version of the above problem involving the weak extent has definitely a negative answer, at least for Hausdorff spaces. Indeed, in [4] it was pointed out that the Katetov's extension $K(\mathbb{N})$ of the discrete space $\mathbb{N}$ is a separable Hausdorff space with a G_δ-diagonal and cardinality $2^{\mathfrak{c}}$. Being separable, such a space has countable weak extent and therefore even the inequality $|X| \leq 2^{we(X)\Delta(X)}$ may fail for Hausdorff spaces.

Acknowledgments. The authors wish to thank the Referee for the useful comments and remarks.

References

[1] A. Bella, *A couple of questions concerning cardinal invariants*, Q&A in General Topology **14** (1996), 139-143.

[2] R. Engelking, *General Topology* Heldermann-Verlag, Berlin (1989).

[3] R. Engelking and M. Karlowicz, *Some theorems of set theory and their topological consequences*, Fund. Math. **57** (1965), 275-285.

[4] J. Ginsburg and G. Woods, *A cardinal inequality for topological spaces involving closed discrete sets*, Proc. Amer. Math. Soc. **64** (1977), 357-360.

[5] R.G. Hodel, *Cardinal functions I*, in *Handbook of set-theoretic topology* (1984), 1-61, K.Kunen and J.E. Vaughan Editors, Elsevier Science Publishers B.V., Amsterdam.

[6] R. G. Hodel, *Combinatorial set theory and cardinal function inequalities*, Proc. Amer. Math. Soc. **111** (1991), 567-575.

[7] E. Michael, *A note on intersection*, Proc. Amer. Math. Soc. **13** (1962), 281-283.

Received June 20, 2005.