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Stabilizing Nonlinear Model Predictive Control in Presence of Disturbances and Off-line Approximations of the Control Law

Settore scientifico-disciplinare ING-INF/04

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A Frosina e alla mia famiglia

Abstract

One of the more recent and promising approaches to control is the Receding Horizon one. Due to its intrinsic characteristics, this methodology, also know as Model Predictive Control, allows to easily face disturbances and model uncertainties: indeed at each sampling instant the control action is recalculated on the basis of the reached state (closed loop). More in detail, the procedure consists in the minimization of an adequate cost function with respect to a control input sequence; then the first element of the optimal sequence is applied. The whole procedure is then continuously reiterated.

In this thesis, we will focus in particular on robust control of constrained systems. This is motivated by the fact that, in practice, every real system is subjected to uncertainties, disturbances and constraints, in particular on state and input (for instance, plants can work without being damaged only in a limited set of configurations and, on the other side, control actions must be compatible with actuators' physical limits).

With regard to the first aspect, maintaining the closed loop stability even in presence of disturbances or model mismatches can result in an essential strategy: moreover it can be exploited in order to design an approximate stabilizing controller, as it will be shown. The control input values are obtained recurring to a Nearest Neighbour technique or, in more favourable cases, to a Neural Network based approach to the exact RH law, which can be then calculated off line: this implies a strong improvement related to the applicability of MPC policy in particular in terms of on line computational burden. The proposed scheme is capable to guarantee stability even for systems that are not stabilizable by means of a continuous feedback control law.

Another interesting framework in which the study of the influence of un-

certainties on stability can lead to significant contributions is the networked MPC one. In this case, due to the absence of physical interconnections between the controller and the systems to be controlled, stability can be obtained only taking into account of the presence of disturbances, delays and data losses: indeed this kind of uncertainties are anything but infrequent in a communication network. The analysis carried out in this thesis regards interconnected systems and leads to two distinct procedures, respectively stabilizing the linear systems with TCP protocol and nonlinear systems with non-acknowledged protocol. The core of both the schemes resides in the online solution of an adequate reduced horizon optimal control problem.

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The first person I would like to thank, is my superb supervisor and great friend, Thomas Parisini; despite being very often busy in very important scientific and managerial businesses, he was - and is - always ready to help (with always focused and effective ideas and remarks) and to support me, both in my scientific and in my hobbies related activities. And it is a pleasure to spend time with him, during working time and during dinners, skiing excursions and laboratory's *rebechin* (which is something similar to a buffet), as well.

Then, there's another great friend and colleague who helped me in my research and studies in the last three years: my friend Gilberto Pin, with whom I spent a lot of time on science production and in relax too (he is the only person who trusted in my capabilities as a soccer player :D).

I would then tank very much Frosina, who has been very understanding and supporting during my PhD course and in particular in the last period. She is a very special person (there's a reason if she is my girlfriend :D). A great tank you to my family, including my just born niece, Sara, who cries each time she sees the uncle! (My sister will get angry as she will read this, but I'm an Engineer, for me causality means :D). The support I had from these persons has been fundamental in becoming what I am.

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My best friends, Nicola Rizzo, Ale Maco, Mauro Casa and their partners, Federica, Manu, Marco Argenton, Tuoni; my flatmates, Claudio Favi, Stefano Ave, Matteo ed Elia, and especially Luisa e Mirella!

My colleagues Giorgio Davanzo, Federico Cervelli, Andrea Petronio, all the PhD students met at Bertinoro's PhD school (among the others, Andrea Ferrise - Corleone, Fabio, Francesco e Vincenzo, Pietro).

Time is running by and I have to deliver the thesis, so I have to call it quits. I'm sorry for who I hadn't time to mention; nevertheless I always remeber you all and I thank you all very much.

(As is can be easily deducted, this is the only part of the thesis I didn't give to anyone to check, so I apologise for the not-so-fluent English in this section :D).

Contents

Chapter 1

Introduction and a glimpse on problem formulation

Model Predictive Control (MPC) is one of the most recently introduced techniques in the framework of controls. It consists in the minimization with respect to a (admissible) sequence of inputs of a suitable cost function related to the plant to be controlled; at each sampling instant only the first element of the optimal sequence is applied to the system. This operation is repeated at each step. Being the optimization carried out over a finite horizon sliding forward of a sample at each sampling instant, the technique is also known as Receding Horizon Control.

Due to the computational burden required for this approach, MPC could be applied only after the introduction of microprocessors (early '70s). Applications in practice appeared in oil refineries (regarding petrochemical industries, system dynamics are often slow enough to admit the application of MPC) and process industries ([94]): in these situations, in which plants are required to work close to the boundary of the admissible operating region in order to maximize the economical efficiency, a dynamic controller (such as an MPC derived, possibly in closed loop with respect to state measurements) turns out to be the best choice. A very interesting survey on industrial MPC applications can be found in [88].

In view of the design of a RH controller, many aspects have to be considered: first of all, the evaluation of the optimal control sequence requires an accurate knowledge of the system dynamics: indeed the future state predictions are calculated by means of an analytical model of the plant. While in some cases a precise description of the system is a reasonable request (such as in the aero-spatial framework, in which the physics of each process is thoroughly analysed), in most applications this is not possible. The presence of these uncertainties can lead to non-efficient stabilization or to instability. Secondly, even when modeling inaccuracies could be considered negligible, in practice it is always necessary to take into account the influence of disturbances on stability.

The above stated considerations enhance the importance of the study of robust stability in the MPC framework. Having said this, this thesis is intended to deepen and analyze the influence of disturbances on the stability of a system controlled by a MPC approach. In particular, the obtained results will be useful when addressing real plants, which very often present nonlinearities, constraints (for instance with regards to actuators) and many different kind of uncertainties (in this regard, often nonlinear plants can be considered as linear with bounded disturbances).

For instance, it will be shown how, under some suitable conditions that will be specified further on, it is possible to approximate the optimal RH control law still preserving the robust stability properties of the controlled plant; the reason for the approximation resides in the need for the reduction of the required on-line calculations.

A recent survey on approaches to robust MPC stability (min-max case) can be found in [91]); nevertheless, the optimization can become a very difficult problem to solve. The minimization of the cost function, especially for nonlinear systems with "fast dynamics", can be indeed a too difficult task to accomplish, not allowing the on-line use of MPC strategy; on the other hand, having the possibility to use a precomputed law will result in a significant increase of the applicability of RH approach. It will be then possible to use the RH control policy even on fast nonlinear systems, also when these are not stabilizable with a continuous feedback controller and in presence of bounded disturbances. As we will see, the main drawback resides in the request of memory: a great number of values of the exact control function can be necessary, depending on the considered plant. Nevertheless, when the function to be approximated is sufficiently smooth, it is possible to resort to neural networks, thus obtaining a controller with reduced needs in terms of both time and storage requirements.

Another interesting field in which robustness can play an important role is the networked control one. In this area of interest, one of the most relevant problems for stability is the presence of delays: it is necessary in practice to take into account this aspect in order to design a stabilizing networked controller. In order to face this problem, two RH based policies will be proposed capable to robustly stabilize networked interconnected systems with both TCP or UDP protocol implementation; moreover, constraints on state and inputs will be robustly enforced by means of a constraint tightening technique.

In the rest of the chapter a simple and short introduction to MPC will be presented, together with some considerations on the most frequent approaches to robust RH stabilization of constrained systems.

1.1 General MPC Formulation for Discrete-Time Systems

Consider the generic discrete-time system:

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t, \mathbf{u}_t), \qquad t = 0, 1, \ldots, \tag{1.1}$$

where $\mathbf{x}_t$ represents the state vector at the time instant t, while $\mathbf{u}_t$ is the input vector (control) at time t. For the moment we don't make any assumption with respect to constraints on states and inputs, i.e.

$$\mathbf{x}_t \in \mathbb{R}^n, \mathbf{u}_t \in \mathbb{R}^m.$$

Assume the case is that of the zero-regulation, $\mathbf{f}(0, 0) = 0$; in other words we consider the origin as an equilibrium state in absence of inputs.

In view of the formulation of the RH control policy, consider now the following optimization problem:

Problem 1.1.1 (Finite Horizon Optimal Control Problem - FHOCP)**.** *Given a positive integer $N_c \geq 1$ (Control Horizon), a vector $\mathbf{x}_t \in \mathbb{R}^n$, a transition cost function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, a terminal cost function $h_f : \mathbb{R}^n \to \mathbb{R}$ and a terminal set $X_f \subseteq \mathbb{R}^n$, indicating with $\mathbf{u}_{t,t+N_c-1}$ a sequence of $N_c - 1$ input variables starting from instant t, the Finite Horizon Optimal Control Problem consists in the minimization of the following cost function with respect to $\mathbf{u}_{t,t+N_c-1}$ at every time instant $t \geq 0$:*

$$J_{FH}(\mathbf{x}_t, \mathbf{u}_{t,t+N_c-1}, N_c) = \sum_{i=t}^{t+N_c-1} h(\mathbf{x}_i, \mathbf{u}_i) + h_f(\mathbf{x}_{t+N_c}) \tag{1.2}$$

subject to

- *the nominal system dynamics,*
- *the terminal state constraints $\mathbf{x}_{t+N_c} \in X_f$,*

with

$$h(\mathbf{0}, \mathbf{0}) = 0, \qquad h_f(\mathbf{0}) = 0.$$

Now, the Receding Horizon algorithm consists in solving at each time instant t the FHOCP, thus finding the optimal control sequence $\mathbf{u}^o_{t,t+N_c-1}$; then the input applied to the system is the first element of $\mathbf{u}^o_{t,t+N_c-1}$:

$$\mathbf{u}_t = \mathbf{u}^o_{t,t} \triangleq \kappa_{RH}(\mathbf{x}_t),$$

where $\kappa_{RH}(\mathbf{x}_t)$ can be used in the closed-loop form of the problem to highlight the fact that the control is obtained as a state-feedback.

The RH policy is graphically depicted in Figure 1.1.

At each sampling instant the minimization of a proper cost function is then required. This task can be viable only in some cases, i.e. when N_c is small, the sampling frequency is very low, the system has slow dynamics and so on. Nevertheless, in most cases the solution of the problem in real time is really difficult.

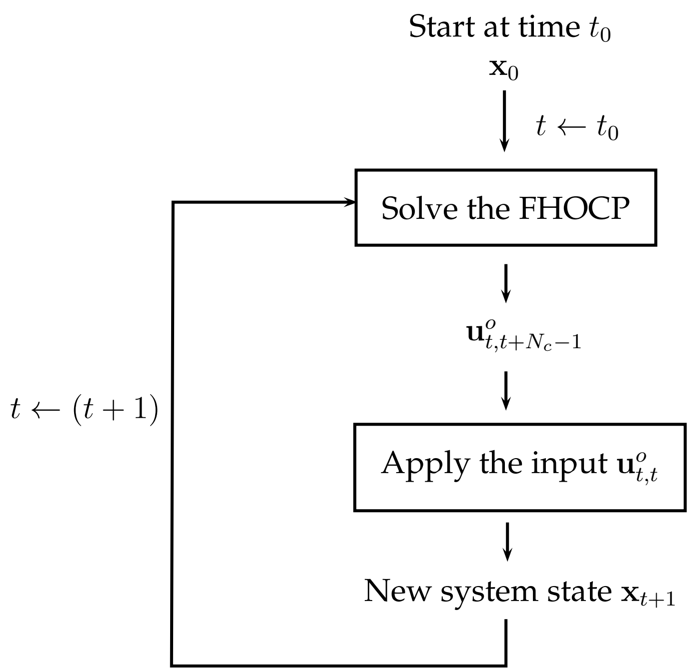

Figure 1.1: Graphical representation of MPC scheme.

Furthermore, the aforementioned formulation is the most general and maybe the simplest one: it refers to unconstrained systems with dynamics exactly described by a precisely known model, in absence of any kind of disturbance. When facing real plants, these requirements easily turn out to be excessive, making this approach unuseful: for example suffice it to think that almost every physical actuator is subjected to saturations and tolerances.

The problem has then to be extended in order to meet the requests of the real cases. The first step is the inclusion in the RHOCP of the constraints: in particular the minimization of (1.2) will be required to be subjected also to:

- $\mathbf{x}_i \in X \subset \mathbb{R}^n \; \forall i = 1, \ldots, N_c$,
- $\mathbf{u}_i \in U \subset \mathbb{R}^m$, $\forall i = 1, \ldots, N_c - 1$,

with X and U compact sets containing the origin in their interior.

The next step is the introduction of uncertainties: this aspect is the argument of the following section.

1.2 Uncertainties in Model Predictive Control

One of the most important aspects to face in the application of the MPC is the presence of uncertainties that typically affect the model of practically relevant systems. Basically these sources of uncertainty can be divided in two main groups:

- Model uncertainties;
- Disturbances affecting the whole system or part of it.

With regard to both classes, we can state that RH techniques represent one of the most effective approaches to robust control. Indeed, the policy is based, on the one hand, on the knowledge of a model of the system to control, and on the other, on the step by step recalculation of the optimal control sequence. This way, if at a certain instant a disturbance occurs, at the next instant the system will modify the previously calculated input sequence considering the new state reached; from this perspective, a model mismatch can be considered in the same way as a persistent disturbance.

Regarding this solution, it is now necessary to introduce in the MPC formulation the *predicted state* $\hat{\mathbf{x}}_i$ and to distinguish between the real system (characterized by its state function $f(\cdot)$) and its *nominal* model used for control design purposes, with state function $\hat{f}(\cdot)$. Moreover, let's consider a positive scalar $N_p \geq N_c$ (*prediction horizon*; see [67]) and an *auxiliary control law* $\mathbf{u}_i = \kappa_f(\mathbf{x}_i), \quad i = t + N_c, \ldots, t + N_p - 1$, to be intended as a control law acting between the instants $t + N_c$ and $t + N_p - 1$, which has to be adequately designed in order to guarantee suitable stabilizing properties for the overall RH control law.

Now, it is possible to state the Finite Horizon Optimal Control Problem in a more general form, considering the recourse to the nominal model, the introduction of the constraints and the presence of uncertainties and disturbances.

Hence, consider the system

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t, \boldsymbol{\nu}_t), \quad \forall t \in \mathbb{Z}_{\geq 0}, \tag{1.3}$$

where $\mathbf{x}_0 = \overline{\mathbf{x}}_0$ and ν_t is an uncertain input vector taking values in the compact set $\Upsilon \subset \mathbb{R}^q$ containing the origin in its interior. Suppose that the following constraints hold:

$$\mathbf{x}_t \in X \subset \mathbb{R}^n \quad \forall t \in \mathbb{Z}_{\geq 0}, \tag{1.4}$$

$$\mathbf{u}_t \in U \subset \mathbb{R}^m \quad \forall t \in \mathbb{Z}_{\geq 0}, \tag{1.5}$$

with $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ compact sets containing the origin in their interior.

Let $\hat{f}(\mathbf{x}, \mathbf{u})$ denote the nominal model of system (1.3), with $\hat{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. By denoting with $\hat{\mathbf{x}}_{t+k|t}$ the k-steps ahead prediction of $\mathbf{x}_t$ by means of $\hat{f}$, starting from instant t once a given input sequence $\mathbf{u}_{t,t+k-1}$ has been applied, we can state the following FHOCP.

Problem 1.2.1 (Finite Horizon Optimal Control Problem - FHOCP)**.** *Given two strictly positive integers N_c and N_p, a state measurement $\mathbf{x}_t \in \mathbb{R}^n$ at time t, a transition cost function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, an auxiliary control law $\kappa_f(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$, a terminal cost function $h_f : \mathbb{R}^n \to \mathbb{R}$ and a terminal set $X_f \subset \mathbb{R}^n$ (compact), the Finite Horizon Optimal Control Problem consists in the minimization of the following cost function with respect to the sequence of control inputs $\mathbf{u}_{t,t+N_c-1}$ at every time instant $t \geq 0$:*

$$J_{FH}(\mathbf{x}_t, \mathbf{u}_{t,t+N_c-1}, N_c, N_p) = \sum_{i=t}^{t+N_c-1} h(\hat{\mathbf{x}}_{i|t}, \mathbf{u}_i) + \sum_{i=t+N_c}^{t+N_p-1} h(\hat{\mathbf{x}}_{i|t}, \kappa_f(\hat{\mathbf{x}}_{i|t})) + h_f(\hat{\mathbf{x}}_{t+N_p|t}) \tag{1.6}$$

subject to

1. *the nominal system dynamics, starting from $\hat{\mathbf{x}}_{t|t} = x_t$;*

2. *the state and control constraints, $\mathbf{u}_{j-1} \in U$ and $\hat{\mathbf{x}}_{t+j|t} \in X$, $\forall j \in \{1, \ldots, N_c\}$;*

3. the terminal state constraints $\hat{\mathbf{x}}_{t+N_p|t} \in X_f$.

Again, at each step the Problem (1.2.1) has to be solved and the first element of the optimal control sequence has to be applied to the system. Notice that, due to the finiteness of the horizon, for the FHOCP the minimum exists if the functions $h(\cdot)$, $h_f(\cdot)$ and $f(\cdot)$ are continuous and all the sets are compact.

Due to the presence of the constraints, the optimization could lead to admissible solutions only for a subset of the states in the domain; the set in which the FHOCP is feasible is called *admissible set* and will be indicated with X_{RH}.

1.3 Setting up the parameters for stability

As already stated, the theoretical approach to the study of Receding Horizon Control policies is more recent than the application of the technique itself; one of the more studied aspects in this framework regards the stability analysis of systems controlled in this way; indeed the RH control strategy does not guarantee the closed loop system to be stable, if some adequate expedients are not applied (see [13]).

In the literature, many approaches to the problem of finding sufficient conditions for closed-loop stability have been proposed, essentially differing in the choice of the parameters composing the cost function (1.6), and in particular of:

- the stage cost $h(\cdot)$;
- the terminal cost $h_f(\cdot)$;
- the terminal set X_f.

A survey on the design of stabilizing control Receding Horizon law can be found for example in [69] and references therein; let's now just recall some important ideas on the more common choices for the three parameters. In the sequel we will refer to nonlinear systems; in the linear case, the following analysis still holds, however some more specific considerations could lead to a dramatic simplification of the feasibility, closed-loop stability and optimization related problems (see for instance [73] and references therein, [74],[25]).

1.3.1 Nominal case without and with constraints

Consider the case of absence of uncertainties and disturbances in an unconstrained system; the more common strategies adopted are the following:

- *Terminal equality constraint*: in this approach the set X_f contains only the point $\{0\}$. This way the state is forced to reach the origin at the end of the control horizon. This idea was firstly proposed in [55], where a stability analysis of this choice is carried out; moreover, another important result of this paper resides in proving that the optimal value of the associated cost function $J_{FH}(\mathbf{x}_t, \mathbf{u}_{t,t+N_c-1}, N_c)$ approaches to that of the infinite horizon problem as the horizon approaches infinity. The terminal equality constraint has been considered and studied as well in some other important works, among the others [27] or [70].

- *Terminal cost*: many authors propose convenient criteria to design this parameter in order to get stability results for the system (for example choosing $h_f(\cdot)$ as a global Lyapunov function, whenever the system is globally stabilizable). Constraints on set X_f fall into decay. Some works with results based on this approach are for example [82], [67], [40]. Often (as proved in the two last cited papers) the prediction horizon N_p has to be sufficiently large in order to guarantee the stability of the controlled system.

- *Terminal constraint set*: the terminal cost term is removed from the cost function, while X_f if chosen as a neighbourhood of the origin. The state is steered to X_f in a finite time and then an auxiliary local stabilizing controller is applied to the system, as proposed for the first time in [72]; this strategy is also known as *Dual Mode* and is often used to provide ISS properties, specially in the case of robust min-max receding horizon controllers (see for example [91] and references therein).

With regards to the constrained case, most of the works have appealed to the imposition of both a terminal cost and terminal constraints, while the state and control constraints have to be added to the optimization problem.

One of the earliest examples of this strategy is the so called *quasi-infinite horizon* predictive control, firstly proposed in [21]. Here, in presence of constraints only on the input set, the authors choose $X_f \in X$ as a positively invariant set for the system and the auxiliary (locally stabilizing) control law such that $\kappa_f(x) \in \mathbb{U}$, $\forall x \in X_f$; regarding h_f, it is chosen as an appropriate local Lyapunov function for the linearized system in a neighbourhood of the origin.

Other interesting works on this framework are for instance [81] or [82], where both the input and the output vectors are subjected to constraints: the proposed auxiliary controller is a locally stabilizing control law, while $h_f(\cdot)$ is a local Lyapunov function for the stabilized system and X_f is chosen as a level set of $h_f(\cdot)$.

As already said, when tackling systems of practical interest it is always necessary to take into account model uncertainties and disturbances: let's now review some important approaches to robust stability in the RH control framework.

1.3.2 Robustness and stability

Guaranteeing stability even in presence of disturbances and uncertainties is one of the most important and recent research topics in the MPC framework. Earlier works (for example [27] or [98]) deal with unconstrained systems and get to estabilish that, given a Lyapunov function for a RH controlled system, if the function preserves its qualities for bounded disturbances, then the system is robust with respect to disturbances within the bound.

When constraints are present, the study of robustness make use of the optimization of a min-max problem (taking the cue from [71]): the idea resides in the minimization of the performance index with respect to the control sequence and at the same time its maximization with respect to the disturbance sequence over the optimization horizon. Then the enforcement of the constraints is imposed for each realization of the uncertainties; as it is easy to imagine, this implies a significant increase of the computational burden. A recent overview on the main stability results for this approach is in [90]: in that work, the authors carry out a unifying analysis on the stabilizing properties of a Min-Max controller, using the concepts of (regional) Input-to-State Stability (ISS) and Input-to-State Practical Stability (ISpS). Notably, standard Min-Max is capable to guarantee only the ISpS property (for example, for state independent disturbances see [63]); in order to achieve the ISS, two approaches have been proposed: the choice of an $\mathcal{H}_\infty$ stage cost (as in [45]) and a dual mode strategy (for ex. [72]).

In order to reduce the computational requests of the Min-Max approach, it is possible to think about another solution: the constraint tightening technique (see for instance [23],[62]). In this case, at each step, constraints set are tightened and the cost function is minimized with respect to the new constraints; this way the stability of the resulting optimal trajectory can be guaranteed with the enforcement of the initial constraints. This implies a dramatic reduction of the computational burden of the optimization; on the other hand, the main drawback of this approach resides in the fact that the strategy is fairly conservative: indeed due to the restrictiveness of the new requirements (only the worst case disturbances are considered), many trajectories could be rejected and furthermore, problems can turn out to be unfeasible. An application of this technique can be found for instance in [62], in which the authors propose some approaches capable of reducing the maximal bound on the one-step prediction of the state in presence of uncertainties (choosing a proper norm, adding a pre-compensator, using local approximations), thus diminishing the conservativeness of the method. In view of the on line application, getting over the limitations of this approach constitutes a challenging field on research ([59], [41]).

In the rest of the thesis robustness will be considered as a fundamental request for the control of systems. This will allow to design off-line RH controllers capable to robustly stabilize systems with fast dynamics.

In the networked case the robustness will be exploited to stabilize both linear and nonlinear systems even in presence of delays. In the next section the thesis structure will be depicted.

1.4 Thesis Structure

This work of thesis is divided in two main parts: the first one regards the approximation of MPC, in order to fasten its practical application to systems without losing stability and robustness properties. The starting point of the proposed approach is to find bounds on admissible model mismatches and disturbances affecting the system to be controlled such that robust stability properties for the RH controlled plant keep holding; based on these bounds, suitable considerations lead to the calculation of the parameters of an adequate grid to be superimposed on the domain (constraint set on state); the value of the first element of the optimal control sequence according to the MPC cost function is then stored for each node of the grid. At this point, if the considered system turns out to be non stabilizable by a continuous feedback control law, at each sampling instant the control input is valued on the basis of the Nearest Neighbour approach on the actual state measurement with respect to the grid points. On the other hand, if the RH control law is sufficiently smooth, the approximated control law can be derived by means of a Neural Network trained on the stored values. A theoretical analysis proves that both approaches assure robust stability for the systems under concern. Simulations results confirm this fact. In addition to this, they endorse the improvements of the method (especially in terms of dramatic requested time reduction) and its applicability.

The second part deals with networked MPC of interconnected systems. In this case, direct (physical) links between the controller and the subsystems are not present anymore. This implies, among others, the drawback of the necessity to handle transmission delays: these kind of uncertainties can indeed easily lead the system to instability. It will be shown that adequate modifications of the RH cost function and control and prediction horizons can face this problem getting the overall plant to a stable equilibrium (input to state); the two proposed schemes are capable to stabilize respectively linear systems with TCP transmission protocol and nonlinear systems with UDP - non acknowledged protocol.

All the proofs of the stated propositions, claims, lemmas and theorems can be found in the Appendix at the end of this Chapter: this solution was adopted in order to improve the overall readability.

1.5 Related articles

The results presented in this Thesis are based on the following publications of the author.

G. Pin, M. Filippo, F. A. Pellegrino, T. Parisini. Approximate Off-Line Receding Horizon Control of Constrained Nonlinear Discrete-Time Systems. *Proc. of the European Control Conference*, pages: 2420-2425, Budapest, Hu, 2009.

G. Pin, M. Filippo, T. Parisini. Networked MPC for constrained linear systems: a recursive feasibility approach. *Proc. of the IEEE Conf. on Decision and Control*, pages: 555-560, Shangai, China, 2009.

G. Pin, M. Filippo, F. A. Pellegrino, G. Fenu, T. Parisini. Approximate Off-Line Receding Horizon Control of Constrained Nonlinear Discrete-Time Systems: Smooth Approximation of the Control Law. *Proc. of the American Control Conference*, Baltimore, Maryland, USA, 2010.

G. Pin, M. Filippo, T. Parisini. A Coordinated Nonlinear Model Predictive Control Scheme Over Non-acknowlegded Networks. *Proc. of 8th IFAC Symposium on Nonlinear Control Systems*, Bologna, Italy, 2010.

G. Pin, M. Filippo, F. A. Pellegrino, G. Fenu, T. Parisini. Approximate Off-Line Receding Horizon Control of Constrained Nonlinear Discrete-Time Systems. Submitted in revised form (as a full paper) to the *IEEE Transactions on Automatic Control.*

Chapter 2

Notations and Basic Definitions

2.1 Notations

In the present Thesis, the following notations will be adopted.

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively.
The Euclidean norm is denoted as $|\cdot|$.

For any discrete-time sequence $\boldsymbol{\varsigma} : \mathbb{Z}_{\geq 0} \to \mathbb{R}^r$, $r \in \mathbb{Z}_{>0}$, let $\|\boldsymbol{\varsigma}\| \triangleq \sup_{t\geq 0}\{|\varsigma_k|\}$ and $\|\boldsymbol{\varsigma}_{[\tau]}\| \triangleq \sup_{0\leq t\leq \tau}\{|\varsigma_k|\}$, where ς_k denotes the value that the sequence $\boldsymbol{\varsigma}$ takes on in correspondence with the index k. The set of discrete-time sequences of $\boldsymbol{\varsigma}$ taking values in some subset $\Upsilon \subset \mathbb{R}^r$ is denoted by $\mathcal{M}_\Upsilon$, while $\Upsilon^{sup} \triangleq \sup_{\varsigma\in\Upsilon}\{|\varsigma|\}$.

The symbol id represents the identity function from $\mathbb{R}$ to $\mathbb{R}$, while $\gamma_1 \circ \gamma_2$ is the composition of two functions γ_1 and γ_2 from $\mathbb{R}$ to $\mathbb{R}$.

Given a set $A \subseteq \mathbb{R}^n$, $\mathrm{int}(A)$ denotes the interior of A, while its boundary will be indicated with ∂A. Given a vector $x \in \mathbb{R}^n$, $\mathrm{d}(x, A) \triangleq \inf\{|\xi - x|, \xi \in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to A. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, $\mathrm{dist}(A, B) \triangleq \inf\{\mathrm{d}(\zeta, A), \zeta \in B\}$ is the minimal set-to-set distance. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A \backslash B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, the Pontryagin difference set C is defined as $C = A \backsim B \triangleq \{x \in \mathbb{R}^n : x + \xi \in A, \forall \xi \in B\}$, while the Minkowski sum set is defined as $S = A \oplus B \triangleq \{x \in \mathbb{R}^n : \exists \xi \in A, \eta \in B, x = \xi + \eta\}$.

Given a vector $\eta \in \mathbb{R}^n$ and a positive scalar $\rho \in \mathbb{R}_{>0}$, the closed ball in $\mathbb{R}^r$ centered in η and of radius ρ , is denoted as $\mathcal{B}^r(\eta, \rho) \triangleq \{\xi \in \mathbb{R}^r : |\xi - \eta| \leq \rho\}$. The shorthand $\mathcal{B}^r(\rho)$ is used when the ball is centered in the origin.

Given m column vectors $v_1 \in \mathbb{R}^{n_1}, \ldots, v_m \in \mathbb{R}^{n_m}$, let $\mathrm{col}[v_1, \ldots, v_m]$ denote the column stacking operator.

The domain of a generic function $\kappa : \mathbb{R}^n \to \mathbb{R}^r$ will be denoted as $\mathrm{dom}(\kappa)$.

2.2 Basic definitions

2.2.1 Comparison functions

The stability properties of nonlinear systems are often characterized in terms of comparison functions: let's define them.

Definition 2.2.1 ($\mathcal{K}$ function)**.** *A function $\alpha(\cdot) : [0, a) \to [0, \infty)$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$.*
If $a = \infty$ and $\lim_{r \to \infty} \alpha(r) = \infty$ then $\alpha(\cdot)$ belongs to class $\mathcal{K}_\infty$.

Definition 2.2.2 ($\mathcal{KL}$ function)**.** *A function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ belongs to class $\mathcal{KL}$ if it is continuous and such that:*

- *for a fixed scalar s, $\beta(r, s)$ is a $\mathcal{K}$-class function with respect to r;*
- *for a fixed scalar r, $\beta(r, s)$ is monotonically decreasing with respect to s, with $\lim_{s \to \infty} \beta(r, s) = 0$.*

It is worth to recall some properties of comparison functions, useful when proving stability for a system.

Given two $\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, a $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and two positive scalars r and q, the following properties hold:

- $\alpha_1^{-1}(\cdot)$ is a $\mathcal{K}$ function;
- $\alpha_1 \circ \alpha_2(\cdot)$ is a $\mathcal{K}$ function;
- $\alpha_1 \circ \beta(\cdot,\cdot)$ is a $\mathcal{KL}$ function;
- $\max(\alpha_1(r), \alpha_2(r))$ and $\min(\alpha_1(r), \alpha_2(r))$ are $\mathcal{K}$ functions;
- $\alpha_1(r+q) \le \alpha_1(2r) + \alpha_1(2q)$;
- $\alpha_1(r) + \alpha_2(q) \le \alpha_1(r+q) + \alpha_2(r+q)$.

2.2.2 Invariance and Input-to-State Stability

Let us consider the discrete-time dynamic system

$$x(t+1) = g(x_t, \varsigma_t),\ t \in \mathbb{Z}_{\ge 0},\ x_0 = \bar{x}\,, \tag{2.1}$$

with $g(0,0) = 0$ and where $x_t \in \mathbb{R}^n$ and $\varsigma_t \in \Upsilon \subset \mathbb{R}^r$ denote the state and the bounded input of the system, respectively. The discrete-time state trajectory of the system (2.1), with initial state $\bar{x}$ and input sequence $\boldsymbol{\varsigma} \in \mathcal{M}_\Upsilon$, $\boldsymbol{\varsigma} = \{\varsigma_t,\, t \in \mathbb{Z}_{\ge 0}\}$, is denoted by $x_t = x(\bar{x}, \boldsymbol{\varsigma}, t),\ t \in \mathbb{Z}_{\ge 0}$.

Definition 2.2.3 (Positively Invariant set [15])**.** *A set $\Xi \subset \mathbb{R}^n$ is said Positively Invariant for system (2.1) if $g(x_t, 0) \in \Xi$, $\forall x_t \in \Xi$.* □

Definition 2.2.4 (Robust Positively Invariant set)**.** *A set $\Xi \subset \mathbb{R}^n$ is said Robust Positively Invariant for system (2.1) if $g(x_t, \varsigma_t) \in \Xi$, $\forall x_t \in \Xi$ and $\forall \varsigma_t \in \Upsilon$.*

□

Definition 2.2.5 (ISS-Lyapunov Function [33, 68])**.** *Given system (2.1) and a pair of compact sets $\Xi \subset \mathbb{R}^n$ and $\Omega \subseteq \Xi$, with $\{0\} \subset \Omega$, a function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is called a (Regional) ISS-Lyapunov function in Ξ, if there exist some $\mathcal{K}_\infty$-functions α_1, α_2, α_3, and a $\mathcal{K}$-function σ such that* *i) the following inequalities hold* $\forall \varsigma \in \Upsilon$

$$V(\xi) \geq \alpha_1(|\xi|), \qquad \forall \xi \in \Xi, \tag{2.2}$$

$$V(\xi) \leq \alpha_2(|\xi|), \qquad \forall \xi \in \Omega, \tag{2.3}$$

$$V(g(\xi, \varsigma)) - V(x) \leq -\alpha_3(|\xi|) + \sigma(|\varsigma|), \ \forall \xi \in \Xi, \tag{2.4}$$

ii) there exists a suitable $\mathcal{K}_\infty$*-function* ρ *(with* ρ *such that* $(id - \rho)$ *is a* $\mathcal{K}_\infty$*-function, too) such that the following compact set*

$$\Theta \subset \{\xi : \xi \in \Omega, d(\xi, \partial\Omega) > c\}$$

can be defined for some arbitrary constant $c \in \mathbb{R}_{>0}$*:*

$$\Theta \triangleq \{\xi : V(\xi) \leq b(\Upsilon^{sup})\},$$

where

$$b(s) \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma(s), \quad \alpha_4 \triangleq \alpha_3 \circ \alpha_2^{-1}.$$

□

Definition 2.2.6 (Regional ISS and ISpS [68])**.** *Given a compact set* $\Xi \subset \mathbb{R}^n$*, if* Ξ *is RPI for (2.1) and if there exist a* $\mathcal{KL}$*-function* β*, a* $\mathcal{K}$*-function* γ *and a positive number* $\eta \in \mathbb{R}_{>0}$ *such that*

$$|x(\bar{x}, \boldsymbol{\varsigma}, t)| \leq \max\left\{\beta(|\bar{x}|, t), \ \gamma(\|\boldsymbol{\varsigma}_{[t]}\|)\right\} + \eta, \quad \forall t \in \mathbb{Z}_{\geq 0}, \forall \bar{x} \in \Xi, \tag{2.5}$$

then the system (2.1), with $\boldsymbol{\varsigma} \in \mathcal{M}_\Upsilon$*, is said to be Input-to-State practically Stable (ISpS) in* Ξ*. If* $\{0\} \in \Xi$ *and inequality (2.5) is satisfied for* $\eta = 0$*, then system (2.1) is said to be Input-to-State Stable (ISS) for initial conditions in* Ξ*.* □

The following important result can be proved: it will be used often in stability proofs.

Theorem 2.2.1 (Regional ISS [68])**.** *If system (2.1) admits an ISS-Lyapunov function in* Ξ*, and* Ξ *is RPI for (2.1), then it is Regional ISS in* Ξ *and*

$\lim_{t \to \infty} \mathrm{d}(x(\bar{x}, \boldsymbol{\varsigma}, t), \Theta) = 0.$ □

Finally, consider the following system:

$$x_{t+1} = f(x_t, u_t, \varsigma_t), \; t \in \mathbb{Z}_{\geq 0}, \; x_0 = \bar{x}\,, \tag{2.6}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ the control vector and $\varsigma_t \in \mathbb{R}^r$ an exogenous disturbance input. It will be often useful to make use of the concept of *k-steps Controllability Set*, in particular with $k = 1$. Let's then introduce the corresponding definition.

Definition 2.2.7 ($\mathcal{C}_1(\Xi)$)**.** *Given a set $\Xi \subset \mathbb{R}^n$ and a set $U \subseteq \mathbb{R}^m$, the (one-step) Controllability Set to Ξ, denoted as $\mathcal{C}_1(\Xi)$, is defined as $\mathcal{C}_1(\Xi) \triangleq \{\xi \in \mathbb{R}^n | \exists u \in U : f(\xi, u) \in \Xi\}$, i.e., $\mathcal{C}_1(\Xi)$ is the set of state vectors that can be steered to Ξ by a control action under $f(\xi, u)$.* □

Chapter 3

Approximated Nonlinear MPC

3.1 Introduction

Due to its intrinsic characteristics, MPC became one of the more studied techniques in controls in the last years: as already said, the necessary solution at each sampling instant of an optimal control problem can be exploited in order to design robust controllers for constrained systems. Maybe the most relevant drawback of such an approach is the great demand in terms of computational time and resources, which renders this policy practically unusable on plants with fast dynamics. For systems not suffering of this problem some recent works propose robust Receding Horizon controllers with discontinuous control laws as well ([32],[17], [60]).

Some fast algorithms for on line optimization can be found in literature, e.g.the recent paper ([109]) regarding linear systems. On the other hand, an efficient on line implementation of the technique could be obtained for instance by recurring to explicit control laws (in [34] is proposed an application of a binary search three to Explicit MPC, without considering robustness): in this sense, some approaches based on quadratic costs in the case of linear systems with linear constraints can be found e.g. in [11, 52], in which authors propose parametric quadratic programming techniques. These methods have been used also for the design of robust controllers by recurring to multi-parametric optimization ([10, 24, 76, 51, 26, 47]): in order to fasten up such techniques, an interesting approach is proposed e.g. in [9]. Suboptimality of the control law is present in [111], in which authors combine explicit MPC with online optimization. When the considered system is piecewise affine an interesting method, based on a polynomial approximation of the explicit PWA - MPC control law, is the one proposed in [58].

Nevertheless, with regards to nonlinear systems the explicit solution of MPC problems is still an open issue; to this aim, efficient formulations have been proposed only for special classes of constrained uncertain nonlinear systems ([108, 77, 56]). Moreover, some authors propose the parameterization of the control law, reducing this way the set of optimization variables, as proposed in [3] and more recently in [53], however obtaining a suboptimal control law. It is worth to mention the recent work in [39], in which an interesting approach to the explicit solution of output feedback NMPC problems is proposed with respect to black-box systems.

Other approaches are based on the development of algorithms for fast on-line optimization (see for example [79, 29, 12, 36]; an interesting survey on efficient numerical methods can be found in [28]) or on off-line RH control law approximation schemes able to guarantee the stability of the closed-loop system despite the related errors. Approximation in this case has to be intended as an adequate estimation of the exact control law, obtained without the recourse to the on line solution of a minimization problem, in order to drastically reduce the computational time required. Approximators that can be used are for example Neural Networks (NN), Nonlinear Set Membership ([20, 82, 81]), piecewise linear approximators (in this regard, in [50] the author proposes a method capable to control systems driven by continuous state functions) and many others.

Although the literature concerning the application of approximate nonlinear RH controllers to plants is rich (see [37, 75, 1, 2, 30, 38]), there is a need for a further investigation toward the effect of approximation errors on the stability and robustness of the closed-loop system, in particular when the dynamics are driven by strong nonlinearities and when state and input variables are subject to hard constraints. Indeed, in the constrained framework, the approximated control law is required to guarantee the stability for the closed-loop system while enforcing the robust constraint satisfaction.

In this regard, some works give interesting results under the assumption that the exact RH control law is smooth: for example in [16] authors propose a Set Membership based approximation starting from Lipschitz continuity hypothesis on the control law with respect to the state variables, with known Lipschitz constant. Nevertheless, in practice the optimal MPC law results very often in being discontinuous, making the hypothesis too restrictive for real implementation (see [70] and [98]).

From the above considerations, it turns out the importance of finding a way to robustly control a generic constrained nonlinear system guaranteeing stability by means of an approximation of the control law even in absence of continuity assumptions on the obtained law. In this Chapter, a RH control policy capable to meet all of these requirements is proposed. The robust stability analysis of the resulting controlled system is carried on both in the case of the use of the exact control law (obtained by a proposed procedure) and the use of an approximated one. Moreover, two different approximators are considered depending on the smoothness of the control law. We show that the Nearest Point (NP) interpolation (see [7] for a complete introduction on this subject) can be effectively used to approximate (possibly discontinuous) RH control laws. On the other hand, we also analyse the possibility to approximate the MPC state-feedback by a smooth function; in particular, following the early papers [82] and [81], we focus on NNs with smooth activation functions for their favourable properties, allowing the reduction of the complexity of the approximation with respect to the NP approach. The use of smooth functions is reasonable when the RH control law is locally Lipschitz (the reader can refer to the recent work [18] and the references therein to get thorough on the sufficient conditions to obtain a smooth MPC feedback); some results in this sense can be found in [83].

Compared to the NP approach, the use of smooth approximators leads to a very significant reduction of the requested storage capacity for on-line implementation; furthermore, the on-line computational time decreases in view of the fact that the extensive norm-distance evaluations required by the Nearest Point search are not needed anymore. The time depends now only polynomially on the number of neurons of the NN.

Bounds on admissible errors, corresponding to the ϵ-tube in network training, are given in order to maintain the practical stability of the closed loop system and to enforce the fulfilment of hard state constraints.
The main features of the proposed approximate RH control design are the following:

i) it removes any "a priori" assumption on the continuity of the RH control law when NP off-line approximators are employed, thus allowing to apply the method on line to systems which are not asymptotically stabilizable by continuous state-feedback;

ii) hard constraints on state and input variables can be robustly enforced;

iii) it allows to compute a (possibly conservative) bound on the quantization of the input command values (due to the numerical implementation of the approximate control law).

iv) for those RH problems in which a Lipschitz state-feedback control law can be found, smooth functional approximators can be used to reduce the complexity of the scheme and the on-line computational needs.

3.2 System Definition and Assumptions

Consider the following nonlinear discrete-time dynamic system with disturbances and constraints on state and inputs:

$$x_{t+1} = f(x_t, u_t, \varsigma_t),\ t \in \mathbb{Z}_{\geq 0},\ x_0 = \bar{x}\,, \tag{3.1}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ the control vector and $\varsigma_t \in \mathbb{R}^r$ an exogenous disturbance input. The state and control variables are subject to the following constraints:

$$x_t \in X, \tag{3.2}$$

$$u_t \in U, \tag{3.3}$$

$\forall t \in \mathbb{Z}_{\geq 0}$, where X and U are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point. Given the system (3.1), let $\hat{f}(x_t, u_t)$, with $\hat{f}(0,0) = 0$, denote the *nominal* model used for control design purposes, such that

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t,\ t \in \mathbb{Z}_{\geq 0},\ x_0 = \bar{x}\,, \tag{3.4}$$

where $d_t \triangleq f(x_t, u_t, \varsigma_t) - \hat{f}(x_t, u_t) \in \mathbb{R}^n$ denotes the discrete-time state transition uncertainty.

In the sequel, the following assumptions are needed.

Assumption 1. *The function $\hat{f} : X \times U \to X$ is Lipschitz continuous w.r.t. $x \in X$, with Lipschitz constant $L_{f_x} \in \mathbb{R}_{>0}$, uniformly in $u \in U$, i.e. for any fixed $u \in U$, it holds that*

$$|\hat{f}(x,u) - \hat{f}(x^{'},u)| \leq L_{f_x}|x - x^{'}|,$$

for all $(x, x^{'}) \in X^2$.

Furthermore, the function $\hat{f}$ is uniformly continuous in u, i.e. there exists a $\mathcal{K}$-function η_u such that

$$|\hat{f}(x,u) - \hat{f}(x,u^{'})| \leq \eta_u(|u - u^{'}|),$$

for all $x \in X$ and for all $(u, u^{'}) \in U^2$. □

Assumption 2 (Uncertainties)**.** *The additive transition uncertainty verifies*

$$|d_t| \leq \mu(|\varsigma_t|),\ \forall t \in \mathbb{Z}_{\geq 0},$$

where μ is a $\mathcal{K}$-function. Moreover, d_t is bounded in a compact ball D, that is

$$d_t \in D \triangleq \mathcal{B}(\overline{d}), \quad \forall t \in \mathbb{Z}_{\geq 0},$$

with $\bar{d} \in \mathbb{R}_{\geq 0}$ finite. □

Assumption 3 (Input-to-state stabilizing controller)**.** *There exist a compact set $\tilde{\Xi} \in X$, with $\{0\} \in \tilde{\Xi}$, and a state-feedback control law (possibly non-*

smooth) $u_t = \kappa(x_t)$, $\kappa(x_t) : \tilde{\Xi} \to U$, *such that the system,*

$$x_{t+1} = \hat{f}(x_t, \kappa(x_t)) + d_t, \ \ t \in \mathbb{Z}_{\geq 0}, \ \ x_0 = \bar{x}, \tag{3.5}$$

enjoys the properties:

i) it is ISS in $\tilde{\Xi}$ *w.r.t. additive disturbances* $d_t \in D$*. In particular, there exists a ISS-Lyapunov function for which Points i) and ii) of Definition 2.2.5 hold.*

ii) the set $\tilde{\Xi}$ *is Robust Positively Invariant (RPI) for system (3.4) with additive disturbances* $d_t \in D$*.*

□

The last assumption plays a key role and can be found in many recent works in the framework of MPC; indeed by recurring to such properties it is possible to easily prove the (even robust) input-to-state stability for the controlled system, as for instance in [62] or [86]. Nevertheless, finding a control law capable to guarantee these requirements can result in a very hard task; due to this, in literature several approaches to the design of the RH cost function and to the choice of the corresponding parameters have been proposed, often specifics for a particular class of systems (see for example [59], in which the system is required to be affine in the disturbance input). From these considerations it turns out the importance of extending the aforementioned RH control design procedures to more general systems: the technique proposed in the following, as it will be proved in the next Section, leads to a control strategy which can stabilize systems even not asymptotically stabilizable by continuous static state feedback. In this perspective, Assumption 3 will be satisfied by recurring to some equivalent suitable conditions.

3.3 Stabilizing RH Control Law

As already said, given a nonlinear system with disturbances (3.1), and a nominal model in the form of (3.4), the control objective consists in designing a state-feedback control law, capable to meet the requirements posed by Assumption 3 and to satisfy state and input constraints in presence of additive uncertainties. On the basis of Assumptions 1 and 2, the RH control policy can be formulated introducing a suitable Finite-Horizon Optimal Control Problem (FHOCP). Let's now introduce the control policy.

Definition 3.3.1 (FHOCP)**.** *Given a positive integer* $N_c \in \mathbb{Z}_{\geq 0}$*, at any time* $t \in \mathbb{Z}_{\geq 0}$*, let* $\mathbf{u}_{t,t+N_c-1|t} \triangleq \mathrm{col}[u_{t|t},\ u_{t+1|t},\ \ldots,\ u_{t+N_c-1|t}]$ *denote a sequence of input variables over the control horizon* N_c*. Moreover, given* x_t *and* $\mathbf{u}_{t,t+N_c-1|t}$*, let* $\hat{x}_{t+j|t}$ *denote the state prediction by means of the nominal model, such that*

$$\hat{x}_{t+j|t} = \hat{f}(\hat{x}_{t+j-1|t}, u_{t+j-1|t}), \hat{x}_{t|t} = x_t, \forall j \in \{1, \ldots, N_c\}\,. \tag{3.6}$$

Then, given a transition cost function h*, an auxiliary control law* κ_f*, a terminal cost function* h_f*, a terminal set* X_f *and a sequence of constraint sets* $\hat{X}_{t+j|t} \subseteq X$*,* $j \in \{1, \ldots, N_c - 1\}$*, to be described later on, the FHOCP consists in minimizing, with respect to* $\mathbf{u}_{t,t+N_c-1|t}$*, the cost function*

$$J_{FH}(x_t, \mathbf{u}_{t,t+N_c-1|t}, N_c) \triangleq \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, u_{l|t}) + h_f(\hat{x}_{t+N_c|t}) \tag{3.7}$$

subject to

i) the nominal dynamics (3.6), with $\hat{x}_{t|t} = x_t$*;*

ii) the control and the state constraints $u_{t+j|t} \in U$*,* $\hat{x}_{t+j|t} \in \hat{X}_{t+j|t}, \forall j \in \{0, \ldots, N_c - 1\}$*;*

iii) the terminal state constraint $\hat{x}_{t+N_c|t} \in X_f$*.* □

The RH control technique can now be stated as follows.

Receding Horizon Control Policy

Given a time instant $t \in \mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t} = x_t$. Find the optimal control sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ by solving the FHOCP at time t. Then apply

$$u_t = \kappa_{RH}(x_t)\,, \tag{3.8}$$

where $\kappa_{RH}(x_t) \triangleq u^{\circ}_{t|t}$ and $u^{\circ}_{t|t}$ is the first element of the optimal control sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ (implicitly dependent on x_t). □

It can be shown that the satisfaction of the original state constraints is ensured, for any admissible disturbance sequence, by imposing to the predicted open-loop trajectories the restricted constraints according to the following lemma.

Lemma 3.3.1 (Constraints tightening [62]). *Assuming to know an upper bound $\bar{d}$ on the uncertainty as specified by Assumption 2, given the state vector x_t at time t, if a control sequence, $\mathbf{u}_{t,t+N_c-1|t}$ is feasible with respect to the state constraints $\hat{X}_{t+j|t}$, where*

$$\hat{X}_{t+j|t} \triangleq X \backsim \mathcal{B}^n\left(\frac{L^j_{f_x}-1}{L_{f_x}-1}\bar{d}\right), \tag{3.9}$$

then, the control sequence $\mathbf{u}_{t,t+N_c-1|t}$, applied to the system (3.1) in open-loop, guarantees that

$$x_{t+j} \in X, \;\; \forall j \in \{1, \ldots, N_c\}.$$

□

The proof of this lemma can be found in [62].

Now, in order to prove the ISS property for the closed loop system, let us introduce the following assumptions, giving some useful directions on the choice of the key parameters involved in the *RHOCP*.

Assumption 4. *The transition cost function h is such that $\underline{h}(|x|) \leq h(x,u)$, $\forall x \in X$, $\forall u \in U$ where $\underline{h}$ is a $\mathcal{K}_\infty$-function. Moreover, h is Lipschitz w.r.t. x, uniformly in u, with Lipschitz constant $L_h > 0$.* □

Assumption 5. *A terminal cost function h_f, an auxiliary control law κ_f, and a set X_f are given such that*

1) $X_f \subset X$, X_f closed, $0 \in X_f$;

2) $\exists \delta > 0 : \kappa_f(x) \in U$, $\forall x \in X_f \oplus \mathcal{B}^n(\delta)$;

3) $\hat{f}(x, \kappa_f(x)) \in X_f$, $\forall x \in X_f \oplus \mathcal{B}^n(\delta)$;

4) $h_f(x)$ is Lipschitz in X, with Lipschitz constant $L_{h_f} > 0$;

5) $h_f(\hat{f}(x, \kappa_f(x))) - h_f(x) \leq -h(x, \kappa_f(x))$, $\forall x \in X_f \oplus \mathcal{B}^n(\delta)$; □

Assumption 6 (X_{κ_f})**.** *Suppose that there exist a compact set $X_{\kappa_f} \supset X_f$ for which $\tilde{\mathbf{u}}_{t,t+N_c-1|t} \triangleq \text{col}[\kappa_f(\hat{x}_{t|t}),\, \kappa_f(\hat{x}_{t+1|t}), \ldots,\, \kappa_f(\hat{x}_{t+N_c-1|t})]$, being $\hat{x}_{t|t} = x_t \in X_{\kappa_f}$, is a feasible control sequence for the FHOCP and for which Points 1), 2) and 5) of Assumption 5 are satisfied.* □

It is important to notice that we do not require neither $\kappa_f(x)$ nor the closed loop map $\hat{f}(x, \kappa_f(x))$ to be Lipschitz continuous for $x \in X_f$. This is a different approach to the design of input-to-state stabilizing RH controllers with respect to previous works [68, 62, 86]: this way it is indeed possible to cope with possibly discontinuous auxiliary control laws, leading to a more general controller.

With respect to this, the following Lemma will play a key role in proving the ISS property in absence of the above-mentioned regularity assumptions; moreover, it will allow us to decouple the estimation of the maximal admissible uncertainty from the particular choice of $\kappa_f(x)$. Finally, by exploiting it, we will show that the robustness of the scheme depends only on the invariant properties of X_f through the computation of the one-step controllability set to X_f.

Lemma 3.3.2 (Technical)**.** *The control law $\kappa_f^*(x) : \mathcal{C}_1(X_f) \to U$ and the function $h_f^*(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as*

$$\kappa_f^*(x) \triangleq \begin{cases} \kappa_f(x), & x \in X_f \oplus \mathcal{B}^n(\delta) \\ \arg\min\limits_{u \in U} \left\{ h_f\left(\hat{f}(x, u)\right)\right\}, & x \in \mathcal{C}_1(X_f) \backslash (X_f \oplus \mathcal{B}^n(\delta)) \end{cases} \tag{3.10}$$

$$h_f^*(x) \triangleq \begin{cases} h_f(x), & x \in X_f \oplus \mathcal{B}^n(\delta) \\ \overline{h}_f + \lambda\, \mathrm{d}(x, X_f), & x \in \mathcal{C}_1(X_f) \backslash (X_f \oplus \mathcal{B}^n(\delta)) \end{cases}$$

with

$$\lambda > \frac{\max\limits_{x \in \mathcal{C}_1(X_f),\, u \in U} \{h(x, u)\}}{\delta}, \tag{3.11}$$

verify the inequality

$$h_f\left(\hat{f}(x, \kappa_f^*(x))\right) + h(x, \kappa_f^*(x)) < h_f^*(x). \tag{3.12}$$

□

Proof in A.1.

Now, it is worth noting that an adequate choice of all the sets and functions defining the $RHOCP$, in accordance with the stated assumptions, can be made as suggested in the following important remark.

Remark 3.3.1. *In order to design a RH controller for a given system, a Lyapunov function W can serve as a terminal cost for the FHOCP, while X_f can be chosen as a sub-level set $X_f = \{x \in X : W(x) \leq \overline{h}_f\}$. Furthermore, it will be necessary to choose δ and h such that both Point 5) of Assumption 5 and inequality (3.12) hold.* □

Under the stated assumptions, the following theorem characterizes the ISS property of the closed loop system with respect to bounded additive uncertainties. This theorem represents the extension of the ISS result presented in [86] to the case of systems which are not asymptotically stabilizable by smooth feedback.

Theorem 3.3.1 (Regional ISS)**.** *Let us denote as $X_{RH} \subset \mathbb{R}^n$ the set of state vectors for which the FHOCP is feasible. Under Assumptions 1,2,4, 5 and 6, the system (3.1), driven by the RH control law (3.8), is regional ISS in X_{RH} with respect to additive perturbations $d_t \in D$, with $D \subseteq \mathcal{B}^n(\overline{d})$ and*

$$\overline{d} \leq L_{f_x}^{1-N_c} \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f). \tag{3.13}$$

□

Proof in A.1.

The scalar $\overline{d}$ represents an upper bound on the admissible uncertainty. As it can be seen, it depends on the invariance properties of X_f. Its introduction is used in the proof of the theorem in order to guarantee the recursive feasibility of the scheme.

In this section it has been shown how to design an input-to-state stabilizing *exact* RH control law κ_{RH} for system (3.4), which renders RPI the set $X_{RH} \subseteq X$ with respect to additive disturbances $d_t \in D$. Therefore, Assumption 3 is verified by the RH controller with $\kappa = \kappa_{RH}$ and $\tilde{\Xi} = X_{RH}$.

The robust stability properties of the designed controller can be now exploited to obtain a fast implementation of the MPC; indeed, Theorem 3.3.1 suggests that the ISS properties of the overall closed loop controlled system are not invalidated by uncertainties (and disturbances) subsuming the indicated bound. Therefore, this uncertainty bound can be suitably employed to formulate a stabilizing approximation of the exact control law, thus not requiring the solution of the optimal control problem at each sampling instant. The technique and the stability proofs will be presented in the following Sections.

3.4 Preservation of stability under approximation - NP case

Consider the following dynamic system:

$$x_{t+1} = \hat{f}(x_t, \kappa^*(x_t)) + w_t,\ t \in \mathbb{Z}_{\geq 0},\ x_0 = \bar{x} \in \Xi \subset \mathbb{R}^n. \tag{3.14}$$

where $w_t \in W \triangleq \mathcal{B}^n(\overline{d}_w)$ is a disturbance input and the function $\kappa^* : \mathbb{R}^n \to \mathbb{R}^m$ is an approximation of the given ISS stabilizing κ satisfying Assumption 3. We will show that the stability properties of (3.14) in a suitable set Ξ can be inferred from those of (3.5) in $\tilde{\Xi}$: in order to obtain this result, it is necessary to conveniently split up the state transition uncertainty in three contributions, affecting the control function, the state vector and the whole model: the error distribution policy has to be made in accordance with the following assumption.

Assumption 7. *Let us define the* $\mathcal{K}_\infty$*-function* $\eta_x(s) = L_{f_x} s + s$ *for* $s \geq 0$

and let $\overline{d}_q \in \mathbb{R}_{\geq 0}$ *and* $\overline{d}_v \in \mathbb{R}_{\geq 0}$ *be two positive scalars satisfying the following inequality*

$$\overline{d}_q + \overline{d}_v + \overline{d}_w \leq \overline{d}. \tag{3.15}$$

Defining $\overline{v} \triangleq \eta_u^{-1}(\overline{d}_v)$ *and* $\overline{q} \triangleq \eta_x^{-1}(\overline{d}_q)$*, there exists* $\lambda \in (0,1) : \forall \xi \in \mathrm{dom}(\kappa),\ \exists\, \zeta_\xi \in \mathcal{B}^n(\xi, \overline{q}) \cap \mathrm{dom}(\kappa)$ *such that*

i) $|\kappa(\zeta_\xi) - \kappa^*(\zeta_\xi)| \leq (1-\lambda)\overline{v}$*;*

ii) $|\kappa^*(\xi) - \kappa^*(\zeta_\xi)| \leq \lambda \overline{v}$*.*

Moreover, let us assume that $\kappa^*(\xi) \in U, \forall \xi \in \mathrm{dom}(\kappa)$*.* □

The just stated conditions allow to distribute the overall disturbances and the uncertainties into three different contributions; this way, the system can be rewritten in a more convenient form. With regard to this, the following proposition can be proved.

Proposition 3.4.1 (Approximation-induced perturbations)**.** *Under Assumption 7, given* $x_t \in \mathrm{dom}(\kappa)$*, there exist two vectors* $q_t \in Q \triangleq \mathcal{B}^n(\overline{q})$ *and* $v_t \in V \triangleq \mathcal{B}^m(\overline{v})$*, with* $q_t \triangleq \zeta_{x_t} - x_t$ *and* $v_t \triangleq \kappa^*(x_t) - \kappa(\zeta_{x_t})$*, such that system (3.14) can be rearranged as*

$$x_{t+1} = \hat{f}(x_t, \kappa(x_t + q_t) + v_t) + w_t, \quad x_0 = \bar{x}, \quad t \in \mathbb{Z}_{\geq 0}, \tag{3.16}$$

with $\kappa(x_t + q_t) + v_t \in U$*.* □

Proof in A.1.

Then, we can state the following important result, concerning the stability properties of the closed loop system driven by the approximate control law κ^*.

Theorem 3.4.1. *Suppose that Assumptions 1-7 hold and let* $\Xi \triangleq \tilde{\Xi} \backsim Q$*. Then, the following statements hold:*

i) The set $\Xi \subset X$ *is RPI for the closed loop system (3.16) with* $v_t \in V, q_t \in Q$ *and* $w_t \in W$*;*

ii) The closed loop system (3.16) is ISS in Ξ *with respect to the perturbations* v_t*,* q_t *and* w_t*.*

□

Proof in A.1.

This is the core theorem for the approximation: then, *given a stabilizing controller obtained by the method proposed in the previous Section, the approximation of that controller keeps its stabilizing properties whenever the approximation-induced errors satisfy Assumption 7.*

In view of the above statement, the following procedure capable to find an adequate robust controller for a given nonlinear system can be formulated.

Off line evaluation of the exact control law

- Set up the RHOCP according to assumptions leading to the exact RH controller (choice of parameters and sets);
- Find three scalars verifying inequality (3.15) and the corresponding $\overline{q}$ and $\overline{v}$;
- Superimpose over the domain a grid in such a way that the distance of each point of the domain from a grid knot is less than $\overline{q}$ (some considerations on this point will be addressed in the following). It is important to notice that gridding implies a remarkable growth of the computational burden;
- For each knot of the grid, store the value of the first step of the optimal control sequence obtained solving the corresponding FHOCP.

At this point, the on line control procedure consists in the application to the system of the input vector obtained by the chosen approximator, capable to verify Assumption 7 and trained on the previously calculated control values, i.e. $\kappa^*(x_t) = \kappa(\zeta_{x_t}) + v_t$.

Once the off line stage has been completed over the domain, the on line part results very much less demanding in terms of evaluation time than a standard MPC on line application: indeed, at each instant, the controller, instead of minimizing a cost function enforcing constraints, has only to retrieve from the approximator the control action to be applied to the system; the requested time depends naturally on the approximation policy and on the complexity of the system.

Due to the importance of the enforcement of the bounds on errors, the choice of an adequate approximator turns out to be fundamental. In particular, in order to easily satisfy the last assumption when RH control law is discontinuous, a Nearest Point (NP) [7] approximator can be used; this approach can be then applied in the case of non smooth control law obtained by RH schemes.

Remarkably, recurring to a NP approximator implies searching among a set of stored values in order to find the point with the minimum distance from the considered element; this operation is a minimization itself, although discrete and easier (if the dimension of the problem is low) of the solution of the RHOCP. Moreover, it is important to notice that gridding the domain implies an exponential growth of the computational burden. These aspects have to be taken into account in view of the application of the proposed method.

Design of the NP approximator

As already said, the parameters characterizing the NP approximator have to be chosen suitably. Let's introduce a possible approach to the design.

First, assuming that a bound on the additive transition uncertainty is given (e.g., $|w_t| \leq \overline{d}_w$) the designer must assign arbitrary values to the scalars $\overline{d}_v$ and $\overline{d}_q$ such that inequality (3.15) holds. The choice of the values depends on the analysis of the case under concern; e.g. a greater value for $\overline{d}_v$ means a growth of the tolerance on the error committed on the estimation of the exact control law in the points of the grid and at the same time an increase of the density of the knots (which depends on $\overline{d}_q$). Similar considerations lead to think to these parameters as tuning knobs for the degree of complexity of κ^*, which finally affects the approximation error introduced by κ^* with respect to κ_{RH}.

Then, as already mentioned, the off-line approximation procedure starts with the construction of a suitable data set (analogous to the "training set" in neural networks applications), by evaluating the RH control law in a finite number (possibly very large) of points (knots) belonging to a (possibly non uniform) grid X_G which covers the whole region X, obtained in accordance with the following assumption.

Assumption 8 (Grid set X_G)**.** *Given the set X and $\overline{d}_q \in \mathbb{R}_{>0}$ satisfying (3.15), let the set X_G verify:*

1. $\forall \xi \in X, \exists \zeta_\xi \in X_G : |\xi - \zeta_\xi| \le \overline{q}_{NP} < \eta_x^{-1}(\overline{d}_q)$;

2. $\exists \psi_{NP} \in \mathbb{R}_{>0} : |\zeta' - \zeta''| \ge \underline{\psi}_{NP}, \forall (\zeta', \zeta'') \in X_G^2$,

where $\overline{q}_{NP}$ and $\underline{\psi}_{NP}$ are the knot density and the knot separation parameters [7]. □

Notice that, being X compact, point 2 implies that X_G is made up of a finite number of knots. The cardinality of the data set grows with the decrease of $\overline{d}_q$, but being a lower limit on this scalar imposed by (3.15), then there exists a finite upper bound on the knot density $\overline{q}_{NP}$.

Once the quantization (to be intended as spatial sampling) of X operated by X_G has been performed, the exact control law must be evaluated at each point of X_G.

Noting that $X_{RH} = \mathrm{dom}\,(\kappa_{RH})$, the NP data are given by the pair $(\mathcal{X}, \mathcal{Y})$, with

$$\mathcal{X} = X_G \cap X_{RH} \subset \mathrm{dom}\,(\kappa_{RH})\,, \;\; \mathcal{Y} \triangleq \bigcup_{\zeta \in \mathcal{X}} \tilde{\kappa}_{RH}(\zeta), \tag{3.17}$$

where $\tilde{\kappa}_{RH}(\zeta) = y\,(\kappa_{RH}\,(\zeta))$ and $y : U \to \mathcal{U} \subset U$ is a quantizer in the command input space which models the error that can be due to the coding of input command values within a finite alphabet. This problem always affects numerical approximation schemes. For a generic approximator in order to meet the requirement posed by Point i) of Assumption 7, the input space quantizer is required to satisfy the following condition in correspondence of points belonging to the training set:

$$|y(\kappa_{RH}(\zeta)) - \kappa_{RH}(\zeta)| \leq \overline{v}_{NP} < (1-\lambda)\overline{v}, \;\; \forall \zeta \in \mathcal{X}. \tag{3.18}$$

Regarding the specific case of the NP, condition (3.18) holds for $\lambda = 0$. Indeed, Point ii) of Assumption 7 is satisfied with $|\kappa^*(\xi) - \kappa^*(\zeta_\xi)| \equiv 0$ by this approximation scheme. The input quantizer is asked to satisfy $\overline{v}_{NP} < \overline{v}$ $(= \eta_u^{-1}(\overline{d}_v))$. Then, given a state measurement $x_t \in X_{RH} \backsim \mathcal{B}^n(\overline{q}_{NP})$ at time t, the NP control law is given by $u_t = \kappa_{NP}(x_t) = \tilde{\kappa}_{RH}(\mathcal{N}_{\mathcal{X}}(x_t))$, where $\mathcal{N}_{\mathcal{X}} : \mathbb{R}^n \to \mathbb{R}^n$ denotes a single-valued Nearest Point search in the data-set $\mathcal{X}$. In view of the NP approximation, $\forall \xi \in X_{RH}$, $\exists \zeta_\xi \in \mathcal{N}_{\mathcal{X}}(\xi) \subseteq \mathcal{X}$: $|\xi - \zeta_\xi| \leq \overline{q}_{NP} < \overline{q}$ $(= \eta_x^{-1}(\overline{d}_q))$. Moreover it holds that $\kappa^*(\xi) \in \kappa^*(\mathcal{N}_{\mathcal{X}}(\xi)) = \tilde{\kappa}_{RH}(\mathcal{N}_{\mathcal{X}}(x_t))$. Finally, in view of Theorem 3.4.1, it is possible to conclude that the closed loop system (3.14), driven by the approximate RH control law $\kappa_{NP}(x_t)$, is regional-ISS in $X_{NP} \triangleq X_{RH} \backsim \mathcal{B}^n(\overline{q}_{NP})$ with respect to the approximation-induced perturbations and the model uncertainty.

Remark 3.4.1. *In order to complete the analysis on the stability properties of the closed loop system under the action of the approximate control law $\kappa^* = \kappa_{NP}$, it is also possible to take in explicit consideration the fact that the perturbations due to the finite knot density $\overline{q}_{NP}$ and to the input quantization $\overline{v}_{NP}$ do not vanish along the system trajectories. From the proof of Theorem 3.4.1 (in Appendix) and in particular from (A.10) and the first inequality in (A.12), it follows that*

$$V(x_{t+1}) - V(x_t) \leq -\alpha_3(|x_t|) + \sigma\Big(2\eta_u(\overline{v}_{NP}) + 2\eta_x(\overline{q}_{NP})\Big) + \sigma(2|w_t|).$$

Hence, the closed loop system driven by the approximate Nearest Point-RH control law κ_{NP} is ISpS in $X_{RH} \backsim \mathcal{B}^n(\overline{q}_{NP})$ with respect to the model uncertainty $w_t \in W$. □

3.5 Simulation Results - Nearest Point case

In this section, an example is proposed concerning a parametrized non-autonomous discrete-time nonlinear system for which it is not possible to stabilize the origin by a continuous state-feedback control law. Conversely, this task can be achieved by a a bounded discontinuous feedback (proved following the approach of [54]).

Consider the following system:

$$\begin{cases} x_{(1)_{t+1}} &= x_{(1)_t}\left[p_1 + \mathrm{sign}(\mathrm{x}_{(1)_t})u_t\right] \\ x_{(2)_{t+1}} &= e^{-2p_3}\left[p_2 - (1-p_2)\dfrac{1}{p_3}u_t\right]x_{(2)_t} \end{cases}, t \in \mathbb{Z}_{\geq 0}. \tag{3.19}$$

subjected to the constraint $|u_t| \leq R$, with $R > 1$ finite. The subscripts (i), $i \in \{1, 2\}$ in (3.19) denote the i-th component of x_t, while p_1, p_2, p_3 are (time-invariant) parameters such that

$$1 < p_1 < R\,, \qquad 0 < p_2 < 1\,, \qquad p_3 \geq p_1\,. \tag{3.20}$$

Proposition 3.5.1. *The nonlinear transition function (3.19) with parameters specified in (3.20), is Lipschitz continuous in x, uniformly in $u \in [-R, R]$.*

Proof in A.1.

Claim 3.5.1. *The parametric system (3.19) does not admit a time-invariant state-feedback function continuous in the state variable capable to locally stabilize the 0-equilibrium.*

Proof in A.1.

On the other hand the following bounded discontinuous feedback is capable to asymptotically stabilize the closed-loop system toward the 0-equilibrium.

$$\kappa_d(x_{(1)}, x_{(2)}) = \begin{cases} 0, & |x_{(1)}| \leq \sqrt{|x_{(2)}|} \\ -p_1 \mathrm{sign}(x_{(1)}) & |x_{(1)}| > \sqrt{|x_{(2)}|} \end{cases}. \tag{3.21}$$

The feedback law κ_d is bounded by $|\kappa_d(x_{(1)}, x_{(2)})| \leq p_1$, $\forall (x_{(1)}, x_{(2)}) \in \mathbb{R}^2$.

Consider now the function $W : \mathbb{R}^2 \to \mathbb{R}_{>0}$, defined as

$$W(x) \triangleq |x_{(1)}| + \frac{p_1 e^{p_3}}{e^{\frac{p_3}{2}} - 1}\sqrt{|x_{(2)}|}. \tag{3.22}$$

W is a Lyapunov function for the closed-loop system; as a matter of fact, W satisfies the inequality

$$W(f(x, \kappa_d(x))) \leq e^{-\frac{p_3}{2}} W(x). \tag{3.23}$$

Indeed, if $|x_{(1)}| \leq \sqrt{|x_{(2)}|}$,

$$\begin{aligned} W(f(x, \kappa_d(x))) &= |x_{(1)} p_1| + \frac{p_1 e^{p_3}}{e^{\frac{p_3}{2}} - 1}\sqrt{|e^{-2p_3} p_2 x_{(2)}|} \\ &= p_1 |x_{(1)}| + \frac{p_1}{e^{\frac{p_3}{2}} - 1}\sqrt{p_2}\sqrt{|x_{(2)}|} \\ &\leq p_1 \left(1 + \frac{1}{e^{\frac{p_3}{2}} - 1}\right)\sqrt{|x_{(2)}|} = p_1 \frac{e^{\frac{p_3}{2}}}{e^{\frac{p_3}{2}} - 1}\sqrt{|x_{(2)}|} \\ &= e^{-\frac{p_3}{2}}\left[p_1 \frac{e^{p_3}}{e^{\frac{p_3}{2}} - 1}\sqrt{|x_{(2)}|}\right] \leq e^{-\frac{p_3}{2}} W(x). \end{aligned}$$

On the other hand, if $|x_{(1)}| > \sqrt{|x_{(2)}|}$,

$$W(f(x, \kappa_d(x))) \quad = \quad \frac{p_1 e^{p_3}}{e^{\frac{p_3}{2}} - 1}\sqrt{\left|e^{-2p_3}\left[p_2 \pm (1 - p_2)\frac{p_1}{p_3}\right] x_{(2)}\right|}.$$

In view of the assumptions $p_3 \geq p_1$ and $0 < p_2 < 1$, we have that

$$W(f(x, \kappa_d(x))) \leq \frac{p_1}{e^{\frac{p_3}{2}} - 1}\sqrt{|x_{(2)}|} \leq \frac{p_1 e^{\frac{p_3}{2}}}{e^{\frac{p_3}{2}} - 1}\sqrt{|x_{(2)}|} \leq e^{-\frac{p_3}{2}} W(x). \tag{3.24}$$

From (3.23) it follows that the candidate Lyapunov function is monotonically strictly decreasing along the closed-loop system trajectories:

$$W(f(x, \kappa_d(x))) - W(x) \leq \left(e^{-\frac{p_3}{2}} - 1\right) W(x). \tag{3.25}$$

Finally, considering that $(e^{-\frac{p_3}{2}} - 1) < 0$, inequality (3.25) implies that the bounded discontinuous state-feedback law (3.21) globally asymptotically stabilizes the system toward the origin.

As previously noticed, a Lyapunov function, in this case W, can serve as a terminal cost for the considered system; moreover X_f will be chosen as a sub-level set of W; on the basis of the just stated considerations, in the proposed case it is possible to choose $\delta = (e^{\frac{p_3}{2}} - 1)\overline{h}_f / L_{h_f}$ and h such that Point 5) of Assumption 5 holds and

$$\sup_{(x,u)\in[C_1(X_f)\setminus(X_f\oplus\mathcal{B}^n(\delta))]\times U} \{h(x,u)\} < (e^{\frac{p_3}{2}} - 1)\overline{h}_f, \tag{3.26}$$

in order to meet inequality (3.12).

To show the effectiveness of the proposed approach, the behaviour of the system proposed under the action of the devised approximate NP control law κ_{NP} has been simulated choosing different starting points inside the feasible domain $X_{RH} \subset X$ (see Figure 3.3). In the considered example, the domain X has been chosen as a square centred in the origin and of side 2; choosing a lager domain would naturally lead to an increase of the number of points for which the first element of the RHOCP problem has to be calculated and stored. Nevertheless, this growth can be slowed down by an opportune choice of other parameters, e.g. X_f or $\overline{h}_f$ (this would indeed result in a change of δ and consequently of $\overline{d}$, in turn implying a new choice of the three scalars in (3.15)), without a dramatic decay of the controller performances. This highlights the fact that the tuning of the proposed method has to be carried out on each specific case.

The parameters chosen for the simulations are $p_1 = 1.01$, $p_2 = 0.97$ and $p_3 = 1.1$. Notice that these values influence directly all the other ingredients of the problem, first of all the Lyapunov function, depicted in Figure 3.1 together with the plane whose height is $\bar{h}_f$. For instance, when p_1 and p_3 change, the shape of the intersection of W with the plane at $\bar{h}_f$ changes (and so does the shape of region X_f), as depicted in Figure 3.2.

In this example we posed $\bar{h}_f = 1$, obtaining a terminal set $X_f = \{x \in X : h_f(x) \leq \bar{h}_f = 1\}$, where the function h_f, used as terminal cost, has been chosen, as said, as the Lyapunov function W for the closed loop system

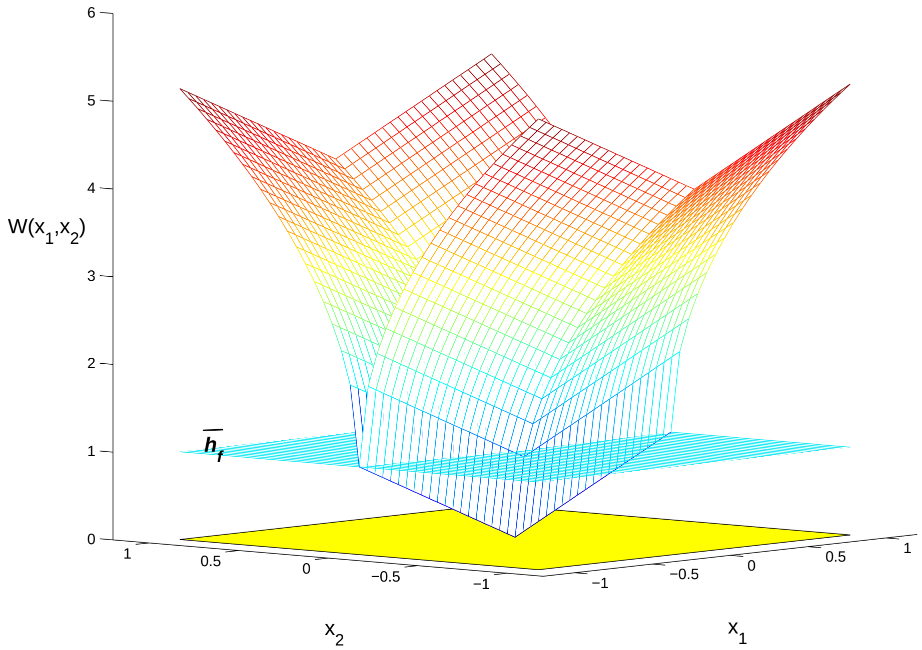

Figure 3.1: Lyapunov function for the considered system with the described parameters choice. Edges are due to the presence of the absolute values in the formulation of W. In cyan, the plane whose height is $\bar{h}_f = 1$.

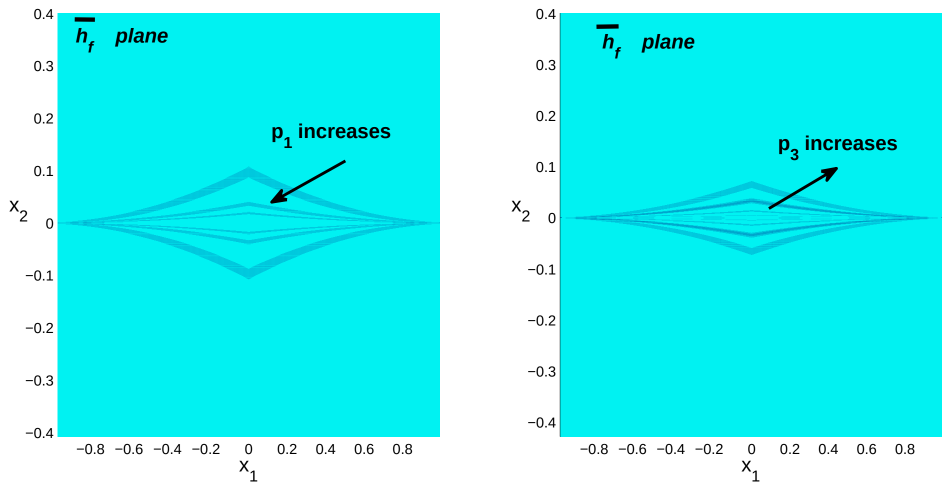

Figure 3.2: The intersection of W with $\overline{h}_f$ plane varies with p_1 and p_3.

under the action of the discontinuous stabilizing auxiliary controller already introduced; notice that the choice criterion for $\bar{h}_f$ and X_f can be somehow inverted, fixing a given radius for the terminal set and obtaining this way a corresponding value for $\bar{h}_f$.
With regard to the control horizon, we chose $N_c = 6$; finally, the input constraint set is $U = [-R, R]$ with $R = 1.0412$; this value is the ratio between p_1 and p_2. No particular reason led to this option but the idea of a parameter somehow connected with the problem itself.

This initialization leads to the calculation of the Lipschitz constants as in the following (see proof of Proposition 3.5.1):

$$L_{f_x} = max\left(p_1 + 3R, e^{-2p_3}\left(p_2 - (1-p_2)\frac{p_1}{p_3}\right)\right) = 4.1337,$$

$$L_{h_f} = p_1 \frac{e^{p_3}}{e^{p_3/2} - 1} = 4.138,$$

and to

$$\delta = (e^{p_3/2} - 1)\frac{\bar{h}_f}{L_{h_f}} = 0.1772$$

and

$$\overline{d} = L_{f_x}^{1-N_c}\delta = 1.47 \cdot 10^{-4}.$$

On the basis of these results, the bounds of the approximation-induced perturbations for the considered system can be chosen as $\overline{q} = 2.57 \cdot 10^{-5}$ and $\overline{v} = 1$.

For the proposed domain, with the mentioned parameter values, the gridding produces a set of about $4 \cdot 10^9$ points in $\mathbb{R}^2$; for each one of these points an optimization problem has to be solved and the first value of the control sequence obtained has to be stored in a vector whose cardinality is naturally the same of the number of nodes of the grid. The calculation of the control values over the domain demanded about four days of CPU time: the problem would have been more demanding (also in terms of memory) for different choices of the parameters, so tuning was a crucial part of the simulations: naturally, facing real systems would require less application to this stage but could result in some cases in almost intractable problems, depending on the characteristics of the system to be controlled and on the RHOCP related choices. In those cases, the off-line campaign has to be carried out in a convenient way, in order to overcome computational difficulties (e.g. memory and time consumption); naturally it would be preferable, for very large data sets, the recourse to other approximators than NP based, such for instance neural networks, if the applicability conditions (to be specified later on) result verified.

The green trajectories in Figure 3.3 show that the system has been effectively steered toward $\{0\}$ by κ_{NP}, while the state has been kept inside the constraint set X. Notably, if the constraints are not tightened in the computation of the approximation control law, it may happen that, due to the approximation-induced perturbations, the approximate controller fails to preserve the state within X. Indeed, the black trajectory in Figure 3.3, generated by an approximate controller without tightening, violates the constraint starting from point (d), as can be seen in Figure 3.4. Finally, the approximate κ_{NP} law obtained by off-line computations over a uniform grid with knot density parameter $\bar{q}_{NP} = 3.01 \cdot 10^{-5}$ is depicted in Figure 3.5, where the discontinuous nature of the RH control law is enhanced.

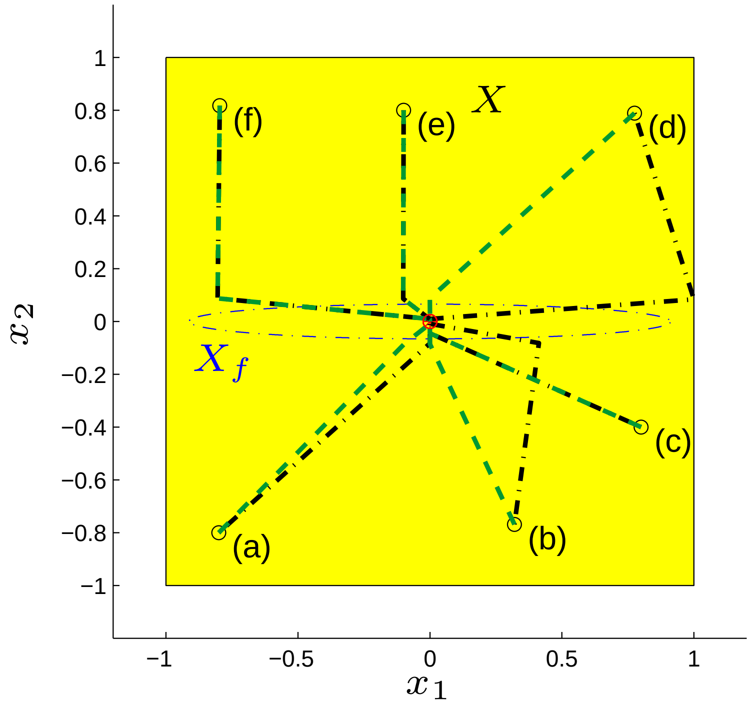

Figure 3.3: Closed loop trajectories of the system under the action of the approximate MPC law κ_{NP}, with starting points: $(a) = [-.80, -.80]$; $(b) = [.32, -.77]$; $(c) = [.80, -.40]$; $(d) = [.77, .79]$; $(e) = [-.10, .80]$; $(f) = [-.80, .82]$. In green, the trajectory obtained with constraint tightening technique, in black the one obtained without tightening. Blue dash-dotted line represents region X_f.

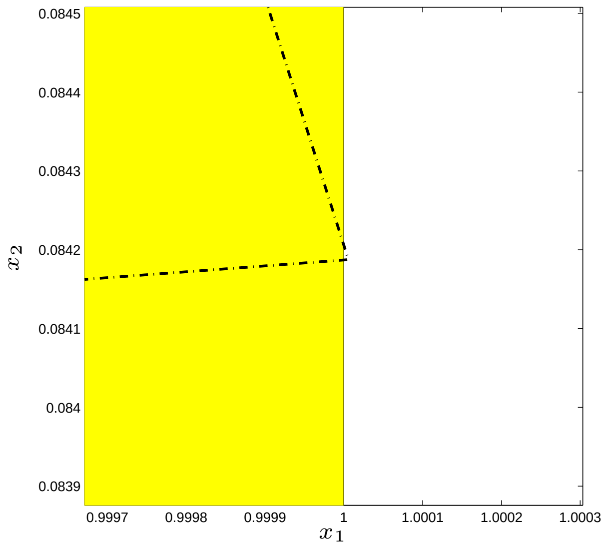

Figure 3.4: Magnification of the trajectory starting from point (d) obtained without constraint tightening technique. The approximate control law without tightening (black) fails in achieving the constraint satisfaction.

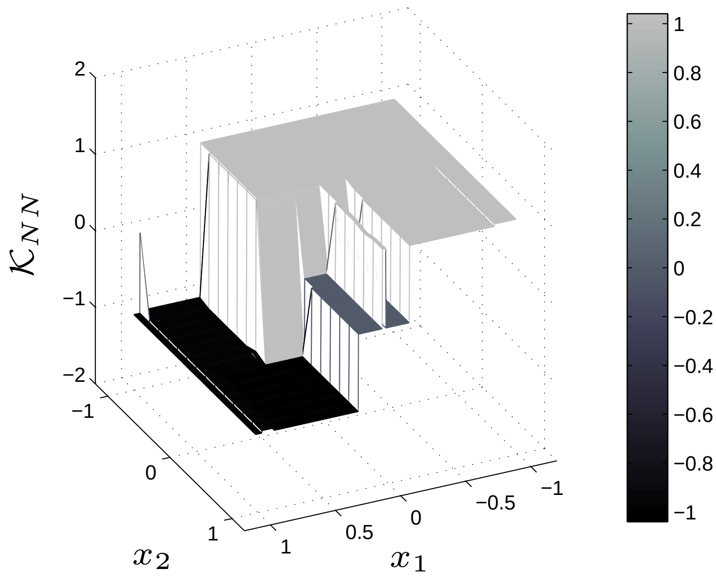

Figure 3.5: The approximate RH control law κ_{NP} in the domain $X_{RH} \subset X$.

3.6 Smooth approximation of the control law

In the previous section, we have shown how to perform the off-line approximation of the RH control law by means of a Nearest Point approach. Although the mechanism underlying the NP search is rather simple and the implementation is straightforward, the major drawback of this method is represented by the large number of points to be stored in order to approximate the RH state-feedback in the whole domain. Furthermore, as the grid dimension grows up, the research of the nearest point can result in a very time demanding task.

In order to improve the sparsity in the representation of the approximate control function, by reducing the number of the parameters of the approximator and consequently the storage memory required for the implementation of the controller, other types of approximators can be used. In the following the case of Neural Networks with smooth activation functions and smooth output function will be considered; the effects of the improvement provided by this approach will be clear in the proposed example.

First of all, in view of the introduced choice, Assumption 7 can be reformulated in a more convenient way, as follows.

Assumption 9 (Bounds on Approximation errors)**.** *Let* $\overline{d}_q$ *and* $\overline{d}_v$ *be two positive scalars satisfying (3.15); being again* $\overline{q} \triangleq \eta_x^{-1}(\overline{d}_q)$*, assume that* $\forall \xi \in \mathrm{dom}(\kappa)$*,* $\exists\, \zeta_\xi \in \mathcal{B}^n(\xi, \overline{q}) \cap \mathrm{dom}(\kappa)$ *such that*

$$\eta_u(|\kappa^*(\xi) - \kappa(\zeta_\xi)|) + \eta_x(|\zeta_\xi - \xi|) + \bar{d}_w \leq \bar{d}(\zeta_\xi), \tag{3.27}$$

where $\overline{d}(\zeta_\xi)$ *is the local uncertainty bound under which the state can be driven from* ζ_ξ *in* Ξ *by the control law* κ*, i.e.*

$$\overline{d}(\zeta_\xi) \triangleq \inf\{c \in \mathbb{R}_{>0} | \exists d \in \mathcal{B}^n(c) : \hat{f}(\zeta_\xi, \kappa(\zeta_\xi)) + d \notin \Xi\}.$$

Following the same lines of the proof of Theorem 3.4.1, it is possible to conclude that if κ is ISS stabilizing in $\tilde{\Xi}$, then any approximating function κ^* verifying (3.27) is ISS stabilizing in $\Xi = \tilde{\Xi} \backsim Q$.

It is worth noting that, being $\overline{d} \leq \overline{d}(\zeta_\xi)$ (by using a local bound on perturbations in place of the semi-global one), the condition (3.27) turns out to be less restrictive than the requirements i) and ii) in Assumption 7, formulated in the non-smooth approximation case.

As in the NP case, in order to check that the approximator verifies (3.27), the domain has to be first gridded according to Assumption 8 in such a way that the error committed in the reconstruction of the control law can be computed for a finite number of points in the training set. *Notice that by using the same training set X_G as for the NP approximator, the relaxation introduced by (3.27) allows for a larger approximation error in correspondence of the grid points.* While in the Nearest Point approach the off-line approximated function does coincide with the exact RH control law on the grid points, in the smooth approximation case we can take full advantage of such a relaxation in order to reduce the number of basis functions (neurons).

Suppose then to have a grid of reference points X_G satisfying Assumption 8 and a set of weights w which correspond to a network realization $\kappa^*(\cdot|w)$; then inequality (3.27) can be rewritten as:

$$\eta_u\Big(|\kappa^*(\zeta_\xi|W) - \kappa^*(\xi|w)| + |\kappa^*(\zeta_\xi|w) - \kappa(\zeta_\xi)|\Big) + \eta_x(|\zeta_\xi - \xi|) \leq \bar{d}(\zeta_\xi) - \overline{d}_w. \tag{3.28}$$

Let the approximating function $\kappa^*(\cdot|w)$ be locally Lipschitz in $\mathrm{dom}(\kappa)$. In particular, assume that for each set of parameters w there exists a function $L_{\kappa^*}(x|w) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$|\kappa^*(x|w) - \kappa^*(x^{'}|w)| \leq L_{\kappa^*}(x|w)|x - x^{'}|, \quad \forall x^{'} \in \mathcal{B}^n(x, \overline{q}).$$

We remark that the smoothness of a Neural Network with a given structure (i.e., having a fixed number of layers, interconnections, number of neurons) depends on the shape of the activation functions and on the weights of the net, that is, on the particular realization. The chosen notation for $\kappa^*(\zeta_\xi|w)$ and $L_{\kappa^*}(\zeta_\xi|w)$ intends to point out such a dependence[1] on the parameters w.

[1]For the case of a two-layer NN with Lipschitz continuous activation function for the

Once fixed the structure of the net (activation function, number of neurons and so on), the problem consists in finding a set of parameters w guaranteeing the fulfillment of Assumption 9 over the grid of ζ points designed according to Assumption 8.

Consider now a point $\xi \in \text{dom}(\kappa)$; from (3.28) we have that for all $\zeta_\xi \in \mathcal{B}^n(\xi, \overline{q})$

$$\begin{aligned}
&\eta_u\Big(|\kappa^*(\zeta_\xi|w) - \kappa^*(\xi|w)| + |\kappa^*(\zeta_\xi|w) - \kappa(\zeta_\xi)|\Big) + \eta_x(|\zeta_\xi - \xi|) \\
&\quad \leq \eta_u\Big(L_{\kappa^*}(\zeta_\xi|w)|\zeta_\xi - \xi| + |\kappa^*(\zeta_\xi|w) - \kappa(\zeta_\xi)|\Big) + \eta_x(|\zeta_\xi - \xi|) \\
&\quad \leq \eta_u(L_{\kappa^*}(\zeta_\xi|w)\overline{q} + |\kappa^*(\zeta_\xi|w) - \kappa(\zeta_\xi)|) + \eta_x(\overline{q}) \\
&\quad \leq \eta_u\Big(L_{\kappa^*}(\zeta_\xi|w)\overline{q} + |\kappa^*(\zeta_\xi|w) - \kappa(\zeta_\xi)|\Big) + L_{f_x}\overline{q} + \overline{q}.
\end{aligned}$$

Conversely, the satisfaction of

$$\eta_u\Big(L_{\kappa^*}(\zeta|w)\overline{q} + |\kappa^*(\zeta|w) - \kappa(\zeta)|\Big) + (L_{f_x} + 1)\overline{q} \leq \overline{d}(\zeta) - \overline{d}_w$$

for some $\zeta \in X_G$ implies that (3.27) is verified for all $\xi_\zeta \in \mathcal{B}(\zeta, \overline{q})$. Therefore, if the last inequality holds for all $\zeta \in X_G$, then κ^* verifies Assumption 9 in the whole domain X_{RH}.

Setting

$$\epsilon(\zeta) = \overline{d}(\zeta) - \overline{d}_w - (L_{f_x} + 1)\overline{q},$$

the last inequality can be rewritten as

$$\eta_u\Big(L_{\kappa^*}(\zeta|w)\overline{q} + |\kappa^*(\zeta|w) - \kappa(\zeta)|\Big) \leq \epsilon(\zeta),$$

which, in turn, can be rearranged as

$$L_{\kappa^*}(\zeta|w)\overline{q} + |\kappa^*(\zeta|w) - \kappa(\zeta_\xi)| \leq \eta_u^{-1}(\epsilon(\zeta)).$$

---

first layer and linear output layer, denoting as $w \in \mathbb{R}^{n_w}$ the overall network parameters, it holds that the approximating function is locally Lipschitz for any value of w. However, the function $L_{\kappa^*}(\xi)$ (which locally bounds the Lipschitz constant) is a function of the parameters.

From a practical point of view, one can first sample the domain with a grid X_G with density parameter $\overline{q}$ and then, posing $\epsilon^{'}(\zeta) \triangleq \eta_u^{-1}(\epsilon(\zeta)),\ \forall\zeta \in X_G$, it will be possible to evaluate the map $\epsilon^{'}(\cdot)$ on the grid points. Then, a sufficient condition on the approximating function to guarantee the ISpS property for the closed-loop system can be expressed in the compact form

$$L_{\kappa^*}(\zeta|w)\overline{q} + |\kappa^*(\zeta|w) - \kappa(\zeta)| \leq \epsilon^{'}(\zeta),\ \forall\zeta \in X_G. \tag{3.29}$$

Remark 3.6.1 (η_u calculation)**.** *In order to compute $\epsilon^{'}(\zeta)$, the $\mathcal{K}-$function $\eta_u(\cdot)$ must be known. In general, for a nonlinear transition map $\hat{f}$ we can easily compute a local linear bound on this function: indeed, the problem of finding a global $\mathcal{K}-$function $\eta_u(\cdot)$ is simplified in that of computing a local Lipschitz bound $L_{f_u}(\zeta)$ such that $|\hat{f}(\xi_\zeta, u) - \hat{f}(\xi_\zeta, u^{'})| \leq L_{f_u}(\zeta)|u - u^{'}|$, $\forall\xi_\zeta \in \mathcal{B}^n(\zeta, \overline{q})$, $\forall(u, u^{'}) \in U^2$. A conservative bound on L_{f_u} can be evaluated as*

$$L_{f_u}(\zeta) = \max_{(\xi_\zeta, u)\in\mathcal{B}^n(\zeta,\overline{q})\times U} \sum_{j=1}^{m}\sum_{i=1}^{n} \left| \frac{\partial \hat{f}_{(i)}}{\partial u_{(j)}}|_{\xi_\zeta, u} \right|, \tag{3.30}$$

where $\hat{f}_{(i)}$ and $u_{(j)}$ are the $i-th$ and the $j-th$ components of $\hat{f}$ and u respectively. □

To conveniently design the NN, it is possible to resort to the method proposed in [81], which allows to optimize the number and the position of the neurons when the tolerance-tube (in our case $\epsilon^{'}(\zeta)$) varies in the domain.

As far as implementation is concerned, it is important to notice that, for most applications, the use of a NN allows to save memory resources compared to the NP approach: the net needs to store in memory just $(num\ of\ neurons)\times$ $(parameters\ per\ neuron)$ elements, whilst the latter requires *(n × (num of points of the grid))×* m elements, where n and m denote the dimensions of the state and input respectively. Since the number of neurons needed to obtain a suitable approximator is usually far smaller than the number of reference points in the training set (which coincides with the grid set of the Nearest Point), then a properly designed Neural Network can reduce consistently both the memory and the time required for on-line computations.

3.7 Simulation Results - Smooth Approximator case

Consider the following system (undamped nonlinear oscillator):

$$\begin{cases} x_{(1)_{t+1}} = x_{(1)_t} + 0.05[-x_{(2)_t} + 0.5(1 + x_{(1)_t})u_t] \\ x_{(2)_{t+1}} = x_{(2)_t} + 0.05[x_{(1)_t} + 0.5(1 - 4x_{(2)_t})u_t] \end{cases} \tag{3.31}$$

subjected to constraints (3.2) and (3.3), with X as depicted in Figure 3.9 and $U \triangleq \{u \in \mathbb{R} : |u| \leq 2\}$.

The Lipschitz constant for the considered system can be obtained numerically by searching for the maximum slope among each pair of an adequately set of points chosen over the domain (in the considered case, a grid): the applicability of this procedure is naturally limited to the case of functions sufficiently regular and restricted in a small domain. If these conditions are not satisfied, a rigorous algebraic analysis of the behaviour of the considered function in the domain will be necessary. For the proposed example, the Lipschitz constant is $L_{f_x} = 1.1390$.

The auxiliary control law can be designed following e.g. [81], obtaining in this case the feedback control law $u_t = \kappa_f(x_t) = k^T x_t$ with $k^T = [0.5955 \;\; 0.9764]$; other choices accomplishing to our Assumptions are $N_c = 8$ and

$$X_f = \left\{ x_t \in \mathbb{R}^2 : x_t^T \begin{bmatrix} 114.21 & -29.45 \\ -29.45 & 208.67 \end{bmatrix} x_t \leq 1 \right\},$$

The stage cost is $h(x, u) = x^T Q x + u^T R u$, while the final cost is $h_f(x) = x^T P x$, with

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \; R = 1, \; P = \begin{bmatrix} 91.56 & -23.61 \\ -23.61 & 167.28 \end{bmatrix}.$$

The exact MPC feedback function is depicted in Figure 3.6. The following bounds on additive uncertainty can be evaluated considering the invariance properties of the chosen terminal set: $\overline{d} = 5.94 \cdot 10^{-4}$ and $\overline{q}_{NP} = \overline{q} = 2.75 \cdot 10^{-4}$.

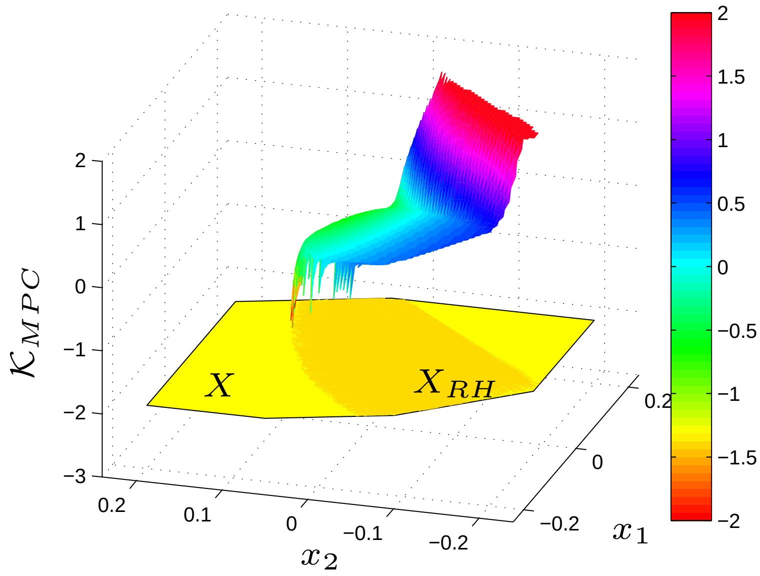

Figure 3.6: The exact nonlinear MPC feedback law.

The feasible training points belonging to the intersection $X_G \cap X_{RH}$ are depicted by darker dots in Figure 3.9). It is worth to notice that the number of points obtained by gridding is around $1.4 \cdot 10^6$; this means that the evaluation of the exact control law has to be done for a very large data set, demanding, for this example on a Xeon 3GHz CPU with 4 GB Ram, a couple of days of calculation.

To carry out the off-line approximation, a NN with two layers, 453 centres (for the distribution of centres see Figure 3.10) and Gaussian activation function is sufficient to fulfil the error bounds (which can be found in Figure 3.8) in the whole training set: the obtained control law can be found in Figure 3.7.

Due to the high number of elements of the training set and to the strictness of the bounds in particular near the border of the domain, a completely automatic tuning of the Neural Network for the proposed example results in a very difficult task: in view of this, the strategy followed to get to the approximation is based on a semi-automatic placement of the centres, which in a first time have been located manually in the zones in which the committed

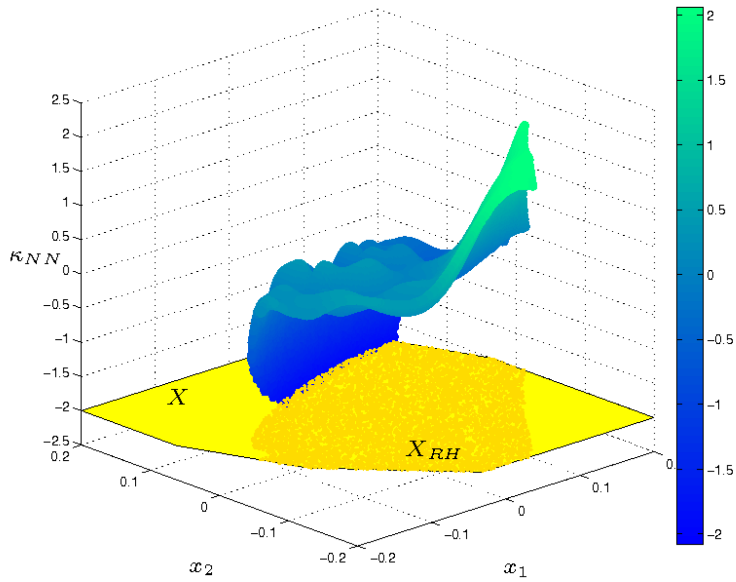

Figure 3.7: The Neural Networks based approximation of the exact feedback law.

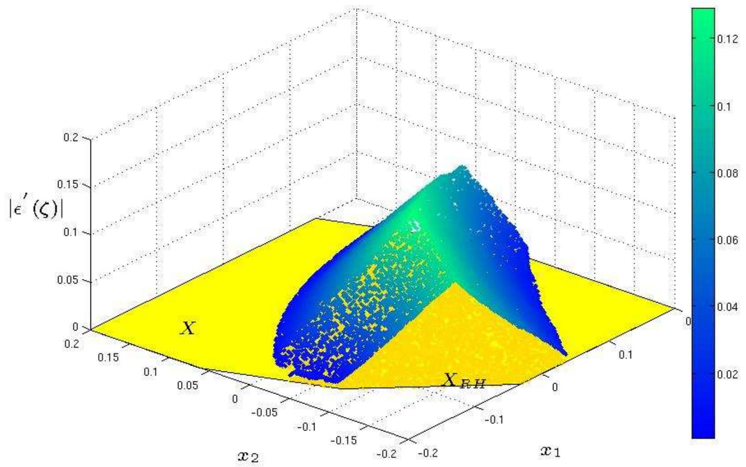

Figure 3.8: Absolute value of the error bounds map: notice that a lower slope of the control law to be reconstructed results in higher values of the corresponding bounds.

error was too high with respect to the admissible one. When adding new centres didn't lead to a significant improvement of the net performances, we switched to the automatic placement of a new center in every point for which the error committed in reconstruction was greater than the bound. This way, there is no certainty that the obtained Neural Network is optimal with respect to the number and locations of the centres, but this approach resulted more practicable than the minimization of the related cost function over a so large domain. The last task could be achieved e.g. by recurring to recently developed efficient optimization methods, such as Genetic Programming (for an overview on this framework, see [8] and references therein). Another interesting approach could be the recourse to Support Vector Machine (see [97] and references therein) based regression. The study of the applicability of both of the last cited approaches represents a subject for possible future researches and improvements of the proposed method.

Despite the above mentioned computational burden, once the off line training has been completed, the storage request for the on line control application goes from $1.4M$ values (Nearest Neighbour case) to 453 values for the location of the centers plus 453 (scalar) values for the variance of each center; furthermore, having recourse to Neural Networks implies a reduction of the on line time requested, since a comparison of the reached state with respect to all the points of the grid is not necessary anymore.
Always in Figure 3.9 some closed loop trajectories are shown: notably, the application of the approximated control renders the system ISpS, that is, the system driven by the approximate feedback is not guaranteed to asymptotically converge to the origin: such a behavior is due to the fact that the approximation errors do not vanish along the trajectories. Nonetheless, a smaller convergence region can be obtained at the cost of increasing the complexity of the network and refining the training of the parameters to further reduce the approximation error.

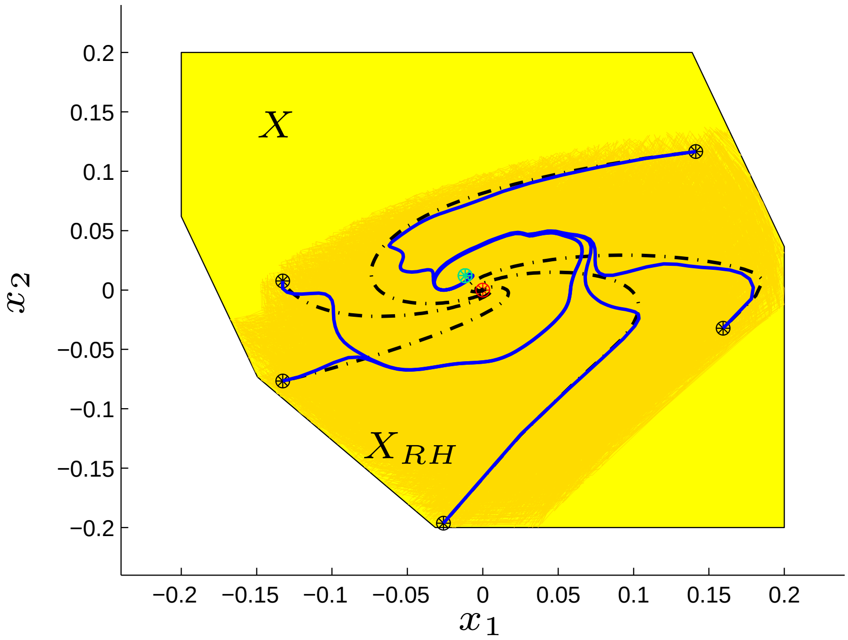

Figure 3.9: The trajectories obtained by the application to five different starting points of the exact control law (black dash-dotted lines) are compared with those given by the approximate Neural Network feedback (blue solid lines). In yellow is depicted the set X, while in light orange it is possible to see the feasible set for the considered MPC problem, X_{RH}. The exact control law achieves the asymptotic convergence of the trajectories to the zero (central asterisk), while the approximate feedback drives the state to the point in green: the closed loop system is ISpS.

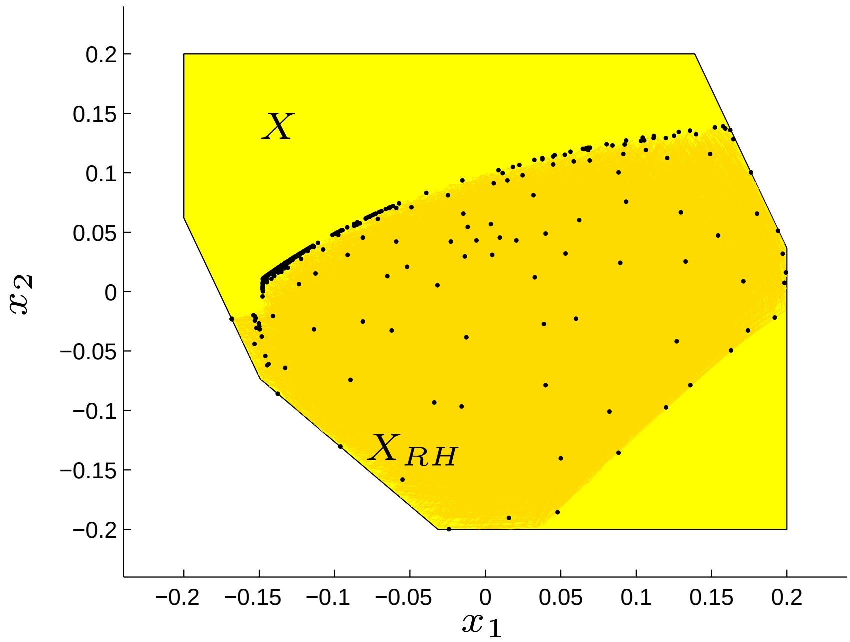

Figure 3.10: A view of the distribution of the centers of the Neural Network basis functions over the feasible region X_{RH}; notice that the density increases in the upper-left zone, where the control function experiences a significant increase of the slope.

Chapter 4

Stabilizing Networked MPC For Interconnected Systems

4.1 Introduction

In the previous chapter, stability analysis has been carried out for nonlinear discrete time systems in presence of disturbances and approximation-induced errors. The controller can be derived by means of the choice of an adequate approximator, depending on the regularity characteristics of the stabilizing MPC law, obtained by solving off line a conveniently set up optimization problem.

The importance of a right setting for the Receding Horizon Optimal Control Problem can be inferred from the carried on treatment; in this section we propose an MPC-based approach to stabilization over a closed-loop networked control architecture in the presence of delays and packet dropouts. Then, robustness analysis will be extended with respect to this source of uncertainty, which is nowadays one of the more investigated due to its importance in applications; new emerging technologies indeed are based more and more on networked communication protocols, and so does control systems. In this connection, an increasing interest in control applications of network technologies can be found in the recent literature (see [5], [6], [46], [112] and the references therein), since the networked control approach permits to remotely control large distributed plants with very simple installation and inexpensive maintenance.

In this class of dynamical systems, usually referenced to as Networked Control Systems (NCS's), the controller is not directly linked to process, but sensors data and input commands are transmitted by using a shared communication network. The main drawback of such a control architecture resides in the poor reliability of networked data transmission: due to the increase of the number of applications sharing computing and communication resources, some inconveniences related to network delay and data loss may occur, in particular when network congestion arises.

It turns out, at this point, that network constraints strongly affect the behavior of NCS's and need to be taken in account in the control design. Furthermore, the system to be controlled is almost always subjected to its own state and input constraints, that need too to be respected in any operating condition, even in presence of disturbances or model uncertainty.

As regards networked linear time invariant control systems, many strategies have been proposed to design effective control schemes ([35], [65], [96], [104]) capable to face with network-induced delays; recent results are focused on stochastic characterization of delays in order to implement LQG control policies ([100], [19], [99],[49]).

Considering the network congestion, an effective way to curb the problems related to this phenomenon consists in using protocols which allow to transmit fewer but more informative packets ([4], [35]); in view of this, the packet structure of most transmission networks has important implications from the control point of view ([107]). In this regards, an improvement can be derived by the use of large data packets to collect data from multiple sensors and to transmit entire sequences of control moves to the actuators. In this framework, MPC techniques play an important role (see [19], [102], [80] when strict bounds on data delays can be assumed, [78] for delays on the measurement channel, and [66] in presence of heterogeneous measurements).

In line with these considerations, based on RH strategy, the future input sequence calculated at each step can be sent to the actuators on a single data packet, without significantly increasing the network load ([87], [103]).

It is then possible to resort to robust control strategies originally conceived for constrained linear system (see [23], [93] and [95], where MPC policies are used) together with the adoption of a Network Delay Compensation (NDC) strategy, based on time-stamping, capable to overcome the limits of the networked communication. In this regards, in [84] it is shown that the MPC succeeds in guaranteeing the fulfilment of state and input constraints under networked packet-based communications.

In this work, we will consider the class of interconnected dynamical systems, both linear and nonlinear, subjected to state and input constraints, to be robustly controlled by a networked predictive control strategy, based respectively on TCP and UDP data transmission protocols. We will assume that each subsystem is pre-compensated by a local linear controller, which exchanges informations with a centralized supervisory controller; the last is in charge to improve robustness for the overall networked system guaranteeing the boundedness of the state trajectories even in presence of disturbances, delays and eventual data losses on the controller-to-subsystem paths. Indeed, in most distributed systems, since the design of local compensators is usually carried out by taking in consideration local state measurements and partial dynamics information, the behaviour of the system emerging from the interconnection may lead to poor control performance, constraint violation, and even instability. Therefore, a networked supervisory controller is needed to coordinate the agents with the final aim of preserving the boundedness of the state trajectories within the prescribed constraints.
A real-world example of such distributed control systems is represented for instance by power networks with distributed generation, in which the load-frequency control of each power unit is first accomplished by a local controller (the so-called primary control loop), while a centralized supervisor (the secondary control loop) provides set-points to the remote units on the basis of nominal closed-loop models of the generators (see [48] for a detailed survey on the topic).

The objective of uniformly bounding the closed-loop trajectories of the overall system will be guaranteed by proving the recursive feasibility of the overall scheme; an extension of the presented approach to ISS stability can be found in very recent paper [85].The robust enforcement of hard state and input constraints will be achieved by using a constraint tightening technique like the one presented in the previous chapter together with a reduction of the control horizon length. We will prove that the system can be controlled by a MPC scheme in which the loop is closed through a packet-based communication network with delays, by assuming that model uncertainties are bounded and transmission delays are subjected to suitable conditions to be specified later on. It is worth to mention that other approaches to stability of networked control systems can be found in literature: for instance, stochastic stability, which e.g. in [64] is obtained when delays can be characterized by Markov chains.

Notice that, being the system affected by disturbances, the recursive feasibility is not an easy task to be achieved; furthermore, the feasibility of hard constraints in networked MPC has not been studied, yet, for locally pre-compensated interconnected system.

In the considered context, with the aim of uniformly bounding the closed-loop trajectories of the interacting agents, the proposed supervisory control policy based on MPC is combined with a network delay compensation (NDC) strategy relying on time-stamps.

Compared with recent contributions on non-acknowledged predictive NCS (see e.g.[105] and [42]), the presented work allows to enforce hard constraints on state and input variables despite bounded transition uncertainty, by exploiting ideas from constraint-tightening nonlinear MPC: this result will be achieved by making the constraint tightening technique dependent on the delays (see [23], [93] and [95] for the non-networked case). By assuming that the transmission delays and the model uncertainties subsume suitable upper bounds, we will prove the recursive feasibility of the scheme and consequently the satisfaction of the specified constraints. Notice that a very recent result for ISS stability for non linear constrained systems can be found in [89], but the proposed approach is capable to face only packet dropouts and not delays (furthermore, a recursive feasibility proof can not be found in that work); on the other side, in [22] it is suggested a way to control linear systems via a wireless network, but there packet dropouts are not considered at all.

In the following, firstly we will propose a procedure capable to robustly control interconnected constrained linear dynamical systems over TCP-like networks, giving directions on how to calculate the maximal uncertainty set; the results will be then extended to the much more general case of systems made of physical interconnection of nonlinear time-invariant subsystems communicating with the supervisor over UDP-like networks.

4.2 Networked MPC - Linear Systems over TCP Networks case

Consider a generic system consisting in the interconnection of $n_s \in \mathbb{Z}_{>0}$, $n_s \geq 2$ linear time-invariant, discrete-time subsystems:

$$\begin{cases} x_{1t+1} = \mathbf{A}_1 x_{1t} + \mathbf{B}_1 u_{1t} + \mathbf{G}_1 v_{1t} + \sum_{j=2}^{n_s} \mathbf{F}_{1,j} x_{jt} \\ \quad\vdots \\ x_{it+1} = \mathbf{A}_i x_{it} + \mathbf{B}_i u_{it} + \mathbf{G}_i v_{it} + \sum_{j=1, j\neq i}^{n_s} \mathbf{F}_{i,j} x_{jt} \\ \quad\vdots \\ x_{n_s t+1} = \mathbf{A}_{n_s} x_{n_s t} + \mathbf{B}_{n_s} u_{n_s t} + \mathbf{G}_{n_s} v_{n_s t} + \sum_{j=1}^{n_s-1} \mathbf{F}_{n_s,j} x_{jt} \end{cases} \tag{4.1}$$

where $x_{i0} = \overline{x}_{i0},\ t \in \mathbb{Z}_{\geq 0},\ i \in \{1, \dots, n_s\}$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $v_i \in \mathbb{R}^{n_i}$.

Each i-th subsystem in (4.1) is pre-compensated by a linear control law with local state feedback, i.e. $u_{it} = \mathbf{K} x_{it} + c_{it}$, where the input corrections $c_{it}, i \in \{1, \dots, n_s\}$ are generated by a networked centralized controller in charge of fulfilling global control objectives.

Suppose that the communication between the subsystems and the centralized controller rely on the transmission of data-packets over a network affected by random bounded delays in both feedback and command channels; on the other side, suppose that the interconnections between the subsystems are deterministic and that the local state feedback policies are not affected by delays.

These considerations lead to the possibility of rearranging the overall system as

$$x_t = \mathbf{A}x_t + \mathbf{B}u_t + \mathbf{G}v_t,\ x_0 = \overline{x}_0,\ t \in \mathbb{Z}_{\geq 0} \tag{4.2}$$

or, by taking in account the presence of local controllers and correction inputs, as

$$x_t = \mathbf{A_K}x_t + \mathbf{B}c_t + \mathbf{G}v_t,\ x_0 = \overline{x}_0,\ t \in \mathbb{Z}_{\geq 0}\,, \tag{4.3}$$

with $x_t \triangleq \mathrm{col}[x_{1t}, \ldots, x_{n_st}] \in \mathbb{R}^n$, where n is the overall system dimension, $v_t \triangleq \mathrm{col}[v_{1t}, \ldots, v_{n_st}] \in \mathbb{R}^n$ and $c_t = \mathrm{col}[c_{1t}, \ldots, c_{n_st}] \in \mathbb{R}^m$, where m is the dimension of the overall correction input vector, which is applied to the subsystems by the local controllers on the basis of the information received from the networked master controller. Here, we have defined

$$\mathbf{A_K} \triangleq \mathbf{A} + \mathbf{BK},$$

where $\mathbf{K}$ is the overall pre-compensation matrix (in general sparse) due to the local linear-feedback laws.

Notice that the presence of local feedbacks gives more generality to the approach: indeed the absence of pre-compensation can be considered, for the stabilizing procedure proposed, as a particular case, i.e. the case with $\mathbf{A_K} = \mathbf{A}$, which implies $c_t \equiv u_t$.

Let us assume that the control input $u_t = \mathbf{K}x_t + c_t$, the state x_t and the disturbance v_t are subjected to hard constraints, i.e.,

$$u \in U, x \in X, v \in V, \tag{4.4}$$

where $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^r$ are (convex, compact) polytopes, containing the origin in their interior, while $X \subset \mathbb{R}^n$ is a (convex) closed polyhedron.

In the following, we will denote as $d_t = \mathbf{G}v_t$ the additive uncertainty vector, and $D \triangleq \{d \in \mathbb{R}^n | d = \mathbf{G}v, v \in V\}$ the set of additive disturbances.

Finally, given an input sequence $\mathbf{c}_{0,i-1}, i \in \mathbb{Z}_{>0}$, and initial condition $\overline{x}_0$, we will denote as $x_i = x_{\mathbf{K}}(i, \overline{x}_0, \mathbf{c}_{0,i-1}, \mathbf{d}_{0,i-1})$ the state of the perturbed system at time i, while $\hat{x}_{i|0} = \hat{x}_{\mathbf{K}}(i, \overline{x}_0, \mathbf{c}_{0,i-1})$ will denote the nominal state prediction obtained with the model (4.3) starting from $t = 0$, assuming $v_t \equiv 0, \ \forall t \in \mathbb{Z}_{\geq 0}$.

It is worth noting that, *the pre-compensator is a local pre-stabilizing feedback, which is not guaranteed to yield the overall stability of the interconnection; the networked controller in this case is needed to stabilize the system. Nonetheless, also when the local controllers alone succeed in stabilizing system, the networked controller may improve the robustness of the scheme and enforce the constraint satisfaction.*

In order to state the networked control objective, the following definition will be essential.

Definition 4.2.1 (UB in X)**.** *System (4.2) with the (possibly time-varying) control policy $u_t = \kappa(t, x_t)$ is said to be Uniformly Bounded (UB) in the set X if there exists an initial condition set $\Xi \subseteq X$, such that for every initial condition $\overline{x}_0 \in \Xi$ and all $\boldsymbol{v} \in \mathcal{M}_V$ (or $\boldsymbol{d} \in \mathcal{M}_D$) we have $x_t \in X$, $u_t \in U$, $\forall t \in \mathbb{Z}_{>0}$.*

Notably, the Uniform Boundedness property is particularly important when the main objective is to keep the state within a prescribed region, facing external perturbations and model uncertainties (see [14]). Furthermore, here we require in addition that the UB is guaranteed despite the presence of communication delays.

As far as the network communication is concerned, a scheme of the NCS topology considered in this work is depicted in Figure 4.1.
It is assumed that at a given time instant a data packet can be sent through the network by a node, while both the sensor-to-master controller and the master-to-distributed controllers links are supposed to be affected by time-varying bounded delays due to the stochastic nature of networked communications.

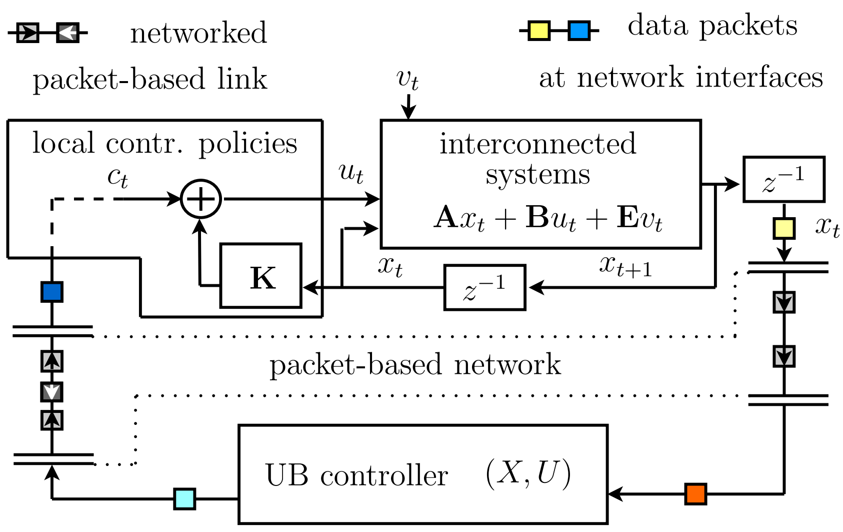

Figure 4.1: Underlying structure of the NCS under consideration. A networked centralized controller is in charge of fulfilling a global control objective (uniform boundedness of closed-loop trajectories) for a system consisting in the interconnection of locally pre-compensated subsystems.

In order to cope with delays, the data packets are assigned a Time-Stamp (TS) containing the information on when they were delivered by the transmitting node. In other words, the data packets sent by the sensor node contain the information on when the transmitted state measurement had been collected. Analogously, the controller node is required to attach to each data packet the time stamp of the state measurement which the computed control action relies on. The time-stamping policy in NCS's requires, in general, that all the nodes of the network have access to a common system's clock, or that a proper clock synchronization service is provided by the network protocol. In our setup, we will assume that perfect clock synchronization is maintained between sensors, actuators and controller, without focusing on the method used to maintain synchronization (to this purpose, see [106], [101], [110] and the references therein).

Moreover, we consider the case of networks with acknowledged communication protocol, also known as *TCP-like* ([49]), in which the destination node sends an acknowledgment packet (ACK) of successful packet reception to the source node.

In a TCP-like scenario, the acknowledgment messages are assumed to have the highest priority among all the routed packets, such that, after each successful packet reception, the source node receives a deterministic notification within a single time-interval.
In this connection, the presence of ACKs in TCP-like networks can be exploited by the controller (which is acknowledged of successful packet reception by the local controllers) to internally reconstruct the true sequence of controls which have been applied to the plant (see [87]) from time instant $t - \tau_c(t)$ to $t-1$, in order to get an estimation of the current state $\hat{x}_{t|t-\tau_c(t)}$ on the basis of the most recent available plant measurement $x_{t-\tau_c(t)}$.

Now, it is necessary to take into account delays and data losses (we will consider the presence of data losses in the controller-to-subsystems path, as specified in the sequel): to formulate a robust stabilizing controller for the setup just introduced we need some assumptions on the network reliability, that will be addressed in the next section.

4.2.1 Network Reliability and Network Delay Compensation

In the sequel, $\tau_{ca\,i}(t)$ and $\tau_{sc\,i}(t)$ will denote the delays occurring respectively in the master-to-i-th-subsystem and in the i-th-subsystem-to-master links, while $\tau_{a_i}(t)$ will represent age (in discrete time instant) of the information used by the i-th local controller (generated from the master at time $t - \tau_{a_i}(t)$) to compute the current input and $\tau_{ci}(t)$ the age of the i-th subsystem state measurement information available at time t at the master node.

Let

$$\tau_c(t) \triangleq \max_{i \in \{1,\ldots,n_s\}} \{\tau_{c_i}(t)\}$$

be the worst case age of subsystem state measurements available at time t.

Finally,

$$\tau_{rti}(t) \triangleq \tau_{a_i}(t) + \max_{i \in \{1,\ldots,n_s\}} \{\tau_{c_i}(t - \tau_{a_i}(t))\}$$

is the so-called *round trip time*, i.e., the age of the oldest subsystem-wise state measurement used to compute the input applied at time t.

Assumption 1 (Network Reliability)**.** *The quantities* $\tau_{sc}(t)$ *and* $\tau_a(t)$ *verify* $\tau_c(t) \leq \tau_{sc}(t) \leq \overline{\tau}_c$ *and* $\tau_a(t) \leq \overline{\tau}_a$, $\forall t \in \mathbb{Z}_{>0}$, *with* $\overline{\tau}_c + \overline{\tau}_a \leq \overline{\tau}_{rt}$, *for some* $\overline{\tau}_{rt} \in \mathbb{Z}_{\geq 0}$ *finite.* □

Notice that in most situations, assuming that the age of the data-packets available at the master and local controller nodes subsume an upper bound is natural.

The Network Delay Compensation strategy adopted in the present work, which relies on the one devised in [87] (originally developed for unconstrained systems), is based on exploiting the time stamps of the data packets in order to retain only the most recent informations at each node.
When a new packet is received from the distributed controllers, if it carries a more recent time-stamp than the one already in the buffer, then an acknowledgment of successful packet reception is sent to the master.

It is worth noting that the TS-based packet arrival management implies that $\tau_{a_i}(t) \leq \tau_{cai}(t)$; *since* $\tau_{cai}(t)$ *is not limited, Assumption 1 allows for the presence of packet dropouts in the controller-to-subsystems paths (considered as infinite transmission delays).*
Conversely, being $\tau_{c_i}(t) \leq \tau_{sci}(t) \leq \overline{\tau}_c$, *the feedback links from the subsystems to the master must not be affected by data losses.* This property can be ensured, for instance, by choosing a suitable communication protocol for acquiring the data collected from the field.

In addition, the proposed NDC strategy comprises a Future Input Buffering (FIB) mechanism, which requires the master to send to each subsystem a packeted sequence of N_c corrections (with $N_c \geq \overline{\tau}_{rt} + 1$), relying on a model-based prediction. A graphical description of FIB can be found in Figure 4.2: initially the buffer contains a sequence of three steps previously received. At time $t = 0$ the controller sends the sequence of N_c control moves by the net, but the control sequence doesn't reach the actuator in time, so to the plant it is applied the element in the buffer relative to $t = 0$. At time $t = 1$ a new sequence is sent, but it doesn't reach the actuator too; the input will be the element buffered for $t = 1$. At $t = 2$ both of the previously computed sequences reach the actuator: the most recent is put in the buffer and the input associated with $t = 2$ is applied to the plant.

At the arrival of a newer time-stamped packet, each i-th local controller can store into an internal buffer an entire sequence $\mathbf{c}_i^b$ of N_c corrections. Then, at each time instant t, it retrieves a time-consistent correction from the buffer and applies to the i-th subsystem the control action

$$u_{it} = c_{it}^{\,b} + \mathbf{K}_i x_{it},$$

where $c_{it}^{\,b}$ is the $\tau_{a_i}(t)$-th element of the locally buffered sequence $\mathbf{c}_{i\,t-\tau_{a_i}(t),t-\tau_{a_i}(t)+N_c-1}^{\,b}$, which is given by

$$\begin{aligned}\mathbf{c}_{i\,t-\tau_{a_i}(t),t+N_c-1}^{\,b} &= \mathrm{col}[c_{i\,t-\tau_{a_i}(t)}^{\,b}, .., c_{it}^{\,b}, .., c_{i\,t-\tau_{a_i}(t)+N_c-1}^{\,b}] \\ &= \mathbf{c}_{i\,t-\tau_{a_i}(t),t+N_c-1|t-\tau_{rt_i}(t)}^{\,c},\end{aligned}$$

where the sequence $\mathbf{c}_{i\,t-\tau_{a_i}(t),t+N_c-1|t-\tau_{rt_i}(t)}^{\,c}$ had been computed at time $t - \tau_{a_i}(t)$ by the master on the basis of the interconnection state measurement collected (considering the worst case sensor-to-master delay) at time $t - \tau_{rt_i}(t)$.

A formal procedure describing in detail the operations to be performed by the master and by the distributed controllers will be given in Section 4.2.3.

Now, we are going to describe the way in which the controller has to compute the sequence of control actions to be forwarded to the subsystem's controllers: the procedure is based on the solution of a suitable Receding Horizon Optimal Control Problem over a reduced horizon.

4.2.2 Networked Predictive Control

The corrections computed at time t by the master controller are based on an overall state measurement performed, considering the worst case sensor-to-

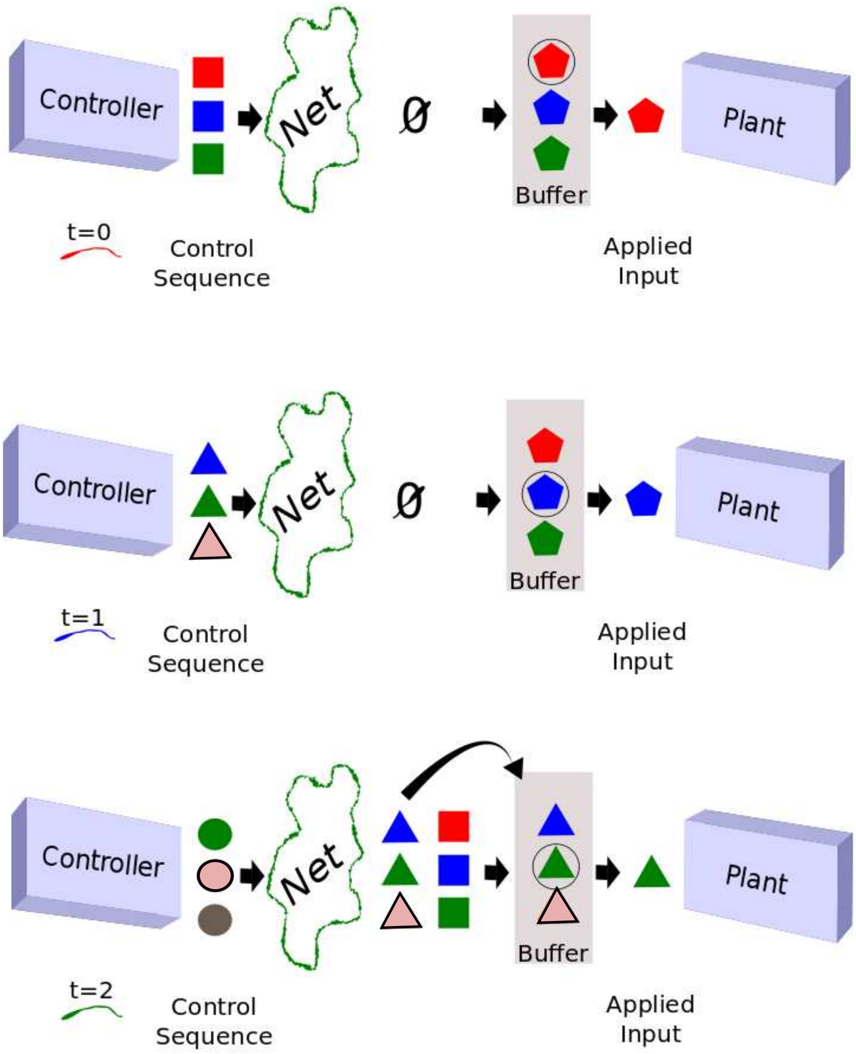

Figure 4.2: Graphical representation of FIB mechanism, with $N_c = 3$; each different color represent a different time instant; moreover, sequences computed in different instants are characterized by different symbols. For instance, the green triangle represents the control action to be applied at time $t = 2$ (green), based on the sequence calculated at time $t = 1$ (triangles).

master delay, at time $t-\tau_c(t)$ (i.e, $x_{t-\tau_c(t)}$). In order to recover the standard MPC formulation, the current state x_t has to be reconstructed with the nominal model (4.3) with $v_t \equiv 0, \forall t \in \{\tau_c(t), \dots, t-\tau_c(t)-1\}$ under the action of the true sequence of corrections $\mathbf{c}_{t-\tau_c(t),t}$ applied to the overall system from time $t-\tau_c(t)$ to $t-1$. In this regard, the benefits due to the use of a state predictor in NCS's are deeply discussed in [87] and in [103, 102].
The sequence $\mathbf{c}_{t-\tau_c(t),t-1}$ can be internally reconstructed by the controller thanks to the acknowledgment-based protocol.

Nevertheless, in presence of delays in the controller-to-subsystems paths, we must consider that the computed correction sequences may not be commanded entirely to the plant, but that the truly applied input sequence may be, in general, made up of pieces of sequences computed in different time instants, if no proper provisions are adopted to recast the problem in a deterministic framework. This could lead the system to instability or constraint violation (see figure 4.3).

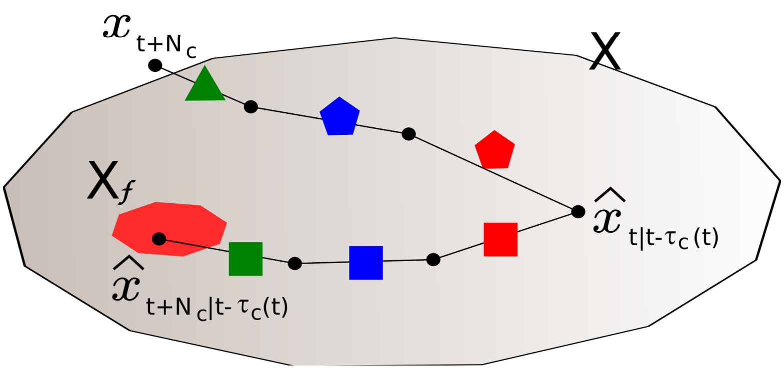

Figure 4.3: Possible effects of the application of a sequence made up of pieces of sequences computed in different time instants, without provision with respect to network induced delays: state could even violate constraints. Symbols and colours has the same meaning than in Figure 4.2.

In order to ensure that the sequence used for prediction would be the same of the one applied in practice to the plant, we can retain, at time t, some of the elements of the input correction sequence computed at time $t-1$ (i.e., the subsequence $\mathbf{c}^b_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}$). Then, the optimization will be carried out only over the remaining elements (i.e. the sequence $\mathbf{c}_{t+\overline{\tau}_a,t+N_c-1}$), initiating the finite horizon optimization with the state prediction $\hat{x}_{t+\overline{\tau}_a}$.

In contrast with the usual MPC setup, in which the number of decision variables of the optimization is equal to length of the horizon N_c, the proposed method relies on the solution, at each time instant t, of a Reduced Horizon Optimal Control Problem (RHOCP): the number of decision variables is then reduced by reusing some elements of the solution obtained in the previous optimization. This feature will allow us to address the problem of delayed communications in the master-to-subsystems paths and, together with the adoption of a constraint tightening technique (see [23] and [95]), will allow us to prove the recursive feasibility of the scheme.

Let's now introduce some definitions useful for our purposes.

Definition 4.2.2 ($\mathcal{C}_1(\Xi|\mathbf{A_K},\mathbf{B},Y)$)**.** *Given a compact set $\Xi \subset \mathbb{R}^n$ and a linear system in the form of (4.2) subjected to the input constraint $c \in Y \subset \mathbb{R}^m$, with Y compact, the controllability set of Ξ under $\mathbf{A_K},\mathbf{B},Y$ is defined as*

$$\mathcal{C}_1(\Xi|\mathbf{A_K},\mathbf{B},Y) \triangleq \{x \in \mathbb{R}^n | \exists c \in Y : \mathbf{A_K}x + \mathbf{B}c \in \Xi\} \ .$$

□

Definition 4.2.3 (d-control invariant set)**.** *Given a compact set $\Xi \subset \mathbb{R}^n$ and a linear system in the form of (4.2) subjected to the input constraint $c \in Y \subset \mathbb{R}^m$, with Y compact, the set Ξ is d-control invariant under $\mathbf{A_K},\mathbf{B},Y$ if $\exists\ \overline{d} \in \mathbb{R}{>}\ 0$ such that*

$$\Xi \subseteq \mathcal{C}_1(\Xi \backsim \mathcal{B}^n(\overline{d})|\mathbf{A_K},\mathbf{B},Y)\,.$$

□

In order to derive the main result concerning the recursive existence of feasible solutions for the MPC, let us consider how the uncertainty affects the nominal prediction. In this connection, the next lemma gives indications on how to evaluate an envelope which bounds all the possible perturbed trajectories.

Lemma 4.2.1 (Uncertainty envelope, [23]). *Given an input sequence* $\mathbf{c}_{0,j-1}$, $j \in \mathbb{Z}_{>0}$ *and an initial condition* $x_0 = \overline{x}_0$*, consider the state trajectory obtained by propagating the state with the nominal model under the action of* $\mathbf{c}_{0,j-1}$*. Then, the perturbed trajectories verify*

$$x_j = x_{\mathbf{K}}(j, \overline{x}_0, \mathbf{c}_{0,j-1}, \mathbf{d}_{0,j-1}) \in \hat{x}_{j|0} \oplus T_{\mathbf{K}j}(D),\ \forall \mathbf{d} \in \mathcal{M}_D$$

where

$$\begin{aligned} &T_{\mathbf{K}0} \triangleq 0, \quad T_{\mathbf{K}1} \triangleq D, \\ &T_{\mathbf{K}j+1} \triangleq T_{\mathbf{K}j}(D) \oplus \mathbf{A}_{\mathbf{K}}^{j} D,\ \forall j \in \{1, \ldots, j-1\}. \end{aligned} \tag{4.5}$$

□

Finally, before introducing the Receding Horizon Optimal Control Problem to be solved at each sampling instant, we need a description on how to tighten the constraints in order to guarantee the stability of the overall controlled system in presence of disturbances, obtaining the recursive enforcement of constraints. In view of this, let's define the following sets.

Definition 4.2.4 (Tightened Constraints). *The tightened state constraint* $X_{\mathbf{K}j}(D)$*, and the tightened input sets* $U_{\mathbf{K}j}(D)$ *are defined as*

$$X_{\mathbf{K}j}(D) \triangleq X \backsim\ T_{\mathbf{K}j}(D),\ \forall j \in \mathbb{Z}_{>0}\,, \tag{4.6}$$

$$U_{\mathbf{K}j}(D) \triangleq U \backsim\ \mathbf{K} T_{\mathbf{K}j}(D),\ \forall j \in \mathbb{Z}_{>0}\,. \tag{4.7}$$

□

Now, we are going to describe the RHOCP solved by the master controller to obtain the overall sequence of corrections

$$\mathbf{c}^c_{t,t+N_c-1} = \mathrm{col}[\mathbf{c}_1{}^c_{t,t+N_c-1}, \ldots, \mathbf{c}_{n_s}{}^c_{t,t+N_c-1}],$$

to be sent to the subsystems, where

$$\mathbf{c}^c_{i\,t,t+N_c-1} = \mathrm{col}[c^c_{i\,t}, \ldots., c^c_{i\,t+N_c-1}],$$

with $i \in \{1, \dots, n_s\}$.

Problem 4.2.1 (RHOCP). *Given a positive integer $N_c \in \mathbb{Z}_{\geq 0}$, at any time $t \in \mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t-\tau_c(t)}$ be the estimate of the current overall state, x_t, obtained with the nominal model from the last available state measurement $x_{t-\tau_c(t)}$ with the controls $\mathbf{c}_{t-\tau_c(t),t-1}$ already applied to the plant. Moreover let $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}$ be the state computed from $\hat{x}_{t|t-\tau_c(t)}$ by extending the prediction using a piece of the overall sequence of corrections computed at time $(t-1)$, i.e. $\mathbf{c}^c_{t,t+\overline{\tau}_a-1}$. Then, given a stage-cost function h, the constraint sets $X_{\mathbf{K}j}(D), j \in \{\tau_c(t)+\overline{\tau}_a+1, \dots, \tau_c(t)+N_c\}$, and the terminal set X_f, the Reduced Horizon Optimal Control Problem (RHOCP) consists in solving, with respect to the input sequence $\mathbf{c}_{t+\overline{\tau}_a,t+N_c-1} \triangleq \mathrm{col}[c_{t+\overline{\tau}_a}, \dots, c_{t+N_c-1}]$ (of $(N_c - \overline{\tau}_a)$-steps) the following minimization problem*

$$J^{\circ}_{FH}(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}, \mathbf{c}^{\circ}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}, N_c - \overline{\tau}_a) \triangleq \min_{\mathbf{c}_{t+\overline{\tau}_a,t+N_c-1}} \left\{ \sum_{l=t+\overline{\tau}_a}^{t+N_c-1} h(\hat{x}_{l|t-\tau_c(t)}, c_l) \right\}$$

subject to the

i) nominal dynamics $\hat{x}_{t+j+1|t-\tau_c(t)} = \mathbf{A_K}\hat{x}_{t+j|t-\tau_c(t)} + c_{t+j}, j \in \{\overline{\tau}_a, \dots, N_c-1\}$;

ii) input constraints $c_{t-\tau_c(t)+j} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in U_{\mathbf{K}j}(D)$, with $j \in \{\tau_c(t)+\overline{\tau}_a, \dots, \tau_c(t)+N_c-1\}$;

iii) restricted state constraints $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_{\mathbf{K}j}(D)$, with $j \in \{\tau_c(t)+\overline{\tau}_a+1, \dots, \tau_c(t)+N_c\}$;

iv) terminal state constraint $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f$.

Finally, the sequence of controls forwarded by the master to the distributed controllers is constructed as

$$\mathbf{c}^c_{t,t+N_c-1|t-\tau_c(t)} \triangleq \mathrm{col}[\mathbf{c}^c_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}, \mathbf{c}^{\circ}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}] \tag{4.8}$$

(i.e., it is obtained by appending the solution of the RHOCP to a piece of the

sequence computed at time $(t-1)$). □

Notice that, under the UB objective, the choice of the stage cost $h(\cdot,\cdot)$ is arbitrary, since a proper formulation of the tightened constraints will suffice in guaranteeing the UB property.

The following definitions will be used to prove the stability properties of the proposed control procedure.

Definition 4.2.5 ($X_{MPC}(\tau)$)**.** *Given an integer $\tau \in \{0,\ldots,N_c\}$, the set containing all the vectors $\overline{x}_0 \in \mathbb{R}^n$ for which there exists a sequence $\overline{\mathbf{c}}_{0,N_c-1}$ of N_c control moves which satisfies all the constraints specified below is said feasible set with τ-delay restriction, and it is denoted with $X_{MPC}(\tau)$.*

$$X_{MPC}(\tau) \triangleq \left\{ \bar{x}_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, \overline{\mathbf{c}}_{0,N_c-1} \in \mathbb{R}^{m\times N_c} : \\ \left\{ \begin{array}{l} \overline{c}_{j-1} \in U_{\mathbf{K}\tau+j-1}(D), \\ \hat{x}_{\mathbf{K}}(j,\overline{x}_0,\overline{\mathbf{c}}_{0,j-1}) \in X_{\mathbf{K}\tau+j}(D), \end{array}\right. \\ \qquad\qquad \forall j \in \{1,\ldots,N_c\} \\ \text{and } \overline{x}_{\mathbf{K}}(N_c,\overline{x}_0,\overline{\mathbf{c}}_{0,N_c-1}) \in X_f \end{array} \right\}$$

□

For the sake of brevity, the set $X_{MPC}(0)$ will be denoted as X_{MPC}.

Now we need to define the feasibility of a sequence for the considered case.

Definition 4.2.6 (Feasible sequence at time t)**.** *Given a delayed state measurement $x_{t-\tau_c(t)}$, available at at time t to the controller, let us consider the prediction $\hat{x}_{t|t-\tau_c(t)}$ of the actual state x_t obtained with the nominal model and with the actual control sequence applied from time $t-\tau_c(t)$ to $t-1$, $\mathbf{c}^*_{t-\tau_c(t),t-1}$, which is known to the controller. Moreover consider a sequence of N_c control moves $\overline{\mathbf{c}}^c_{t,t+N_c-1}$ and its two subsequences $\overline{\mathbf{c}}^c_{t,t+\overline{\tau}_a-1}$ and $\overline{\mathbf{c}}^c_{t+\overline{\tau}_a,t+N_c-1}$ such that $\overline{\mathbf{c}}^c_{t,t+N_c-1} = \mathrm{col}[\overline{\mathbf{c}}^c_{t,t+\overline{\tau}_a-1},\overline{\mathbf{c}}^c_{t+\overline{\tau}_a,t+N_c-1}]$.*

The input sequence $\overline{\mathbf{c}}^c_{t,t+N_c-1}$ is said feasible at time t if

1) the subsequence $\overline{\mathbf{c}}^c_{t,t+\overline{\tau}_a-1}$ *yields to* $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_{\mathbf{K}j}(D)$, $\forall j \in \{\tau_c(t)+1,\ldots,\tau_c(t)+\overline{\tau}_a\}$ *and* $\overline{c}^c_{t-\tau_c(t)+j} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in U_{\mathbf{K}j}(D)$, $\forall j \in \{\tau_c(t),\ldots,\tau_c(t)+N_c-1\}$*;*

2) the second subsequence satisfies all the constraints of the RHOCP initiated with $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)} = \hat{x}_{\mathbf{K}}(\overline{\tau}_a, x_{t-\tau_c(t)}, \mathbf{c}^*_{t-\tau_c(t),t+\overline{\tau}_a-1})$

□

Remark 4.2.1. *Note that what we call feasible sequence in t is not just an input sequence which satisfies the state constraints of the RHOCP (specified in the horizon $\{t+\overline{\tau}_a+1,\ldots,t+N_c\}$), but it is required to keep the nominal trajectories inside the restricted constraints for a larger horizon of N_c steps, from $t+1$ to $t+N_c$.*

It is possible now to show that the recursive feasibility of the scheme can be guaranteed for all $t \in \mathbb{Z}_{>0}$, also in presence of norm-bounded additive transition uncertainties and network delays depending on the choice of the terminal constraint set X_f. The following assumption gives a concrete suggestion in this sense.

Assumption 2 (Ω_f, X_f)**.** *There exists a convex set $\Omega_f \subset X$, containing the origin as interior point, such that $\mathbf{K}\Omega_f \subseteq U$ and $A_{\mathbf{K}}\Omega_f \backsim \mathbf{B}(U-\mathbf{K}\Omega_f) \subset \Omega_f$. Then the terminal set is chosen as $X_f = A_{\mathbf{K}}\Omega_f \backsim \mathbf{B}(U-\mathbf{K}\Omega_f)$.* □

Notice that X_f can be calculated using the algorithm proposed in [57] to compute the maximal invariant sets with finite number of iterations.

Finally, before the formalization of the control scheme, we state and prove the following Lemma: it ensures that the original state constraints can be satisfied by imposing to the nominal trajectories in the RHOCP the restricted constraints introduced in Definition 4.2.4.

Lemma 4.2.2 (Robust Constraint Satisfaction)**.** *Any feasible control sequence at time t, $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$, applied in open-loop to the perturbed system from time t to $t+N_c-1$, guarantees that the true (networked/perturbed) state trajectory will satisfy $x_{t+j} \in X$, and $\overline{c}_{t+j-1}+\mathbf{K}x_{t+j-1} \in U$, $\forall j \in \{1,\ldots,N_c\}$.*

□

Proof in A.2.

4.2.3 Formalization of the networked MPC–NDC scheme

The proposed control scheme, which uses the MPC technique to compute the control sequences and a NDC strategy to compensate for network delays, will be address as MPC-NDC scheme.

The overall networked control policy discussed can be formally described by the Procedure 4.2.1 below, which gives the sequence of operations that have to be performed by the NCS components[1].

In the sequel, we will denote as $\mathbf{P}_{sc_i}$ and $\mathbf{P}_{ca_i}$ the data packets sent by to the i-th subsystem to the master and by the master to the i-th distributed controller respectively, while $\mathbf{P}_{ack_i}$ will represent the acknowledgment (which is, in turn, a data packet) transmitted by the subsystem controller to the master. For the sake of clarity, all the packets will be addressed to as data structures of the form $\mathbf{P} = \{\ \mathbf{P}.data,\ \ \mathbf{P}.time\ \}$, containing a data field and a time stamp field.

Moreover, the sensors nodes, the master and the distributed controller are in charge of processing informations and forming suitably structured data packets, by using some internal storage buffers and computational resources.

Denoting as $\mathbf{S}_i$ the local storage memory of the i-th subsystem controller, we assume that $\mathbf{S}_i$ is structured in buffers: $i)$ $\mathbf{S}_i.\mathbf{c} \in \mathbb{R}^m \times N_c$, which is used to store a sequence of N_c future control actions and $ii)$ $\mathbf{S}_i.T \in \mathbb{Z}_{\geq 0}$, which contains the time stamp (i.e., the age)of the information stored in $\mathbf{S}_i.\mathbf{c}$.

The storage memory of the master controller, $\mathbf{M}$, in turn, is structured as follows:

- n_s First-In-First-Out (FIFO) buffers $\mathbf{M}.\mathbf{C}_i \in (\mathbb{R}^m \times N_c) \times \overline{\tau}_a$, used to store the correction sequences forwarded to each subsystem in the past $\overline{\tau}_a$ time instants (each element of $\mathbf{M}.\mathbf{C}_i$ is a sequence);

[1]The low-level TCP–like communication protocol, in charge for packet routing and synchronization, is considered as a service provided by the network transparently to the components of the NCS

- n_s sequence buffers $\mathbf{M}.\mathbf{c}_i \in \mathbb{R}^m \times \overline{\tau}_c$, that are used to store the local corrections applied to the the subsystems from time $t - \overline{\tau}_c$ to $t-1$ (each element is a control move);

- n_s array buffers $\mathbf{M}.X_i \in \mathbb{R}^n \times \overline{\tau}_c$, in which the received state measurement are stored for equalizing[2] the delays;

- n_s scalar buffers $\mathbf{M}.T_i \in \mathbb{Z}_{\geq 0}$, which contain the time stamp relative to most recent measurement stored in $\mathbf{M}.X_i$;

- $2n_s$ counters $\mathbf{M}.k_{seq_i} \in \mathbb{Z}_{\geq 0}$ and $\mathbf{M}.k_{u_i} \in \mathbb{Z}_{\geq 0}$.

Let us denote with $\leftarrow$ a data assignment operation. Given a sequence buffer $\mathbf{B}$ containing N elements (vectors), let us denote as $\mathbf{B}(j)$ the j-th element of the array, with $j \in \{1, \ldots, N\}$. Given a buffer $\mathbf{B}$ containing M sequences of N elements each (such as $\mathbf{M}.\mathbf{C}_i$), let us denote as $\mathbf{B}(l, j)$ the j-th element (vector) of the l-th sequence, with $l \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$. Then, the following procedure can be outlined.

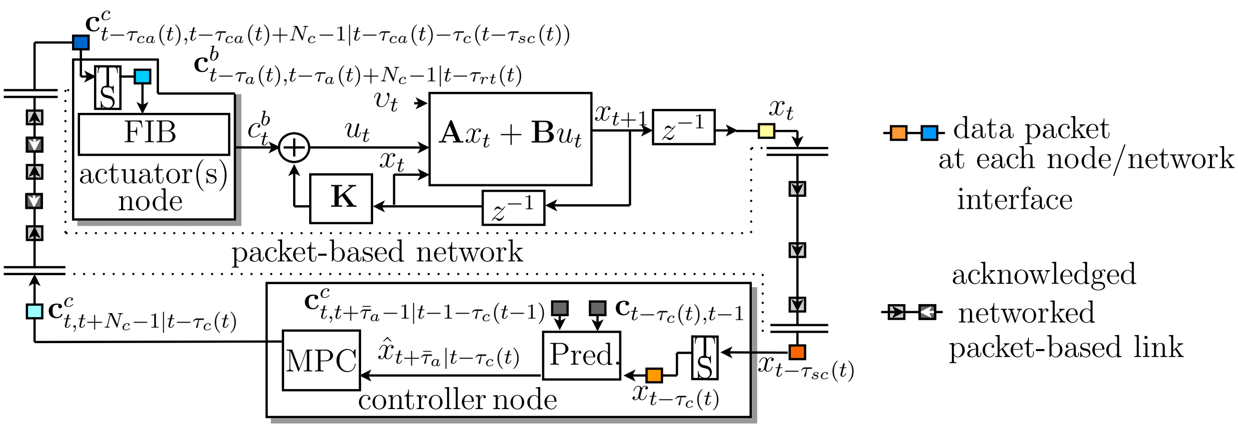

Figure 4.4: Scheme of the MPC-NDC strategy

[2]In the networked framework with multiple subsystems, we must account for the different arrival time of state measurement from the remote sensors. Therefore the buffer $\mathbf{M}.X_i$ is used to retrieve the last entirely available overall state measurement $x_{t-\tau_c(t)}$, which is guaranteed to be found within the past $\overline{\tau}_c$ time instants, that is, the state value at time instant $(t - \overline{\tau}_c)$, i.e., $x_{t-\overline{\tau}_c}$, is always available at the master node for performing the predictions.

Procedure 4.2.1 (MPC–NDC scheme for TCP–like networks)**.** *Assume that, starting from time instant* $t = 0$*, the initial condition* $x_0 = \overline{x}_0 = \mathrm{col}[\overline{x}_{10}, \ldots, \overline{x}_{n_s 0}]$ *is known.*

Initialization

for $i \in \{1, \ldots, n_s\}$

  Let $\mathbf{M}.X_i(0) \leftarrow \overline{x}_{i0}$*;*

  $\mathbf{S}_i.\mathbf{c} = \mathbf{M}.\mathbf{c}_i = \mathbf{M}.\mathbf{C}_i(1) \leftarrow \bar{\mathbf{c}}_{i\,0,N_c-1}$*,*

  with $\bar{\mathbf{c}}_{0,N_c-1}$ *feasible in* $\overline{x}_0$*;*

  $\mathbf{S}_i.T = \mathbf{M}.T_i \leftarrow 0$*;*

  $\mathbf{M}.k_{seq_i} = \mathbf{M}.k_{u_i} \leftarrow 0$*.*

Sensor node of the i-th sybsystem

for $t \in \mathbb{Z}_{\geq 0}$

  form the packet $\begin{cases} \mathbf{P}_{sc_i}.x \leftarrow x_t \\ \mathbf{P}_{sc_i}.T \leftarrow t \end{cases}$ *;*

  send $\mathbf{P}_{sc_i}$*.*

Master Controller node

for $t \in \mathbb{Z}_{\geq 0}$

  for $i \in \{1, \ldots, n_s\}$

    if a packet $\mathbf{P}_{sc_i}$ *arrived from the* i*-th subsystem*

      if $\mathbf{P}_{sc_i}.T > \mathbf{M}.T_i$

        $\mathbf{M}.X_i(t - \mathbf{M}.T_i + 1) \leftarrow \mathbf{P}_{sc_i}.x; (= x_{i\,t-\tau_{c_i}(t)})$

        $\mathbf{M}.T_i \leftarrow \mathbf{P}_{sc_i}.T;\ (= t - \tau_{c_i}(t)\,)$

    if the acknowledgment $\mathbf{P}_{ack_i}$ *arrived*

      $\mathbf{M}.k_{seq_i} \leftarrow t - \mathbf{P}_{ack_i}.T + 1$*;*

      $\mathbf{M}.k_{u_i} \leftarrow t - \mathbf{M}.T_i + 1$*;*

    else

      $\mathbf{M}.k_{seq_i} \leftarrow \mathbf{M}.k_{seq_i} + 1$*;*

      $\mathbf{M}.k_{u_i} \leftarrow \mathbf{M}.k_{u_i} + 1$*;*

13 $\mathbf{M}.\mathbf{c}_i \leftarrow \mathrm{col}[\,\mathbf{M}.\mathbf{c}_i(2), \ldots, \mathbf{M}.\mathbf{c}_i(\overline{\tau}_c),$
$\mathbf{M}.\mathbf{C}_i(\mathbf{M}_c.k_{seq_i}, \mathbf{M}.k_{u_i})\,]$;

14 *being* $\mathbf{M}.X_i(\tau_c(t)+1) = x_{i\,t-\tau_c(t)}$, *consider the last entirely available overall state meas.* $x_{t-\tau_c(t)}$ *and compute the prediction* $\hat{x}_{t|t-\tau_c(t)}$, *using the nominal model and*
$\mathbf{c}_{t-\tau_c(t),t} = \mathrm{col}[\mathbf{M}.\mathbf{c}_i(\overline{\tau}_c - \tau_c(t)+1), \ldots, \mathbf{M}.\mathbf{c}_i(\overline{\tau}_c)]$
where $\tau_c(t) = \max_{i\in\{1,\ldots,n_s\}}(t - \mathbf{M}.T_i)$ *(see line 4) ;*

15 *starting from* $\hat{x}_{t|t-\tau_c(t)}$, *compute the prediction* $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}$ *by using the nominal model and the input sequence* $\mathbf{c}^c_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}$, *which can be retrieved from* $\mathbf{M}.\mathbf{C}_i(1)$ *(see line 9);*

16 *solve the RHOCP initiated with* $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}$, *obtaining* $\mathbf{c}^\circ_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}$;

17 *form the sequences of corrections to be forwarded*
$\mathbf{c}_{i\,t,t+N_c-1|t-\tau_c(t)}^{c}$
$= \mathrm{col}[\mathbf{c}_{i\,t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}^{c}, \mathbf{c}_{i\,t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}^{\circ}]$;

18 *shift by one position the sequences in the register* $\mathbf{M}.\mathbf{C}_i$ *and store* $\mathbf{M}.\mathbf{C}_i(1) \leftarrow \mathbf{c}_{i\,t,t+N_c-1|t-\tau_c(t)}^{c}$;

19 *form the packet* $\begin{cases} \mathbf{P}_{ca_i}.\mathbf{c} \leftarrow \mathbf{c}_{i\,t,t+N_c-1|t-\tau_c(t)}^{c} \\ \mathbf{P}_{ca_i}.T \leftarrow t \end{cases}$;

20 *send* $\mathbf{P}_{ca_i}$.

Local controller of the i-th subsystem

1 *for* $t \in \mathbb{Z}_{\geq 0}$

2 *if a packet* $\mathbf{P}_{ca_i}$ *arrived*

3 *if* $\mathbf{P}_{ca_i}.T > \mathbf{S}_i.T$

4 $\mathbf{S}_i.\mathbf{c} \leftarrow \mathbf{P}_{ca_i}.\mathbf{c}$;

$\mathbf{S}_i.T \leftarrow \mathbf{P}_{ca_i}.T$; $(= t - \tau_a(t)\ )$

form the packet $\mathbf{P}_{ack_i}.T \leftarrow \mathbf{S}_i.T$;

send $\mathbf{P}_{ack_i}$;

apply the control action

$u_{it} = \mathbf{S}_i.\mathbf{c}(t - \mathbf{S}_i.T + 1) + \mathbf{K}x_{it}$.

□

Now, the robust stability properties of the described control scheme will be analysed in presence of transmission delays and model uncertainty; to this purpose, the main theorem, stating the recursive feasibility of the combined MPC–NDC scheme, is the following.

Theorem 4.2.1 (Invariance of the feasible set)**.** *Assume that the overall sequence of corrections computed by the master controller (comprising all the sequences to be forwarded to the subsystems),* $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$*, is feasible at time* t*. Moreover, let* $S_V \subset \mathbb{R}^w$ *be an arbitrary convex polytope containing the origin as interior point. If the additive uncertainty set verifies the inclusion*

$$D \subseteq \beta^\circ \mathbf{G} S_V, \tag{4.9}$$

with $\beta^\circ = \max\{\beta \in \mathbb{R}_{\geq 0}\}$*, such that*

1) $X_f \subseteq \mathcal{C}_1\left(X_f | \mathbf{A_K}, \mathbf{B}, U \backsim \mathbf{K}(X_f \oplus T_{\mathbf{K}r+N_c}(\beta \mathbf{G} S_V))\right.$

$\left.\backsim \mathbf{A_K}^{N_c+r} T_{\mathbf{K}s-r}(\beta \mathbf{G} S_V)\right),$

$\forall r \in \{-1, \overline{\tau}_c - 1\}$, $\forall s \in \{\max(r, 0), \ldots, \overline{\tau}_c\}$;

2) $X_f \oplus T_{\mathbf{K}N_c}(\beta \mathbf{G} S_V) \subseteq X$,

then, the recursive feasibility of the scheme in ensured for every time instant $t + i, \forall i \in \mathbb{Z}_{>0}$ *and* X_{MPC} *is robust positively invariant.* □

Proof in A.2.

The connection between the robust recursive feasibility of the scheme and the UB in X of the resulting closed-loop system can be established by some simple observations, as stated in the following remark.

Remark 4.2.2 (Recursive feasibility and UB in $X_{MPC} \subseteq X$)**.** *Given a delayed state measurement $x_{t-\tau_c(t)}$, if there exists a feasible sequence at time t, $\bar{\mathbf{c}}_{t,t+N_c-1}$, we have that $\hat{x}_{t|t-\tau_c(t)}$ verifies $\hat{x}_{t|t-\tau_c(t)} \in X_{MPC}(\tau_c(t))$, since $\bar{\mathbf{c}}_{t,t+N_c-1}$ satisfies all the constraints specified in (4.2.5). Thus, proving that the scheme is recursively feasible (that is, given a feasible sequence at time t, there exists a feasible sequence at time $t+1$), would prove that $\hat{x}_{t+1|t+1-\tau_c(t+1)}$, will belong to $X_{MPC}(\tau_c(t+1))$, whatever be the value of $\tau_c(t+1)$ in the set $\{0, \ldots, \overline{\tau}_c\}$. Without loss of generality, assume that $\tau_c(t+1) = 0$, then it holds that $x_{t+1} = \hat{x}_{t+1|t+1} \in X_{MPC}$.*

Now, assuming that the initial condition $\bar{x}_0$, at time $t = 0$, is known to the controller (i.e.,$\tau_c(0) = 0$) and that the sequence stored in the FIB's is feasible, by induction it follows that

$$x_t \in X_{MPC}(t) \subseteq X_{MPC}, \ \forall t \in \mathbb{Z}_{\geq 0}. \tag{4.10}$$

We can conclude that the NCS's trajectories, driven by the MPC-NDC scheme, are bounded in X_{MPC}. Being $X_{MPC} \subseteq X$, the UB in X property follows. □

Then, from what has been said earlier, the NCS is in charge of enforcing the constraint satisfaction for the pre-compensated system. In order to apply Theorem 4.2.1, it is necessary to compute the maximal admissible uncertainty set D. In the following, we propose a strategy to accomplish this task by setting up an adequate linear programming problem.

4.2.4 Maximal admissible uncertainty set D computation

Assume that the "a priori" specified pre-compensator $\mathbf{K}$ is known at the NCS design stage. Then, supposing that a d-invariant set $\Omega_f \subseteq X$ has been determined (see e.g., [57],[92] and the references therein for an overview on available methods) and that X_f is chosen as specified in Assumption 2, a sufficient condition for Point 1) of Theorem 4.2.1 to hold is that the additive uncertainty set $D = \beta \mathbf{G} S_V$ (parametrized by a polytope $S_V \subset \mathbb{R}^w$ specified by the designer and by a scalar $\lambda \in \mathbb{R}_{\geq 0}$ to be determined) verifies both

$$T_{\mathbf{K}r+N_c}(\beta \mathbf{G} S_V) \subseteq \Omega_f \backsim X_f \tag{4.11}$$

and

$$\mathbf{A}_{\mathbf{K}}^{N_c+r} T_{\mathbf{K}s-r}(\beta \mathbf{G} S_V) \subseteq \Omega_f \backsim X_f \tag{4.12}$$

$\forall r \in \{-1, \ldots, \overline{\tau}_c - 1\}$, $\forall s \in \{\max(r, 0), \ldots, \overline{\tau}_c\}$. Indeed, under (4.11) and (4.12) it holds that

$$\begin{aligned} X_f \;\; &= A_{\mathbf{K}} \Omega_f \backsim \mathbf{B}(U \backsim \mathbf{K}\Omega_f) \\ &\subseteq \Omega_f \backsim (\Omega_f \backsim X_f) \\ &= A_{\mathbf{K}}^{-1}(X_f \oplus \mathbf{B}(U \backsim \mathbf{K}\Omega_f)) \backsim (\Omega_f \backsim X_f) \\ &= \mathcal{C}_1(X_f | A_{\mathbf{K}}, B, U \backsim \mathbf{K}\Omega_f) \backsim (\Omega_f \backsim X_f) \end{aligned} \tag{4.13}$$

Thanks to (4.11) we have that $U \backsim \mathbf{K}\Omega_f \subseteq U \backsim (X_f \oplus T_{\mathbf{K}r+N_c}(\beta \mathbf{G} S_V))$, which implies

$$\begin{aligned} &\mathcal{C}_1(X_f | A_{\mathbf{K}}, B, U \backsim \mathbf{K}\Omega_f) \\ &\quad \subseteq \mathcal{C}_1(X_f | A_{\mathbf{K}}, B, U \backsim (X_f \oplus T_{\mathbf{K}r+N_c}(\beta \mathbf{G} S_V)))\,. \end{aligned} \tag{4.14}$$

Finally, (4.12), (4.13) and (4.14) together yield

$$\begin{aligned} X_f \subseteq \;\; &\mathcal{C}_1(X_f | A_{\mathbf{K}}, B, U \backsim (X_f \oplus T_{\mathbf{K}r+N_c}(\beta \mathbf{G} S_V))) \\ &\qquad \backsim \mathbf{A}_{\mathbf{K}}^{N_c+r} T_{\mathbf{K}s-r}(\beta \mathbf{G} S_V). \end{aligned} \tag{4.15}$$

Now, we will set up a linear program to maximize the estimated set of admissible uncertainties. Assume that $\Omega_f \backsim X_f$ can be described by the inequalities

$$\Omega_f \backsim X_f = \left\{ x \in \mathbb{R}^n | \delta_l^T x \leq g_l,\ l \in \{1, \cdots, n_\delta\} \right\}.$$

Denoting as $p_{j|r} \in \mathbb{R}^n$, $j \in \{1, \ldots, n_p\}$ and $q_{k|r,s} \in \mathbb{R}^w$, $k \in \{1, \ldots, n_q\}$ the numerable vertexes of $T_{\mathbf{K}r+N_c}(\mathbf{G} S_V)$ and $\mathbf{A}_{\mathbf{K}}^{N_c+r} T_{\mathbf{K}s-r}(\mathbf{G} S_V)$ respectively, then the two conditions (4.11) and (4.12) are both satisfied if

$$\begin{cases} \beta\delta_l^T p_{j|r} \leq g_l \\ \beta\delta_l^T q_{k|r,s} \leq g_l \end{cases} \tag{4.16}$$

$\forall r \in \{-1, \ldots, \overline{\tau}_c - 1\}$, $\forall s \in \{\max(r, 0), \ldots, \overline{\tau}_c\}$, $\forall l \in \{1, \ldots, n_\delta\}$, $\forall j \in \{1, \ldots, n_p\}$, $\forall k \in \{1, \ldots, n_q\}$. Remarkably, conditions (4.11) and (4.12) together with $\Omega_f \subseteq X$ imply Point 2) of Theorem 4.2.1. Then the maximal uncertainty set $V = \beta^\circ S_V$ can be obtained by solving the Linear Program

$$\beta^\circ = \max\{\beta \in \mathbb{R}_{\geq 0}\} \tag{4.17}$$

subject to (4.16).

In the following an example of D calculation is presented.

Example of D set calculation

Consider the following interconnected systems

$$x_{at+1} = \mathbf{A}_a x_{at} + \mathbf{B}_a u_{at} + \mathbf{G}_a v_{at} + \mathbf{F}_a x_{bt} \tag{4.18}$$

with $x_a \in \mathbb{R}^2$, $u_a \in \mathbb{R}$ and $v_a \in \mathbb{R}^2$, and

$$x_{bt+1} = \mathbf{A}_b x_{bt} + \mathbf{B}_b u_{bt} + \mathbf{G}_b v_{bt} + \mathbf{F}_b x_{at} \tag{4.19}$$

where $x_b \in \mathbb{R}$, $u_b \in \mathbb{R}$ and $v_{bt} \in \mathbb{R}$. Assume that

$$\mathbf{A}_a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{B}_a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{G}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{F}_a = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}$$

and

$$\mathbf{A}_b = \begin{bmatrix} 1 \end{bmatrix}, \mathbf{B}_b = \begin{bmatrix} 1 \end{bmatrix}, \mathbf{G}_b = \begin{bmatrix} 1 \end{bmatrix}, \mathbf{F}_b = \begin{bmatrix} 0.1 & 0 \end{bmatrix}$$

The state variables x_a,x_b are constrained in the set X depicted in Figure 4.5 and the inputs u_a, u_b are subjected to

$$u_a \in U_a = [-0.95, 0.95], \;\; u_b \in U_b = [-0.95, 0.95],$$

Assume that the two subsystem are locally pre-stabilized (neglecting the interconnection) by the linear control laws,

$$\mathbf{K}_a = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix}, \;\; \mathbf{K}_b = [-0.1].$$

Notably, the overall interconnected system is unstable. In order to enforce the constraints on x and u, and thus the UB property of the state trajectories, a networked predictive controller, to which all the state vector is available through delayed communication channels, can be designed as described in previous sections.

Assume that $\overline{\tau}_c = \overline{\tau_a} = 5$. Then, in view of Theorem 4.2.1, the proposed predictive networked control strategy guarantees the UB of the closed-loop trajectories for suitably small uncertainties, provided that that the specified restricted constraints are imposed on the nominal trajectories during the optimization as prescribed in Problem 4.2.1. For the system under concern, a terminal constraint set X_f satisfying Assumption 2 has been determined, as shown in Figure 4.5.

Thanks to the algorithm discussed in Section 4.2.4, we can provide an estimate of the maximal admissible uncertainty set. Choosing $N_c = 12$ and parameterizing the set $V = \beta S_V$, with $S_V = [-0.1, 0.1] \times [-1, 1]^2$, the solution of the Linear Program (4.17) yields to $\beta^{\circ} = 1.8 \cdot 10^{-3}$.

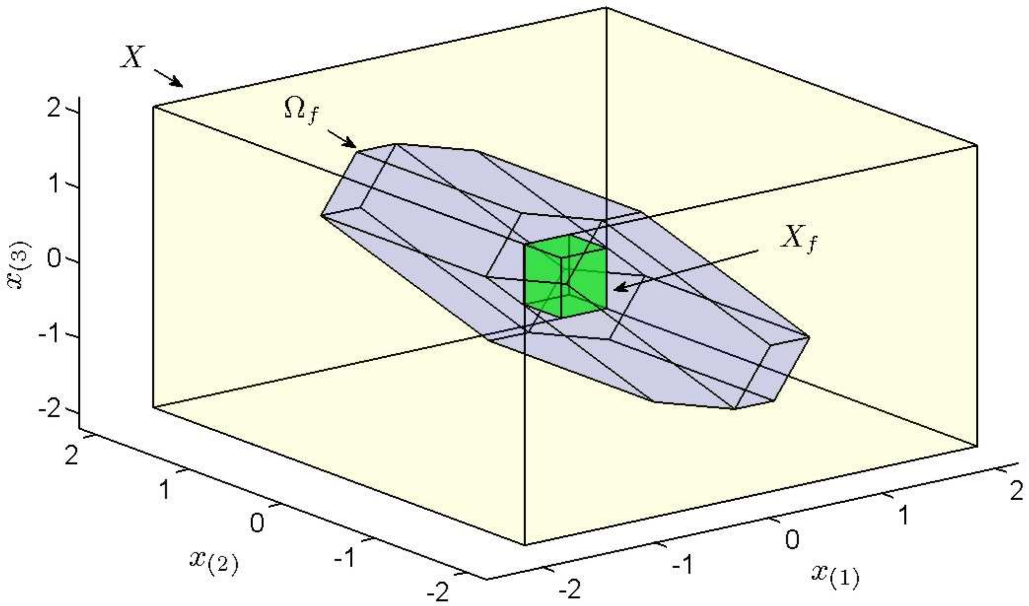

Figure 4.5: State constraint set X and terminal set X_f for the system in the example

4.3 Networked MPC - Nonlinear Systems over UDP Networks case

The results obtained for the linear - TCP case will be now extended to nonlinear systems having mutual physical interconnections and communicating over a UDP based network ([49]). Indeed, as we will see, by suitable modifications of the just stated assumptions and theorems, it is possible to describe a new procedure robustly stabilizing such systems. Robustness will be referred to with respect to both model uncertainties and delays occurring due to networked transmission.

It is worth noting that this scheme can be very useful in practice: beyond the fact that most real systems present nonlinearities, disturbances, model mismatches and, in the networked case, possible delays in the communication channels, the choice of a procedure based on UDP instead of TCP protocol can represent a remarkable improvement. Indeed, in the considered case, the controller is not required to be informed by the actuator of successful packet delivery. Although many control-theoretic works postulate that in TCP-like networks, after a successful packet receipt, the source node receives a deterministic notification within a single time-interval (see i.e. [87]), this assumption is very hard to verify in practice. Therefore, the analysis of a UDP-like scenario can lead to much more realistic results.

The system considered now has the following form, and consists in the interconnection of $n_s \in \mathbb{Z}_{>0}$, $n_s \geq 2$ *nonlinear* time-invariant, discrete-time subsystems:

$$
\begin{cases}
x_{1t+1} = f_1(x_{1t}) + \mathbf{B}_1 u_{1t} + d_{1t} + \sum_{j=2}^{n_s} f_{1,j}(x_{jt}) \\
\quad\vdots \\
x_{it+1} = f_i(x_{it}) + \mathbf{B}_i u_{it} + d_{it} + \sum_{j=1, j\neq i}^{n_s} f_{i,j}(x_{jt}) \\
\quad\vdots \\
x_{n_st+1} = f_{n_s}(x_{n_st}) + \mathbf{B}_{n_s} u_{n_st} + d_{n_st} + \sum_{j=1}^{n_s-1} f_{n_s,j}(x_{jt})
\end{cases}
\tag{4.20}
$$

with $x_{i0} = \overline{x}_{i0}, t \in \mathbb{Z}_{\geq 0}$, where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $d_i \in \mathbb{R}^{n_i}$ are respectively the state, the input and the additive disturbance affecting the $i-$th subsystem for all $i \in \{1, \ldots, n_s\}$.

Again, each i-th subsystem in (4.20) is pre-compensated by a linear control law with local state feedback, i.e. $u_{it} = \mathbf{K}_i x_{it} + c_{it}$.

The networked centralized controller generates the input corrections c_{it}, $i \in \{1, \ldots, n_s\}$ in order to fulfil the global control objectives.

Assuming that the local state feedback policies are not affected by delays and taking into account the presence of local controllers and correction inputs, this time the overall system can be rearranged as

$$
x_{t+1} = f_{\mathbf{K}}(x_t) + \mathbf{B}c_t + d_t, \tag{4.21}
$$

$$
= g(x_t, c_t) + d_t,\ x_0 = \overline{x}_0,\ t \in \mathbb{Z}_{\geq 0}\,, \tag{4.22}
$$

with $x_t \triangleq \text{col}[x_{1t}, \ldots, x_{n_st}] \in \mathbb{R}^n$, where n is the overall system dimension, $f_{\mathbf{K}}(x_t) \triangleq f(x_t) + \mathbf{BK}x_t$, $\mathbf{K}$ is the overall pre-compensation matrix (in general sparse) due to the local linear-feedback laws and $g(\cdot, \cdot)$ is the nominal model function used for state prediction based on the input c_t, that is $g(x_t, c_t) = f_{\mathbf{K}}(x_t) + \mathbf{B}c_t$.
Moreover, we have that $d_t \triangleq \text{col}[d_{1t}, \ldots, d_{n_st}] \in \mathbb{R}^n$ and $c_t = \text{col}[c_{1t}, \ldots, c_{n_st}]$, $c_t \in \mathbb{R}^m$ where m is the dimension of the overall correction input, which is applied to the subsystems by the local controllers on the basis of the information received from the networked master controller.

In addition, let us assume that the control input $u_t = \mathbf{K}x_t + c_t$, the state

x_t and the disturbance d_t are subjected to hard constraints, i.e.,

$$u \in U, x \in X, d \in D, \tag{4.23}$$

where $U \subset \mathbb{R}^m$, $D \subset \mathbb{R}^r$, $X \subset \mathbb{R}^n$.

In order to face with system nonlinearities, we have now to make a Lipschitz continuity assumption on the local maps of the subsystems.

Assumption 3 (Lipschitz)**.** *The local maps $f_i(x)$ and $f_{i,j}(x)$ are Lipschitz continuous with respect to $x \in X$.*

Notice that Assumption 3 implies that also the global function $g(x, c)$ is Lipschitz continuous w.r.t. x, uniformly in c, for each possible choice of the local feedbacks K_i. Let L_g denote the Lipschitz constant for $g(x, c)$, which can be opportunely reduced by means of a convenient choice of the local feedback gains.

In the considered case, the following assumption regarding disturbances is needed.

Assumption 4 (Disturbances)**.** *The disturbances affecting the system take values from the compact ball $D \triangleq \mathcal{B}^r(\overline{d})$, with $\overline{d} \in \mathbb{R}_{\geq 0}$.*

A scheme of the NCS topology considered in this (UDP - nonlinear) case is depicted in Figure 4.6.

As far as the network dynamics and the communication protocol are concerned, also in this case both the sensor-to-controller and the controller-to-actuator links are supposed to be affected by delays due to the unreliable nature of networked communications (a recent stability analysis in this case can be found in [44]). In order to cope with these eventuality, a Time-Stamping policy identical to the one used in the linear - TCP case is adopted.

The next section will describe how the TS mechanism can be used to compensate for transmission delays.

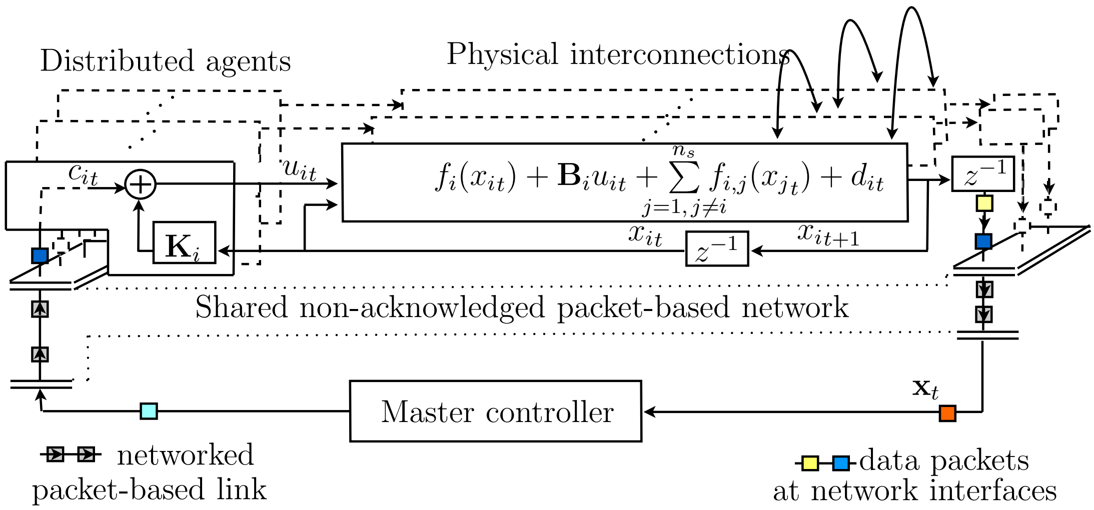

Figure 4.6: Underlying structure of the NCS under consideration. A networked centralized controller is in charge of fulfilling a global control objective (uniform boundedness of closed-loop trajectories) for a system consisting in the interconnection of locally pre-compensated subsystems.

4.3.1 Network Delay Compensation

Using a notation similar to the one of the linear-TCP case, with $\tau_{ca}(t)$ and $\tau_{sc}(t)$ we will denote the delays in the controller-to-actuator and in the sensor-to-controller links. Notice that we are now requiring that all the subsystems are *physically* interconnected: this permits to refer to global delay values instead of considering for each τ its maximum value among the subsystems. Also in this case, $\tau_a(t)$ will represent the "age" in discrete time instants of the control sequence used by the actuator to compute the current input and $\tau_c(t)$ the age of the state measurement which had been used by the controller at time t to compute the control corrections to be sent to the actuator. Finally, $\tau_{rt}(t) \triangleq \tau_a(t) + \tau_c(t - \tau_a(t))$ is the so called *round trip time*, representing the age of the state measurement used to compute the input applied at time t.
By means of the time stamps of the data packets, only the most recent informations will be retained at the actuator nodes (this approach is similar to

the one proposed in [87] and originally developed for unconstrained systems nominally stabilized by a generic nonlinear controller): when a novel packet is received, if it carries a more recent time-stamp than the one already in the buffer, then it replaces the older one, while the time-stamps are used on controller's side to order the measurements received at different instants from the distributed sensors. The TS-based packet arrival management implies that the information bounds subsume the constraints $\tau_a(t) \leq \tau_{ca}(t)$ and $\tau_c(t) \leq \tau_{sc}(t)$. Also the NDC strategy here adopted comprises a Future Input Buffering (FIB) mechanism (also known as "play-back buffer", see [61] for details), which requires that the controller node send a packeted sequence of N_c (with $N_c \in \mathbb{Z}_{>0}$) control actions to the actuator node; such a sequence must be long enough to accommodate the worst case delay or the maximum number of successive packet losses on the uplink channel.

Each i-th local controller is provided of an internal buffer to store an entire sequence $\mathbf{c}_i^b$ of N_c corrections each time a newer time-stamped packet arrives. Then, at each time instant t, it retrieves a time-consistent correction from the buffer and applies to the i-th subsystem the control action $u_{it} = c_{it}^b + \mathbf{K}_i x_{it}$, where c_{it}^b is the $\tau_{a_i}(t)$-th element of the locally buffered sequence $\mathbf{c}_{i t-\tau_{a_i}(t), t-\tau_{a_i}(t)+N_c-1}^b$, given by

$$\begin{aligned}\mathbf{c}_{i t-\tau_{a_i}(t), t+N_c-1}^b &= \mathrm{col}[c_{i t-\tau_{a_i}(t)}^b, \ldots, c_{it}^b, \ldots, c_{i t-\tau_{a_i}(t)+N_c-1}^b] \\ &= \mathbf{c}_{i t-\tau_{a_i}(t), t+N_c-1|t-\tau_{rt_i}(t)}^c.\end{aligned}$$

where the sequence $\mathbf{c}_{i t-\tau_{a_i}(t), t+N_c-1|t-\tau_{rt_i}(t)}^c$ is computed at time $t-\tau_{a_i}(t)$ by the master on the basis of the interconnection state measurement collected (considering the worst case sensor-to-master delay) at time $t-\tau_{rt_i}(t)$.

Also in this case, we will assume that the age of the data-packets available at the master and local controller nodes admits an upper bound; suppose then that Assumption 1 keeps holding.
This assumption allows for the use of a finite length buffer for each actuator. Again, not imposing bounds on $\tau_{ca}(t)$, we allow the presence of packet losses (infinite delay) on the actuators links.

Notice that in the UDP case the controller is not informed by the actuator of successful packet delivery.

Let us now describe the mechanism used by the controller to compute the sequence of control corrections to be forwarded to the distributed agents.

4.3.2 MPC with Delay-Dependent Constraint Tightening

The reconstruction of the current (possibly unavailable) state x_t is made by recurring to the nominal model, i.e. (4.22), without considering the influence of disturbances under the action of the true sequence of corrections $\mathbf{c}_{t-\tau_c(t),t}$ applied from time $t-\tau_c(t)$ to $t-1$, $\mathbf{c}_{t-\tau_c(t),t-1}$. This sequence must be internally reconstructed by the controller by exploiting the control actions computed in the previous time instants. In this regard, the problem of delayed arrival of packeted sequences to the actuator represents a major source of uncertainty. Indeed, in presence of delays in the controller-to-subsystems paths, the computed correction sequences may not be commanded entirely to the plant (some approaches to solve this problem can be found i.e. in [31],[43]): the truly applied input sequence may be in general made up of pieces of sequences computed in different time-instants.

With respect to the acknowledged case, here we don't know whether packets arrived to destination or not: in this case, then, considering the worst case implies an enlargement of the horizon over which compute the optimal control sequence, and in particular it is necessary to consider the whole round trip time instead of the age of the state measurement τ_c.

In order to ensure that the sequence used for prediction would coincide with the one truly applied to the plant, then, this time we will retain, at time t, the subsequence $\mathbf{c}^b_{t,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$, where, as specified in Assumption 1, $\overline{\tau}_{rt}$ is the upper bound for the round trip time. Then, we will optimize only over the remaining elements, i.e. the sequence $\mathbf{c}_{t+\overline{\tau}_{rt},t+N_c-1}$, initiating the finite horizon optimization with the state prediction $\hat{x}_{t+\overline{\tau}_{rt}}$.

The RHOCP will be then modified in the following taking into account of these considerations. Nevertheless, before proceeding with the new description of such problem, it is necessary to adequately adapt the definitions of one step controllability set and of tightened constraints to the considered setup.

Definition 4.3.1 ($\mathcal{C}_1(\Xi|X,U,g)$)**.** *Given a compact set $\Xi \subset X$, the one step controllability set to Ξ, $\mathcal{C}_1(\Xi|X,U,g)$, is the set of the states that can be steered to Ξ by a single control correction c, under the nominal global map $g(\cdot,\cdot)$, subject to constraints (4.23), i.e.*

$$\mathcal{C}_1(\Xi|X,U,g) \triangleq \left\{ x_0 \in X \;\middle|\; \begin{array}{l} \exists\, \overline{c} \in \mathbb{R}^m : \\ \left\{ \begin{array}{ll} \mathbf{K}x_0 + \overline{c} \in U & \textit{and} \\ g(x_0,\overline{c}) \in \Xi \subseteq X & \end{array} \right. \end{array} \right\}$$

The shorthand $\mathcal{C}_1(\Xi)$ will be used instead of $\mathcal{C}_1(\Xi|X,U,g)$.

Definition 4.3.2 (Tightened Constraints)**.** *Under Assumptions 3 and 4, suppose[3], without loss of generality, $L_g \neq 1$. The tightened sets are defined as:*

$$X_i(\overline{d}) \triangleq X \backsim \mathcal{B}^n \left(\frac{L_g^i - 1}{L_g - 1} \overline{d} \right), \tag{4.24}$$

$$U_i(\overline{d}) \triangleq U \backsim \mathbf{K}\mathcal{B}^n \left(\frac{L_g^i - 1}{L_g - 1} \overline{d} \right), \tag{4.25}$$

for all $i \in \mathbb{Z}_{\geq 0}$.

Now, we are able to describe the RHOCP to be solved by the master controller in order to obtain the overall sequence of corrections $\mathbf{c}^c_{t,t+N_c-1}$.

Problem 4.3.1 (RHOCP)**.** *Given a positive integer $N_c \in \mathbb{Z}_{\geq 0}$, at any time $t \in \mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t-\tau_c(t)}$ be the estimate of the current overall state, x_t, obtained with the nominal model from the last available state measurement $x_{t-\tau_c(t)}$ with the controls $\mathbf{c}_{t-\tau_c(t),t-1}$ already applied to the plant.*
Let $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$ be the state computed from $\hat{x}_{t|t-\tau_c(t)}$ by extending the prediction using a piece of the overall sequence of corrections computed at time $t-1$, $\mathbf{c}^c_{t,t+\overline{\tau}_{rt}-1}$.

[3]The very special case $L_g = 1$ can be trivially addressed by a few suitable modifications to definition 4.3.2.

Then, given a stage-cost function h, the constraint sets $X_j(\overline{d}) \subseteq X$, $U_j(\overline{d}) \subseteq U$, $j \in \{\tau_c(t) + \overline{\tau}_{rt} + 1, \dots, \tau_c(t) + N_c\}$, *and the terminal set* X_f, *the RHOCP consists in solving, with respect to a* $(N_c - \overline{\tau}_{rt})$*-steps input sequence,*

$$\mathbf{c}_{t+\overline{\tau}_{rt},t+N_c-1} \triangleq \text{col}[c_{t+\overline{\tau}_{rt}}, \dots, c_{t+N_c-1}],$$

the following minimization problem

$$J^\circ_{FH}(\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}, \mathbf{c}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}, N_c - \overline{\tau}_{rt}) \triangleq \min_{\mathbf{c}_{t+\overline{\tau}_{rt},t+N_c-1}} \left\{ \sum_{l=t+\overline{\tau}_{rt}}^{t+N_c-1} h(\hat{x}_{l|t-\tau_c(t)}, c_l) \right\}$$

subject to the

i) nominal dynamics $\hat{x}_{t+j+1|t-\tau_c(t)} = f_{\mathbf{K}}(\hat{x}_{t+j|t-\tau_c(t)}) + \mathbf{B}c_{t+j}$, $j \in \{\overline{\tau}_{rt}, \dots, N_c - 1\}$;

ii) restricted input constraints $c_{t-\tau_c(t)+j} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in U_j(\overline{d})$, *with* $j \in \{\tau_c(t) + \overline{\tau}_{rt}, \dots, \tau_c(t) + N_c - 1\}$;

iii) restricted state constraints $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_j(\overline{d})$, *with* $j \in \{\tau_c(t) + \overline{\tau}_{rt} + 1, \dots, \tau_c(t) + N_c\}$;

iv) terminal state constraint $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f$.

The sequence of controls forwarded by the master to the distributed controllers will be then constructed as

$$\mathbf{c}^c_{t,t+N_c-1|t-\tau_c(t)} \triangleq \text{col}[\mathbf{c}^c_{t,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}, \mathbf{c}^\circ_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}] \tag{4.26}$$

(i.e., it is obtained by appending the solution of the RHOCP to a piece of the sequence computed at time $(t-1)$ *).* □

The choice of the stage cost $h(\cdot, \cdot)$ is still arbitrary, as in the TCP case addressed in the previous section.

While the definition of the feasible set X_{MPC} can be easily derived from Definition 4.2.5, for the sake of clarity we propose here the definition of feasible sequence for the considered case: indeed, as in RHOCP we have now to consider the round trip time in order to face the absence of the acknowledgements.

Definition 4.3.3 (Feasible sequence at time t)**.** *Given a delayed state measurement $x_{t-\tau_c(t)}$, available at time t to the controller, let us consider the prediction $\hat{x}_{t|t-\tau_c(t)}$ of the actual state x_t obtained with the nominal model and with the actual control sequence applied from time $t-\tau_c(t)$ to $t-1$, $\mathbf{c}^*_{t-\tau_c(t),t-1}$, which is known to the controller. Moreover consider a sequence of N_c control moves $\overline{\mathbf{c}}^c_{t,t+N_c-1}$ and its two subsequences $\overline{\mathbf{c}}^c_{t,t+\overline{\tau}_{rt}-1}$ and $\overline{\mathbf{c}}^c_{t+\overline{\tau}_{rt},t+N_c-1}$ such that $\overline{\mathbf{c}}^c_{t,t+N_c-1} = \mathrm{col}[\overline{\mathbf{c}}^c_{t,t+\overline{\tau}_{rt}-1}, \overline{\mathbf{c}}^c_{t+\overline{\tau}_{rt},t+N_c-1}]$.*
The input sequence $\overline{\mathbf{c}}^c_{t,t+N_c-1}$ is said feasible at time t if

1) the subsequence $\overline{\mathbf{c}}^c_{t,t+\overline{\tau}_{rt}-1}$ yields to

$$\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_j(\overline{d}), \quad \forall j \in \{\tau_c(t)+1, \ldots, \tau_c(t)+\overline{\tau}_{rt}\}$$

and

$$\overline{c}^c_{t-\tau_c(t)+j} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in U_j(\overline{d}), \quad \forall j \in \{\tau_c(t), \ldots, \tau_c(t)+N_c-1\};$$

2) the second subsequence satisfies all the constraints of the RHOCP initiated with

$$\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)} = \hat{x}_{\mathbf{K}}(\overline{\tau}_{rt}, x_{t-\tau_c(t)}, \mathbf{c}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1})$$

□

Due to the nonlinearity of the subsystems, it is necessary to reformulate Lemma 4.2.2 in accordance with the stated assumptions on the restricted constraints.

Lemma 4.3.1 (Robust Constraint Satisfaction)**.** *Under Assumptions 3 and 4, if sets $X_j(\overline{d})$ and $U_j(\overline{d})$ are computed as in (4.24) and (4.25), any feasible control sequence at time t, $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$, applied in open-loop to the perturbed system from time t to $t+N_c-1$, guarantees that the true (net-*

worked/perturbed) state trajectory will satisfy $x_{t+j} \in X$, *and*

$$\overline{c}_{t+j-1} + \mathbf{K}x_{t+j-1} \in U, \quad \forall j \in \{1, \ldots, N_c\}.$$

□

Proof in A.2.

4.3.3 Formalization of the networked MPC scheme - UDP case

We can now formulate the procedure giving the sequence of operations that have to be performed by the NCS components.

Similarly to the previous procedure, we will denote as $\mathbf{P}_{sc_i}$ and $\mathbf{P}_{ca_i}$ the data packets sent by the i-th subsystem to the master and by the master to the i-th distributed controller respectively. All the packets will be addressed to as data structures of the form $\mathbf{P} = \{\ \mathbf{P}.data,\ \ \mathbf{P}.time\ \}$, containing a data field and a time stamp field.
The local storage memory of the i-th subsystem controller, $\mathbf{S}_i$, is structured as follows:

- $\mathbf{S}_i.\mathbf{c} \in \mathbb{R}^m \times N_c$, which is a buffer containing N_c control actions
- $\mathbf{S}_i.T \in \mathbb{Z}_{\geq 0}$, which contains the time stamp (i.e., the age) of the information stored in $\mathbf{S}_i.\mathbf{c}$.

The storage memory of the master controller $\mathbf{M}$ has:

- n_s buffers $\mathbf{M}.\mathbf{c}_i \in \mathbb{R}^m \times \overline{\tau}_c$, $i \in \{1, \ldots, n_s\}$, storing the local corrections applied to the the subsystems from $t - \overline{\tau}_c$ to $t-1$ (each element of $\mathbf{M}.\mathbf{c}_i$ is a control move);
- n_s array buffers $\mathbf{M}.X_i \in \mathbb{R}^n \times \overline{\tau}_c$, in which the received state measurement are stored for equalizing[4] the delays;

[4]In the networked framework with multiple subsystems, we must account for the different arrival time of state measurement from the remoted sensors. Therefore the buffer

- n_s scalar buffers $\mathbf{M}.T_i \in \mathbb{Z}_{\geq 0}$, containing the time stamp of the most recent measurement in $\mathbf{M}.X_i$.

Given a buffer $\mathbf{b}$ containing N elements (vectors), $\mathbf{b}(j)$ is the j-th element of the array, with $j \in \{1, \dots, N\}$. Now the procedure can be outlined.

Procedure 4.3.1 (MPC-NDC scheme for UDP-like networks)**.** *Assume that, starting from time instant* $t = 0$*, the initial condition* $x_0 = \overline{x}_0 = \mathrm{col}[\overline{x}_{10}, \dots, \overline{x}_{n_s 0}]$ *is known.*

Initialization

set $\mathbf{M}.\mathbf{c}_i \leftarrow \bar{\mathbf{c}}_{i\,0,N_c-1}$,

with $\bar{\mathbf{c}}_{i\,0,N_c-1}$ *feasible in* $\overline{x}_0$;

for $i \in \{1, \dots, n_s\}$

  Let $\mathbf{M}.X_i(0) \leftarrow \overline{x}_{i\,0}$;

  $\mathbf{S}_i.\mathbf{c} \leftarrow \bar{\mathbf{c}}_{i\,0,N_c-1}$;

  $\mathbf{S}_i.T = \mathbf{M}.T_i \leftarrow 0$.

Sensor node of the i-th sybsystem

for $t \in \mathbb{Z}_{\geq 0}$

  form the packet $\begin{cases} \mathbf{P}_{sc_i}.x \leftarrow x_{it} \\ \mathbf{P}_{sc_i}.T \leftarrow t \end{cases}$;

  send $\mathbf{P}_{sc_i}$ *to the supervisory controller.*

Master Controller node

for $t \in \mathbb{Z}_{\geq 0}$

  for $i \in \{1, \dots, n_s\}$

    if a packet $\mathbf{P}_{sc_i}$ *of sensor measurement*

    arrived from the i-th subsystem

      $\mathbf{M}.X_i(t - \mathbf{M}.T_i + 1) \leftarrow \mathbf{P}_{sc_i}.x; (= x_{i\,t-\tau_{c_i}(t)})$

$\mathbf{M}.X_i$ is used to retrieve the last entirely available overall state measurement $x_{t-\tau_c(t)}$, which is guaranteed to be found within the past $\overline{\tau}_c$ time instants, that is, the state value at time instat $(t - \overline{\tau}_c)$, i.e., $x_{t-\overline{\tau}_c}$, is always available at the master node for performing the predictions.

5 $\mathbf{M}.T_i \leftarrow \mathbf{P}_{sc_i}.T;\ (= t - \tau_{c_i}(t)\,)$

6 *being* $\mathbf{M}.X_i(\overline{\tau}_c(t) - \tau_c(t) + 1) = x_{i\,t-\tau_c(t)}$, *consider the last entirely available overall state measurement* $x_{t-\tau_c(t)}$, *where* $\tau_c(t) = \max_{i \in \{1,\ldots,n_s\}}(t - \mathbf{M}.T_i)$ *(see line 4) and compute the prediction* $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$, *using the nominal model with the overall sequence*

$$\mathbf{c}_{\,t-\tau_c(t),t+\overline{\tau}_{rt}-1} = \mathrm{col}[\mathbf{c}_{1\,t-\tau_c(t),t+\overline{\tau}_{rt}-1}, \ldots, \mathbf{c}_{n_s\,t-\tau_c(t),t+\overline{\tau}_{rt}-1}]$$

where

$$\mathbf{c}_{i\,t-\tau_c(t),t+\overline{\tau}_{rt}-1} = \mathrm{col}[\mathbf{M}.\mathbf{c}_i(\,\overline{\tau}_c - \tau_c(t) + 1), \ldots, \mathbf{M}.\mathbf{c}_i(\overline{\tau}_{rt})];$$

7 *then, solve the RHOCP initiated with* $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$, *obtaining the overall control sequence*

$$\mathbf{c}^{\circ}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)} = \mathrm{col}[\mathbf{c}^{\circ}_{1\,t-\tau_c(t),t+\overline{\tau}_{rt}-1}, \ldots, \mathbf{c}^{\circ}_{n_s\,t-\tau_c(t),t+\overline{\tau}_{rt}-1}];$$

8 *for* $i \in \{1, \ldots, n_s\}$

9 *shift by one position the elements in the buffers* $\mathbf{M}.\mathbf{c}_i$ *and set*

10 $\mathbf{M}.\mathbf{c}_i(\overline{\tau}_{rt}, \ldots, N_c - 1) \leftarrow c^{o}_{i\,t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$;

11 *extract the sequences to be forwarded to the single agents*

$$\mathbf{c}^{c}_{i\,t,t+N_c-1|t-\tau_c(t)} = \mathbf{M}.\mathbf{c}_i(\overline{\tau}_c, \overline{\tau}_{rt});$$

12 *form the packets* $\begin{cases} \mathbf{P}_{ca_i}.\mathbf{c} \leftarrow \mathbf{c}^{c}_{i\,t,t+N_c-1|t-\tau_c(t)} \\ \mathbf{P}_{ca_i}.T \leftarrow t \end{cases}$;

13 *send* $\mathbf{P}_{ca_i}$.

Local controller of the i-th agent

1 *for* $t \in \mathbb{Z}_{\geq 0}$

if a packet $\mathbf{P}_{ca_i}$ *arrived*

if $\mathbf{P}_{ca_i}.T > \mathbf{S}_i.T$

$\mathbf{S}_i.\mathbf{c} \leftarrow \mathbf{P}_{ca_i}.\mathbf{c};$

$\mathbf{S}_i.T \leftarrow \mathbf{P}_{ca_i}.T;\ (= t - \tau_a(t)\ \ )$

apply the control action

$u_{it} = \mathbf{S}_i.\mathbf{c}(t - \mathbf{S}_i.T + 1) + \mathbf{K}_i x_{it}.$

□

The robust stability properties of the networked supervisory scheme formally described by the Procedure 4.3.1 can be characterized by the following result.

Theorem 4.3.1 (Invariance of the feasible set)**.** *Assume that the overall sequence of corrections computed by the master controller (comprising all the sequences to be forwarded to the subsystems),* $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$*, is feasible at time* t*. Then, under the stated assumptions, if the norm bound on the uncertainties verifies*

$$\overline{d} \leq \min_{k\in\{0,\dots,\overline{\tau}_c\}} \left\{ \min\left[\frac{L_g - 1}{L_g^{N_c+k} - L_g^{N_c-1}} dist(\mathbb{R}^n \setminus \mathcal{C}_1(X_f), X_f), \frac{L_g - 1}{L_g^{N_c+k} - 1} dist(\mathbb{R}^n \setminus X_{N_c+k}(\overline{d}), X_f) \right] \right\},$$

then, the recursive feasibility of the scheme in ensured for every time instant $t + i, \forall i \in \mathbb{Z}_{>0}$ *and* X_{MPC} *is robust positively invariant.* □

Proof in A.2.

With the same considerations in Remark 4.2.2, it is possible to state that the NCS's trajectories, driven by the MPC-NDC scheme just proposed are bounded in X_{MPC}. Being $X_{MPC} \subseteq X$, the UB in X property follows also in the UDP case.

The obtained theoretical results are now being validated by means of simulations on an illustrative example.

4.3.4 Example

The following constrained interconnected nonlinear system with 3 agents has been used to test the devised method:

$$\begin{cases} \begin{cases} x_{(1,1)_{t+1}} = x_{(1,1)_t} + 0.3\sin(x_{(1,2)_t}) + u_{(1)_t} \\ \qquad +0.2x_{(2,1)_t} + 0.1x^4_{(1,2)_t} + d_{(1,1)_t}; \\ x_{(1,2)_{t+1}} = 0.4x^2_{(1,2)_t} + u_{(1)_t} - 0.3x_{(3,1)_t} + d_{(1,2)_t} \end{cases} \\ \left\{ x_{(2,1)_{t+1}} = e^{-x^2_{(2,1)_t}} x_{(2,1)_t} + u_{(2)_t} + 0.5x^2_{(1,2)_t} + d_{(2,1)_t} \right. \\ \begin{cases} x_{(3,1)_{t+1}} = -\left(1 - e^{-|x_{(3,1)_t}|}\right) x_{(3,1)_t} + u_{(3)_t} \\ \qquad -0.3\left(\sin(x_{(1,1)_t}) + x_{(2,1)_t}\right) + d_{(3,1)_t} \end{cases} \end{cases} \tag{4.27}$$

with $u_{(i)_t} \in [-1, 1]$, $\forall i \in \{1, 2, 3\}$, $\forall t \in \mathbb{Z}_{\geq 0}$. The following local linear feedback gains are assigned:

$$\mathbf{K}_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \quad \begin{array}{l} \mathbf{K}_2 = -0.4 \\ \mathbf{K}_3 = 0.4 \end{array}$$

Note that, with the specified local feedback laws, the overall system is not stable in the origin.
A conservative bound for the Lipschitz constant of the closed-loop system with the specified local controllers in the compact constraint set can be computed obtaining $L_g = 1.7$.
Denoting as $x_t = (x_{(1,1)_t}, x_{(1,2)_t}, x_{(2,1)_t}, x_{(3,1)_t})$ and $d_t = (d_{(1,1)_t}, d_{(1,2)_t}, d_{(2,1)_t}, d_{(3,1)_t})$ the overall system's state and the additive disturbance, suppose that the following constraint $x_t \in [-1, 1]^4$ has to be fulfilled by the networked control architecture, for all d_t such that $|d_t| \leq 1 \cdot 10^{-4}$.

An horizon $N_c = 12$ has been chosen for the MPC in order to face network delays up to $\overline{\tau}_c \leq 5$ and $\overline{\tau}_a \leq 5$.

The following ellipsoidal terminal constraint has been imposed

$$x^T_{t+N_c|t} P_{X_f} x_{t+N_c|t} \leq 1,$$

with

$$P_{X_f} = 10^3 \begin{bmatrix} 0.8856 & 0.1155 & 0.0651 & 0.1035 \\ 0.1155 & 0.6413 & 0.0555 & 0.0051 \\ 0.0651 & 0.0555 & 1.1579 & -0.1853 \\ 0.1035 & 0.0051 & -0.1853 & 0.8237 \end{bmatrix}$$

Sample closed-loop trajectories, under the action of the networked master MPC controller, are depicted in Figure 4.7 for initial condition $x_0 = (0.9, 0.9, -0.9, -0.9)$. The input corrections commanded by the networked master controller are shown in Figure 4.8.

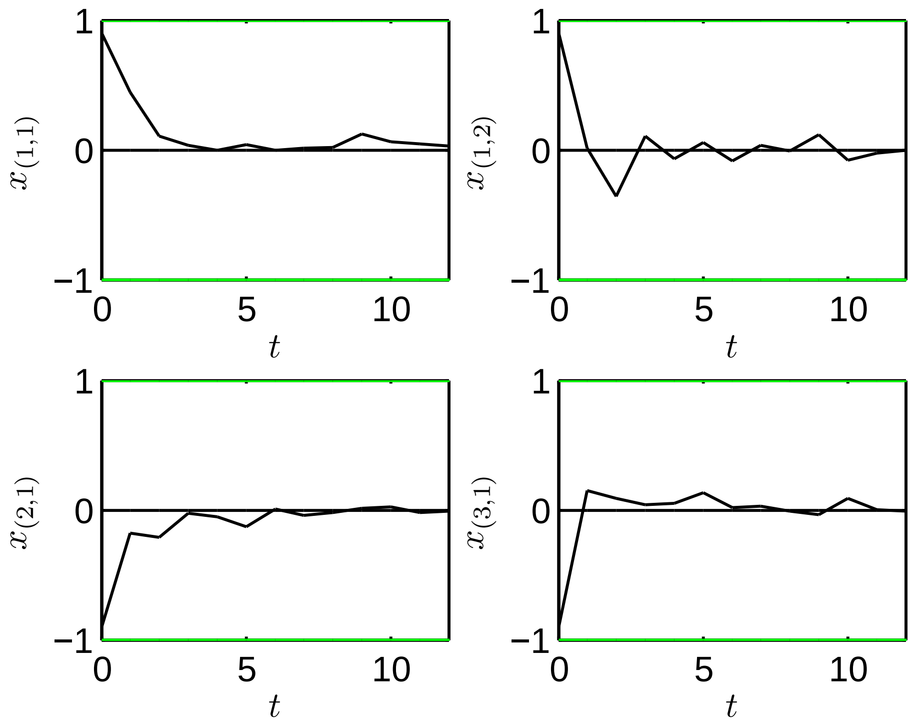

Figure 4.7: Closed-loop state evolution under the networked control policy. The Uniform Boundedness within the constraints is guaranteed in spite of bounded disturbances.

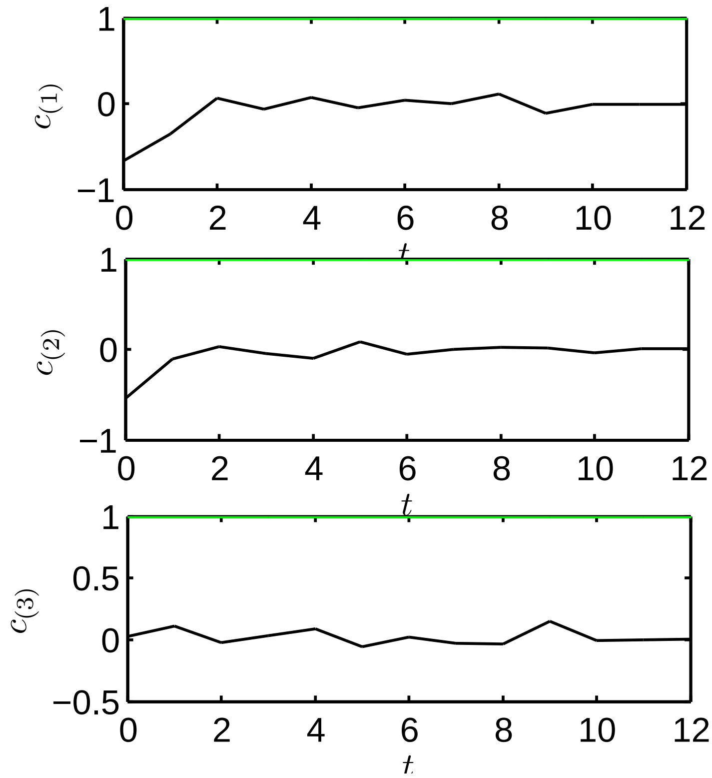

Figure 4.8: Input corrections applied by the local controllers in addition to local state-feedback, on the basis of networked packets transmitted by the centralized controller.

Chapter 5

Concluding remarks

The capabilities of MPC, in particular in terms of robustness, caused in recent years an increasing interest for this approach. Nevertheless, the applicability of this technique is strongly limited by the on-line computational burden required to calculate the control actions.

In order to overcome this limitation, in this work the robust stability analysis of the RH control policy was firstly extended, with respect to the literature, to a very generic class of constrained nonlinear systems (even not admitting a continuous stabilizing feedback control law) and then combined with a suitable approximation method for the improvement of the applicability of the Receding Horizon approach.

From these considerations, two distinct strategies have been followed, both based on the exploitation of the optimal control value for the RHOCP computed on the points of a grid superimposed on the domain. The first one is based on a Nearest Point policy: this kind of approximator is perhaps the simplest one and able to guarantee the applicability of the proposed method to a wide class of systems; nevertheless, as pointed out in the discussion, finding the nearest point requires a minimization whose computational burden grows up with the size of the considered set of points. Due to the fact that the requested gridding density is often high, this approach could turn out to be, in some cases, very difficult to be applied, both for the memory storage requirements and the neighbour searching time demand.

The second strategy is based on Neural Networks. In this case, difficulties can arise in the training stage: being in some cases the number of training points very large and the error bounds very restrictive, finding a suitable net could be a really hard assignment. Nevertheless, once this task has been carried out (for instance in a semi-automatic way, as in the considered case), the resulting on line control strategy is very efficient, being the number of parameters to be stored in memory and the number of operations to be performed to get to the approximated control input very much lower than in the NP case.

Despite the above mentioned possible difficulties, depending on the characteristics of the system under concern, the two approximation strategies proposed allow one to cope with systems which are not stabilizable by continuous static state feedback and are capable to guarantee the satisfaction of hard constraints and to allow efficient on-line computations. The closed-loop system has been proved to be Input-to-State Practically stable, with respect to approximation-induced errors, in a subset of the feasible region of the exact RH controller.

Extensive simulation results on the two cases have been reported showing the effectiveness of the devised methods.

Furthermore, MPC have been exploited in order to control both linear and nonlinear interconnected constrained systems, communicating respectively over TCP and UDP like networks; in these cases, in addition to model uncertainties, in the design of a centralized controller, network induced delays and packet dropouts have to be taken into account as well. From this point of view, besides the already mentioned capabilities in terms of robustness, MPC turns out to be a very useful approach: indeed, for each sampling instant the technique allows sending to the net an entire set of future control inputs, which involves a reduction of network congestion related control problems.

The proposed procedures have been proven to be able guarantee the Uniform Boundedness of the closed-loop trajectories in presence of bounded disturbances, even enforcing constraints; furthermore, the recursive feasibility of the scheme has been established by recurring to a constraint tightening based approach.

Possible future research efforts can be devoted, in the approximated MPC case, to reduce the off-line computational burden (for instance looking for less conservative bounds, in more specific cases) or to look for more efficient approximators (in the training stage, as regards Neural Networks, for example recurring to Genetic Programming) capable to satisfy the stated assumptions; an interesting alternative could be represented e.g. by Support Vector Machines.

In order to exploit the capabilities of this approach, it would be important to extend the technique e.g. to the cases of cooperative or large-scale distributed control, for which communication network related problems arise: then resorting to approximators can be combined with the proposed networked procedures. In this connection, improvement margins can be represented for instance by the removal of the assumption about the synchronization of each component of the plant or the study of the cases in which state measurements are not all available.

Bibliography

[1] B.M. Åkesson, J.B. Waller H.T. Toivonen, and R.H. Nyström. Neural network approximation of a nonlinear model predictive controller applied to a ph neutralization process. *Computers and Chemical Engineering*, 29:323–335, 2005.

[2] B.M. Åkesson and H.T. Toivonen. A neural network model predictive controller. *Journal of Process Control*, 16:937–946, 2006.

[3] M. Alamir and A. Murilo. Swing-up and stabilization of a twin-pendulum under state and control constraints by a fast NMPC scheme. *Automatica*, 44:1319–1324, 2008.

[4] G. Alldredge, M. Branicky, and V. Liberatore. Play-back buffers in networked control systems: Evaluation and design. In *Proceedings of the American Control Conference, Seattle*, pages 3106–3113, 2008.

[5] P. Antsaklis and J. Baillieul. Guest editorial: Special issue on networked control systems. *IEEE Trans. on Automatic Control*, 49:1421 – 1423, 2004.

[6] P. Antsaklis and J. Baillieul. Guest editorial: Special issue on networked control systems. *IEEE Transaction on Automatic Control*, 2006.

[7] A. Arya, D.M. Mount, N.S. Netanyahu, R. Silverman, and A.Y. An optimal algorithm for approximate nearest neighbor searching fixed dimensions. *Journal of Difference Equations and Applications*, 45(6):891 – 923, 1998.

[8] W. Banzhaf, P. Nordin, R.E. Keller, and F.D. Francone. *Genetic Programming: An Introduction*. Morgan Kaufmann, 1997.

[9] Farhad Bayat, Tor Arne Johansen, and Ali Akbar Jalali. Using hash tables to manage the time-storage complexity in a point location problem: Application to explicit model predictive control. *Automatica*, In Press, Corrected Proof, 2011.

[10] A. Bemporad, F. Borrelli, and M. Morari. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control,*, 48(9):1600–1606, 2003.

[11] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38:3–20, 2002.

[12] D. P. Bertsekas. Separable dynamic programming and approximate decomposition methods. *Automatic Control, IEEE Transactions on*, 52(5):911–916, 2007.

[13] R. R. Bitmead, M. Gevers, and V. Wertz. *Adaptive Optimal Control*. Prentice Hall, 1990.

[14] F. Blanchini. Ultimate boundedness control for discretetime uncertain systems via set-induced lyapunov functions. *IEEE Trans. on Automatic Control*, 39:428–433, 1994.

[15] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

[16] M. Canale, L. Fagiano, and M. Milanese. Fast implementation of predictive controllers using sm approximation methodologies. In *Proc. of the IEEE Conf. on Decision and Control*, pages 1361 – 1367, New Orleans, La, 2007.

[17] M. Canale, L. Fagiano, and M. Milanese. Fast nonlinear model predictive control via set membership approximation: An overview, in

nonlinear model predictive control. *Lecture Notes in Control and Information Sciences*, 384:461–470, 2009.

[18] M. Canale, L. Fagiano, and M. Milanese. Set membership approximation theory for fast implementation of model predictive control laws. *Automatica*, 45(1):45 – 54, 2009.

[19] A. Casavola, F. Mosca, and M. Papini. Predictive teleoperation of constrained dynamic system via internet-like channels. *IEEE Trans. Control Systems Technology*, 14:681–694, 2006.

[20] L. Cavagnari, L.Magni, and R.Scattolini. Neural network implementation of nonlinear receding-horizon control. *Neural Computing and Applications*, 8:86–2, 1999.

[21] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 14(10):1205–1217, 1998.

[22] Jian Chen, G.W. Irwin, and A. McKernan. Packet-based robust mpc for wireless networked control using co-design. In *American Control Conference 2010*, pages 1829 –1834, 2010.

[23] L. Chisci, J. A. Rossiter, and G. Zappa. Systems with persistent disturbances:predictive control with restriceted constraints. *Automatica*, 37:1019–1028, 2001.

[24] D. Chu, T.Chen, and H.J.Marquez. Explicit robust model predictive control using recursive closed-loop prediction. *Int. J. Robust Nonlinear Control*, 16:519–546, 2006.

[25] D.W. Clarke and R. Scattolini. Constrained receding horizon predictive control. In *IEE Proceedings, Part D*, volume 138, pages 347–354, 1991.

[26] D. Muñoz de la Peña, A. Bemporad, and C. Filippi. Robust explicit MPC based on approximate multi-parametric convex programming. *IEEE Trans. Automatic Control*, 51(8):1399–1403, 2006.

[27] G. De Nicolao, L. Magni, and R. Scattolini. On the robustness of receding horizon control with terminal constraints. *IEEE Trans. on Automatic Control*, 41:451–453, 1996.

[28] M. Diehl, H.J. Ferreau, and N. Haverbeke. Efficient numerical methods for nonlinear mpc and moving horizon estimation. In *Nonlinear Model Predictive Control: Towards New Challenging Applications*, 384:391–418, Series: Lecture Notes in Control and Information Sciences. Magni, L.; Raimondo, D. M.; Allgöwer, F.(Eds.). Springer, 2009.

[29] M. Diehl, L. Magni, and G. De Nicolao. Online NMPC of unstable periodic systems using approximate infinite horizon closed loop costing. *Annual Reviews in Control*, 28:37–45, 2004.

[30] P. Dua, K. Kouramas, V. Duab, and E.N. Pistikopoulos. MPC on a chip - recent advances on the application of multi-parametric model-based control. *Computers & Chemical Engineering*, 32:754–765, 2008.

[31] R. Findeisen and P. Varutti. Stabilizing nonlinear predictive control over nondeterministic communication networks. In *Nonlinear Model Predictive Control: Towards New Challenging Applications*, 384:167–179, Series: Lecture Notes in Control and Information Sciences. Magni, L.; Raimondo, D. M.; Allgöwer, F.(Eds.). Springer, 2009.

[32] F.A.C.C. Fontes and L. Magni. Min-max model predictive control of nonlinear systems using discontinuous feedbacks. *IEEE Transaction on Automatic Control*, 48:1750–1755, 2003.

[33] E. Franco, L. Magni, T. Parisini, M. Polycarpou, and D. M. Raimondo. Cooperative constrained control of distributed agents with nonlinear dynamics and delayed information exchange: A stabilizing receding-horizon approach. *IEEE Trans. on Automatic Control*, 53(1):324–338, 2008.

[34] A.N. Fuchs, C.N. Jones, and M. Morari. Optimized decision trees for point location in polytopic data sets - application to explicit mpc. In *American Control Conference, 2010*, pages 5507 –5512, 2010.

[35] D. Georgiev and D. Tilbury. Packet-based control: The h2-optimal solution. *Automatica*, 42:127–144, 2006.

[36] Reza Ghaemi, Jing Sun, and Ilya V. Kolmanovsky. An integrated perturbation analysis and sequential quadratic programming approach for model predictive control. *Automatica*, 45(10):2412 – 2418, 2009.

[37] A. Grancharova, T. A. Johansen, and J. Kocijan. Explicit model predictive control of gas-liquid separation plant via orthogonal search tree partitioning. *Computers and Chemical Engineering*, 28:2481–2491, 2004.

[38] A. Grancharova and T.A. Johansen. Computation, approximation and stability of explicit feedback min-max nonlinear model predictive control. *Automatica*, 45(5):1134–1143, 2009.

[39] A. Grancharova, J. Kocijan, and T. A. Johansen. Explicit output-feedback nonlinear predictive control based on black-box models. *Engineering Applications of Artificial Intelligence*, 24(2):388 – 397, 2011.

[40] G. Grimm, M.J. Messina, S.E. Tuna, and A.R. Teel. Model predictive control: for want of a local control Lyapunov function, all is not lost. *IEEE Transaction on Automatic Control*, 50:546–558, 2005.

[41] G. Grimm, M.J. Messina, S.E. Tuna, and A.R. Teel. Nominally robust model predictive control with state constraints. *IEEE Transaction on Automatic Control*, 52:1856–1870, 2007.

[42] L. Grüne, J. Pannek, and K. Worthmann. A networked unconstrained nonlinear mpc scheme. In *Proc. European Control Conference*, pages 371–376, Budapest, Hu, 2009.

[43] L. Grúne, J. Pannek, and K. Worthmann. A prediction based control scheme for networked systems with delays and packet dropouts. In *Proc. of the IEEE Conf. on Decision and Control*, pages 537–542, Shanghai, China, 2009.

[44] V. Gupta and N.C. Martins. On stability in the presence of analog erasure channel between the controller and the actuator. *IEEE Transactions on Automatic Control*, 55(1):175–179, 2010.

[45] E. Gyurkovics and T. Takacs. Quadratic stabilisation with $\mathcal{H}_\infty$-norm bound of non-linear discrete-time uncertain systems with bounded control. *Systems and Control Letters*, 50:277–289, 2003.

[46] J. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. In *Proc. of IEEE Special Issue on Technology of Networked Control Systems*, volume 95, pages 138–162, 2007.

[47] M. Hovd, F. Scibilia, J. Maciejowski, and S. Olaru. Verifying stability of approximate explicit mpc. In *Proceedings of the 48th IEEE Conference on Decision and Control*, pages 6345 –6350, December 2009.

[48] K. Ibraheem and D.P. Kothari. Recent philosophies of automatic generation control strategies in power systems. *IEEE Transactions on Power Systems*, 21(1):346–357, 2005.

[49] O. Imer, S. Yuksel, and T.Basar. Optimal control of lti systems over unreliable communication links. *Automatica*, 42:1429–1439, 2006.

[50] T. A. Johansen. Approximate explicit receding horizon control of constrained nonlinear systems. *Automatica*, 40:293–300, 2004.

[51] C.N. Jones, P. Grieder, and S.V. Raković. A logarithmic-time solution to the point location problem for parametric linear programming. *Automatica*, 42:2215–2218, 2006.

[52] C.N. Jones and M. Morari. The double description method for the approximation of explicit MPC control laws. In *Proc. of the 47th IEEE Conf. on Decision and Control*, pages 4724–4730, Cancun, 2008.

[53] C.N. Jones and M. Morari. Polytopic approximation of explicit model predictive controllers. *Automatic Control, IEEE Transactions on*, 55(11):2542–2553, 2010.

[54] I. Karafillis and S. Kotsios. Necessary and sufficient conditions for robust global asymptotic stabilization of discrete-time systems. *Journal of Difference Equations and Applications*, 12:741–768, 2006.

[55] S. S. Keerthi and E. G. Gilbert. Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: stability and moving-horizon approximations. *J. of Optimization Theory and Applications*, 57:265–293, 1988.

[56] E.C. Kerrigan and D.Q. Mayne. Optimal control of constrained piecewise affine systems with bounded disturbances. In *Proc. IEEE Conf. Decision and Control*, pages 1552–1557, 2002.

[57] I. Kolmanovky and E.G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Math.Probl. in Engineering*, 35(11):1747–1767, 1999.

[58] M. Kvasnica, J. Lofberg, M. Herceg, L. Cirka, and M. Fikar. Low-complexity polynomial approximation of explicit mpc via linear programming. In *Proc. of the American Control Conference 2010*, pages 4713–4718, 2010.

[59] M. Lazar and W. Heemels. Predictive control of hybrid systems: Input-to-state stability results for sub-optimal solutions. *Automatica*, 45:180–185, 2009.

[60] M. Lazar, W.P.M.H. Heemels, A. Bemporad, and S. Weiland. Discrete-time non-smooth nonlinear mpc: Stability and robustness, in assessment and future directions of nonlinear model predictive control. *Lecture Notes in Control and Information Sciences*, 358:93–103, 2007.

[61] V. Liberatore. Integrated play-back, sensing and networked control. In *Proceedings of INFOCOM*, pages 1–12, 2006.

[62] D. Limón, T. Alamo, and E. F. Camacho. Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive

uncertainties. In *Proc. IEEE Conf.Decision Control*, pages 4619–4624, 2002.

[63] D. Limón, T. Alamo, F. Salas, and E. F. Camacho. Input to state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties. *Automatica*, 42:797–803, 2006.

[64] A. Liu, L. Yu, and W.A. Zhang. One-step receding horizon $\mathcal{H}_\infty$ control for networked control systems with random delay and packet disordering. *ISA Transactions*, 50:44–52, 2011.

[65] G.P. Liu, Y. Xia, J. Che, D. Rees, and W. Hu. Networked predictive control of systems with random network delays in both forward and feedback channels. *IEEE Trans. Industrial Electronics*, 54:1282–1297, 2007.

[66] J. Liu, D. Muńoz de la Peńa, B.J. Ohran, P.D. Christofides, and J.F. Davis. A two-tier architecture for networked process control. *Chemical Engineering Science*, 63(22):5394–5409, 2008.

[67] L. Magni, G. De Nicolao, L. Magnani, and R. Scattolini. A stabilizing model-based predictive control algorithm for nonlinear systems. *Automatica*, 37(9):1351–1362, 2001.

[68] L. Magni, D. M. Raimondo, and R. Scattolini. Regional input-to-state stability for nonlinear model predictive control. *IEEE Trans. on Automatic Control*, 51(9), 2006.

[69] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Skokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36:789–814, 2000.

[70] E.S. Meadows, M.A. Henson, J.W. Eaton, and J.B. Rawlings. Receding horizon control and discontinuous state feedback stabilization. *Int. J. Control*, 62:1217–1229, 1995.

[71] H. Michalska and D.Q. Mayne. A minimax control problem for sampled linear systems. *IEEE Trans. on Automatic Control*, 13(1):5–21, 1968.

[72] H. Michalska and D.Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Trans. on Automatic Control*, 38(11):1623–1633, 1993.

[73] M. Morari and J.J. Lee. Model predictive control: past, present and future. *Computers and Chemical Engineering*, 23:667–682, 1999.

[74] E. Mosca, J.M. Lemos, and J. Zhang. Stabilizing i/o receding horizon control. In *Proc. of the IEEE Conf. on Decision and Control, Honolulu*, pages 2518–2523, 1990.

[75] D. Muñoz de la Peña, D.R. Ramirez, E.F. Camacho, and T. Alamo. Application of an explicit min-max mpc to a scaled laboratory process. *Control Engineering Practice*, 13:1463–1471, 2005.

[76] D. Muñoz de la Peña, D.R. Ramirez, E.F. Camacho, and T. Alamo. Explicit solution of min-max mpc with additive uncertainties and quadratic criterion. *Systems and Control Letters*, 55:266–274, 2006.

[77] M. Mukai, T. Azuma, A. Kojima, and M. Fujita. Approximate robust receding horizon control for piecewise linear systems via orthogonal partitioning. In *Proc. European Control Conference*, 2003.

[78] D. Muńoz de la Peńa and P.D. Christofides. Lyapunov-based model predictive control of nonlinear systems subject to data losses. *IEEE Transactions on Automatic Control*, 53(9):2076–2089, 2008.

[79] Z. Nagy, S. Agachi, F. Allgower, R. Findeisen, M. Diehl, H.G. Bock, and J.P. Schloder. Using genetic algorithm in robust nonlinear model predictive control. *Computer Aided Chemical Engineering*, 1:711–716, 2009.

[80] Carlos Vivas P. Millán, I. Jurado and Francisco R. Rubio. Networked predictive control of systems with data dropouts. In *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, 2008.

[81] T. Parisini, M. Sanguineti, and R. Zoppoli. Nonlinear stabilization by receding horizon neural regulators. *Int. J. Control*, 70(3):341–362, 1998.

[82] T. Parisini and R. Zoppoli. A receding-horizon regulator for nonlinear systems and a neural approximation. *Automatica*, 31(10):1443–1451, 1995.

[83] G. Pin, M. Filippo, F. A. Pellegrino, G. Fenu, and T. Parisini. Approximate off-line receding horizon control of constrained nonlinear discrete-time systems: Smooth approximation of the control law. In *Proc. of the American Control Conference 2010*, Baltimore, Maryland, USA, 2010.

[84] G. Pin and T. Parisini. Stabilization of networked control systems by nonlinear model predictive control: A set invariance approach. In *Nonlinear Model Predictive Control: Towards New Challenging Applications*, 384:195–204, Series: Lecture Notes in Control and Information Sciences. Magni, L.; Raimondo, D. M.; Allgöwer, F.(Eds.). Springer, 2009.

[85] G. Pin and T. Parisini. Networked predictive control of uncertain constrained nonlinear systems: Recursive feasibility and input-to-state stability analysis. *IEEE Transaction on Automatic Control*, 56(1):72–87, 2011.

[86] G. Pin, T. Parisini, L.Magni, and D. Raimondo. Robust receding - horizon control of nonlinear systems with state dependent uncertainties: an input-to-state stability approach. In *Proc. American Control Conference*, 2008 (to appear).

[87] I. Polushin, P. Liu, , and C. Lung. On the model-based approach to nonlinear networked control systems. In *Proc. American Control Conference*, 2007.

[88] S.J. Qin and T.A. Badgwell. A survey of industrial model predictive control technology. *Control Engineering Practice*, 11:733–764, 2003.

[89] D.E. Quevedo and D. Nesic. Input-to-state stability of packetized predictive control over unreliable networks affected by packet-dropouts. *Automatic Control, IEEE Transactions on*, 56(2):370–375, 2011.

[90] D. M. Raimondo, T. Alamo, D. Limon, and E. F. Camacho. Towards the practical implementation of min-max nonlinear model predictive control. In *Proc. of the 46th IEEE Conference on Decision and Control*, New Orleans, USA, 2007.

[91] D.M. Raimondo, D. Limon, M. Lazar, L. Magni, and E. Camacho. Min-max model predictive control of nonlinear systems: A unifying overview on stability. *European Journal of Control*, 15(1):5–21, 2009.

[92] S.V. Raković, E.C. Kerrigan, D.Q. Mayne, and K.I. Kouramas. Optimized robust control invariance for linear discrete-time systems: Theoretical foundations. *Automatica*, 43:831–841, 2007.

[93] S.V. Raković, A. R. Teel, and A. Astolfi. Simple robust control invariant tubes for some classes of nonlinear discrete time systems. In *Proc. of the IEEE Conf. on Decision and Control*, pages 6392–6402, 2006.

[94] J. Richalet and D. O'Donovan. *Predictive Functional Control*. Springer, London, 2009.

[95] A. Richards, M.J. Messina, A.R. Teel, and S.E. Tuna. Model predictive control when a local control lyapunov function is not available. In *Proc. of the American Control Conference*, pages 1557 – 1562, Denver, CO, 2003.

[96] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, , and S. Sastry. Foundations of control and estimation over lossy networks. In *Proceedings of the IEEE*, volume 95, pages 163–187, 2007.

[97] B. Schölkopf and A. J. Smola. *Learning with Kernels*. MIT Press, 2002.

[98] P.O.M. Scokaert, J.B. Rawlings, and E.S. Meadows. Discrete-time stability with perturbations: application to model predictive control. *Automatica*, 33(3):463–470, 1997.

[99] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, , and S. Sastry. Optimal linear lqg control over lossy networks without packet acknowledgment. *Asian Journal of Control*, 10(1):3–13, 2008.

[100] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, and S.S. Sastry. Optimal linear LQG control over lossy networks without packet acknowledgment. *Asian Journal of Control*, 10(1):3–13, 2003.

[101] B. Sundararaman, U. Buy, and A. Kshemkalyani. Clock synchronization for wireless sensor networks: a survey. *Ad Hoc Networks*, 3(3):281–323, 2008.

[102] P.L. Tang and C. W. de Silva. Stability validation of a constrained model predictive networked control system with future input buffering. *International Journal of Control*, 80:1954–1970, 2007.

[103] P.L. Tang and C.W. de Silva. Compensation for transmission delays in an ethernet-based control network using variable horizon predictive control. *IEEE Trans. on Control Systems Technology*, 14:707–716, 2006.

[104] Y. Tipsuwan and M. Chow. Control methodologies in networked control systems. *Control Engineering Practice*, 11:1099–1111, 2003.

[105] P. Varutti and R. Findeisen. Compensating network delays and information loss by predictive control methods. In *Proc. of the European Control Conference*, pages 1722–1727, Budapest, Hu, 2009.

[106] P. Varutti and R. Findeisen. On the synchronization problem for the stabilization of networked control systems over nondeterministic networks. In *Proc. of the American Control Conference*, pages 2216–2221, St. Louis, 2009.

[107] G. Walsh and Y. Hong. Scheduling of networked control systems. *IEEE Control Systems Magazine*, 21(1):57–65, 2001.

[108] Z. Wan and M. Kothare. An efficient off-line formulation of robust model predictive control using linear matrix inequalities. *Automatica*, 39:837–846, 2003.

[109] Yang Wang and S. Boyd. Fast model predictive control using online optimization. *Control Systems Technology, IEEE Transactions on*, 18(2):267 –278, 2010.

[110] S. Yoon and M. L. Sichitiu. Tiny-sync: Tight time synchronization for wireless sensor networks. *ACM Transactions on Sensor Networks*, 3(2), 2007.

[111] M. Zeilinger, C. Jones, and M. Morari. Real-time suboptimal model predictive control using a combination of explicit mpc and online optimization. *Automatic Control, IEEE Transactions on*, PP(99), 2011.

[112] W. Zhang, M. Branicky, and S. Phillips. Stability of networked control systems. *IEEE Control Systems Magazine*, 21(1):84–99, 2001.

Appendix A

Proofs of the stated Theorems, Lemmas and Propositions

A.1 Proofs of Chapter 3

Proof of Lemma 3.3.2

Consider the following facts:
i) the control law $\kappa_f^*(x)$ steers the state from $\mathcal{C}_1(X_f)$ to X_f with a single admissible control action (i.e., $\hat{f}(x, \kappa_f^*(x)) \in X_f, \kappa_f^*(x) \in U$, $\forall x \in \mathcal{C}_1(X_f)$);
ii) it holds that for all $x \in \mathcal{C}_1(X_f) \backslash (X_f \oplus \mathcal{B}^n(\delta))$ the following inequality holds: $\mathrm{d}(x, X_f) > \delta$, which yields to $h_f^*(x) > \overline{h}_f + \lambda\delta$.

If we choose λ according to (3.11), then we have that

$$\begin{aligned} h_f^*(x) \ &> \overline{h}_f + \max_{x \in \mathcal{C}_1(X_f),\, u \in U} \Big\{ h(x,u) \Big\} > \overline{h}_f + \max_{x \in \mathcal{C}_1(X_f)} \Big\{ h(x, \kappa_f^*(x)) \Big\} \\ &> h_f \left(\hat{f}(x, \kappa_f^*(x)) \right) + h(x, \kappa_f^*(x)),\ \forall x \in \mathcal{C}_1(X_f), \end{aligned}$$

which finally implies (3.12).

Proof of Theorem 3.3.1

The proof will be carried out in two steps. The first step is aimed to prove the recursive feasibility of the scheme under the prescribed bound on uncertainties, thus establishing the robust positive invariance of the feasible set X_{RH} w.r.t. $d_t \in D$. Next step consists in showing that $V(x_t) = J_{FH}(x_t, \mathbf{u}^{\circ}_{t,t+N_c-1|t}, N_c)$ is an ISS-Lyapunov function for the closed loop system in X_{RH}.

i) First, by Assumption 6, the set X_{RH} is not empty. In fact, for any $x_t \in X_{\kappa_f}$, a feasible control sequence for FHOCP is given by

$$\tilde{\mathbf{u}}_{t,t+N_c-1|t} \triangleq \mathrm{col}\big[\kappa_f(\hat{x}_{t|t}),\ \kappa_f(\hat{x}_{t+1|t}),\ \ldots,\ \kappa_f(\hat{x}_{t+N_c-1|t})\big].$$

Then $X_{RH} \supseteq X_{\kappa_f} \supseteq X_f$. Moreover, since $d_{t+j} \in D, \forall j \in \mathbb{Z}_{\geq 0}$, with $D \subseteq \mathcal{B}^n(\overline{d})$ and $\overline{d}$ such that (3.13) holds, by using standard arguments [62, 86] it is also possible to show that, if the FHOCP at time t is feasible, then the recursive feasibility of the scheme is guaranteed w.r.t. the restricted constraints. Furthermore, it possible to show that, under the stated assumption on $\overline{d}$, also the recursive feasibility w.r.t. the terminal constraint set can be guaranteed. Indeed, from the assumption $x_t \in X_{RH}$, it follows that the predicted state $\hat{x}_{t+N_c|t}$, obtained with the optimal sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$, verifies $\hat{x}_{t+N_c|t} \in X_f$.
Now we claim that at time $t+1$, given $x_{t+1} = \hat{f}(x_t, u^{\circ}_{t|t}) + d_t$, there exists a feasible input sequence $\bar{\mathbf{u}}_{t+1,t+N_c|t+1}$, based on the optimal sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ at time t, such that $\hat{x}_{t+N_c+1|t+1} \in X_f$. Indeed, let us pick

$$\bar{\mathbf{u}}_{t+1,t+N_c|t+1} = \mathrm{col}[u^{\circ}_{t+1|t}, u^{\circ}_{t+2|t}, \ldots, u^{\circ}_{t+N_c-1|t}, \bar{u}],$$

where $\bar{u} \in U$ is a feasible control action to be specified later on. From the Lipschitz continuity of $\hat{f}(x,u)$ w.r.t. x, it follows that

$$|\hat{x}_{t+j|t} - \hat{x}_{t+j|t+1}| \leq L_{f_x}^{j-1}\bar{d},\ \forall j \in \{1, \ldots, N_c\}.$$

Then, in view of (3.13), it holds that $\hat{x}_{t+N_c|t+1} \in \mathcal{C}_1(X_f)$, which implies the existence of a feasible $\bar{u} \in U$ such that

$$\hat{x}_{t+N_c+1|t+1} = \hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}) \in X_f.$$

Thus, we can conclude that X_{RH} is RPI w.r.t. $d_t \in D$.

ii) Suppose[1] that $L_{f_x} \neq 1$; then, in view of Point 5) of Assumption 5, for all $x_t \in X_{\kappa_f}$ it holds

$$\begin{aligned} V(x_t) &\leq J_{FH}(x_t, \tilde{\mathbf{u}}_{t,t+N_c-1|t}, N_c) = \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, \kappa_f(\hat{x}_{l|t})) + h_f(\hat{x}_{t+N_c|t}) \\ &\leq \sum_{l=t}^{t+N_c-1} [h_f(\hat{x}_{l|t}) - h_f(\hat{x}_{l+1|t})] + h_f(\hat{x}_{t+N_c|t}) \leq h_f(|x_t|). \end{aligned}$$

Hence, there exists a $\mathcal{K}$-function $\alpha_2(s) = h_f(s)$ such that $V(x_t) \leq \alpha_2(|x_t|)$, $\forall x_t \in X_{\kappa_f}$. The lower bound on $V(x_t)$ can be easily obtained by using Assumption 4, $V(x_t) \geq \underline{h}(|x_t|)$, $\forall x_t \in X_{RH}$. Then, inequalities (2.2) and (2.3) hold respectively with $\Xi = X_{RH}$ and $\Omega = X_{\kappa_f}$.

Given the optimal control sequence at time t, $\mathbf{u}^{\circ}_{t,t+N_c-1|t+1}$, consider now the sequence $\bar{\mathbf{u}}_{t+1,t+N_c|t+1} \triangleq \mathrm{col}\big[u^{\circ}_{t+1|t}, \ldots, u^{\circ}_{t+N_c-1|t}, \kappa^*_f(\hat{x}_{t+N_c|t+1})\big]$.

Note that $\bar{\mathbf{u}}_{t+1,t+N_c|t+1}$ is a feasible (in general, suboptimal) control sequence for the FHOCP at time $t+1$, with cost

$$\begin{aligned} J_{FH}(x_{t+1}, &\bar{\mathbf{u}}_{t+1,t+N_c|t+1}, N_c) = V(x_t) - h(x_t, u^{\circ}_{t|t}) + \\ &+ \sum_{l=t+1}^{t+N_c-1} [h(\hat{x}_{l|t+1}, u^{\circ}_{l|t}) - h(\hat{x}_{l|t}, u^{\circ}_{l|t})] + h(\hat{x}_{t+N_c|t+1}, \kappa^*_f(\hat{x}_{t+N_c|t+1})) \\ &+ h_f\left(\hat{f}(\hat{x}_{t+N_c|t+1}, \kappa^*_f(\hat{x}_{t+N_c|t+1}))\right) - h_f(\hat{x}_{t+N_c|t}). \end{aligned} \tag{A.1}$$

In view of Assumptions 5 and Lemma 3.3.2, and considering that $\hat{x}_{t+N_c|t+1} \in \mathcal{C}_1(X_f)$, and that $(\hat{x}_{t+N_c|t} \in X_f) \Rightarrow h^*_f(\hat{x}_{t+N_c|t}) - h_f(\hat{x}_{t+N_c|t}) = 0$, then the

[1] The very special case $L_{f_x} = 1$ can be trivially addressed by a few suitable modifications to the proof of Theorem 3.3.1.

following inequalities hold

$$
\begin{aligned}
& h(\hat{x}_{t+N_c|t+1}, \kappa_f^*(\hat{x}_{t+N_c|t+1})) + h_f\left(\hat{f}(\hat{x}_{t+N_c|t+1}, \kappa_f^*(\hat{x}_{t+N_c|t+1}))\right) - h_f(\hat{x}_{t+N_c|t}) \\
& \leq h_f^*(\hat{x}_{t+N_c|t+1}) - h_f(\hat{x}_{t+N_c|t}) \\
& \leq |h_f^*(\hat{x}_{t+N_c|t+1}) - h_f^*(\hat{x}_{t+N_c|t})| + |h_f^*(\hat{x}_{t+N_c|t}) - h_f(\hat{x}_{t+N_c|t})| \\
& \leq |h_f^*(\hat{x}_{t+N_c|t+1}) - h_f^*(\hat{x}_{t+N_c|t})| \leq L_{h_f^*} L_{f_x}^{N_c-1} \mu(|\upsilon_t|).
\end{aligned}
\tag{A.2}
$$

where $L_{h_f^*} \triangleq \max\{L_{h_f}, \lambda\}$, with λ defined in (3.11). Moreover, Assumption 4 implies that

$$
|h(\hat{x}_{t+j|t+1}, u^\circ_{t+j|t}) - h(\hat{x}_{t+j|t}, u^\circ_{t+j|t})| \leq L_h L_{f_x}^{j-1} |d_t|, \qquad \forall j \in \{1, \ldots, N_c - 1\}
\tag{A.3}
$$

Now, in view of (A.1), (A.2), (A.3) and Assumption 4, it is possible to conclude that the optimal cost $V(x_{t+1})$ satisfies

$$
\begin{aligned}
V(x_{t+1}) & \leq J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t+1}, N_c) \\
& \leq V(x_t) - \underline{h}(|x|) + \left(L_h \frac{L_{f_x}^{N_c} - 1}{L_{f_x} - 1} + L_{h_f^*} L_{f_x}^{N_c-1}\right) |d_t|.
\end{aligned}
\tag{A.4}
$$

Finally, inequality (A.4) implies the existence of two $\mathcal{K}$-functions

$$\alpha_3(s) = \underline{h}(s)$$

and

$$\sigma(s) = [(L_h L_{f_x}^{N_c} - 1)/(L_{f_x} - 1) + L_{h_f^*} L_{f_x}^{N_c-1}]s,$$

such that

$$V(x_{t+1}) - V(x_t) \leq -\alpha_3(|x_t|) + \sigma(|d_t|). \tag{A.5}$$

Notice that under the stated assumptions, and in particular under Assumption 3, with the suggested choices point ii) of definition 2.2.5 results to be automatically satisfied.

Proof of Proposition 3.4.1

The proof can be easily obtained by noticing some properties induced by Assumption 7; indeed, due to this assumption, $\forall x_t \in \mathrm{dom}(\kappa)$ there exists a vector $q_t \in \mathcal{B}^n(\overline{q})$ such that $\zeta_{x_t} = q_t + x_t$. Then it holds that

$$\kappa^*(x_t) = \kappa^*(x_t) - \kappa(\zeta_{x_t}) + \kappa(\zeta_{x_t}) = v_t + \kappa(x_t + q_t).$$

Now, let us show that $v_t \in V \triangleq \mathcal{B}^m(\overline{v})$. Indeed

$$\begin{aligned} |v_t| \leq |\kappa^*(x_t) - \kappa(\zeta_{x_t})| &= |\kappa^*(x_t) - \kappa(\zeta_{x_t}) + \kappa^*(\zeta_{x_t}) - \kappa^*(\zeta_{x_t})| \\ &\leq |\kappa^*(x_t) - \kappa^*(\zeta_{x_t})| + |\kappa(\zeta_{x_t}) - \kappa^*(\zeta_{x_t})|; \end{aligned}$$

finally, thanks to Points i) and ii) of Assumption 7, it immediately follows that $|v_t| \leq \overline{v}$.

Proof of Theorem 3.4.1

Points i) and ii) of Theorem 3.4.1 will be addressed separately in the following

i) Let $x_t \in \Xi, q_t \in Q$ and $w_t \in W$. Now we will prove that Ξ is RPI for (3.16), showing that for all $q_t \in Q$, we have

$$\hat{f}(x_t, \kappa(x_t + q_t) + v_t) + w_t + q_t \in \tilde{\Xi}. \tag{A.6}$$

Substituting $\tilde{x}_t \triangleq x_t + q_t$ in (A.6), we obtain

$$\begin{aligned} \hat{f}(x_t, \kappa(\tilde{x}_t) + v_t) + w_t + q_t = \hat{f}(x_t, \kappa(\tilde{x}_t) + v_t) - \hat{f}(x_t, \kappa(\tilde{x}_t)) + \\ + \hat{f}(x_t, \kappa(\tilde{x}_t)) + \hat{f}(\tilde{x}_t, \kappa(\tilde{x}_t)) - \hat{f}(\tilde{x}_t, \kappa(\tilde{x}_t)) + w_t + q_t, \end{aligned}$$

which can be written in compact form as follows

$$\hat{f}(x_t, \kappa(\tilde{x}_t) + v_t) + w_t + q_t = \hat{f}(\tilde{x}_t, \kappa(\tilde{x}_t)) + d_t \tag{A.7}$$

where

$$d_t \triangleq \hat{f}(x_t, \kappa(\tilde{x}_t) + v_t) - \hat{f}(x_t, \kappa(\tilde{x}_t)) + \hat{f}(x_t, \kappa(\tilde{x}_t)) - \hat{f}(\tilde{x}_t, \kappa(\tilde{x}_t)) + w_t + q_t. \tag{A.8}$$

Using Assumption 1, we note that, for all $q_t \in Q$, for all $w_t \in W$ and for all $\tilde{x}_t \in \tilde{\Xi}$, the following inequalities hold:

$$\begin{aligned}|d_t| &= |\hat{f}(x_t, \kappa(\tilde{x}_t) + v_t) - \hat{f}(x_t, \kappa(\tilde{x}_t)) + \hat{f}(x_t, \kappa(\tilde{x}_t)) - \hat{f}(\tilde{x}_t, \kappa(\tilde{x}_t)) + w_t + q_t| \\ &\leq |\hat{f}(x_t, \kappa(\tilde{x}_t) + v_t) - \hat{f}(x_t, \kappa(\tilde{x}_t))| + \\ &\qquad + \left|\hat{f}(x_t, \kappa(\tilde{x}_t)) - \hat{f}(\tilde{x}_t, \kappa(\tilde{x}_t))\right| + |w_t| + |q_t| \\ &\leq \eta_u\left(|v_t|\right) + L_{f_x}\left|q_t\right| + |w_t| + |q_t|.\end{aligned} \tag{A.9}$$

Considering the restriction at time t of the admissible perturbations sequences $\boldsymbol{v}_{[t]} \in \mathcal{M}_{\mathcal{B}^m(\eta_u^{-1}(\overline{d}_v))}, \boldsymbol{w}_{[t]} \in \mathcal{M}_{\mathcal{B}^n(\overline{d}_w)}$ and $\boldsymbol{q}_{[t]} \in \mathcal{M}_{\mathcal{B}^n(\eta_x^{-1}(\overline{d}_q))}$, it follows that

$$\begin{aligned}|d_t| &\leq \eta_u\left(||\boldsymbol{v}_{[t]}||\right) + L_{f_x}||\boldsymbol{q}_{[t]}|| + ||\boldsymbol{w}_{[t]}|| + ||\boldsymbol{q}_{[t]}|| \leq \eta_u\left(\overline{v}\right) + L_{f_x}\overline{q} + \overline{q} + \overline{d}_w \\ &= \eta_u\left(\eta_u^{-1}(\overline{d}_v)\right) + L_{f_x}\eta_x^{-1}(\overline{d}_q) + \eta_x^{-1}(\overline{d}_q) + \overline{d}_w = \overline{d}_v + \overline{d}_q + \overline{d}_w \leq \overline{d}.\end{aligned} \tag{A.10}$$

In view of Point ii) of Assumption 3, (A.7) and (A.10) together imply that

$$\hat{f}(x_t, \kappa(x_t + q_t) + v_t) + w_t \in \Xi,\ \forall x_t \in \Xi, \forall q_t \in Q, \forall w_t \in W. \tag{A.11}$$

Finally, thanks to Assumptions 1 and 7, it holds that for any $x_t \in \Xi, t \in \mathbb{Z}_{\geq 0}$, the state trajectory of the closed loop system in presence of disturbances satisfies $x_{t+j} \in \Xi \subseteq X$, $\forall j \in \mathbb{Z}_{>0}$.

ii) The ISS property for the closed loop system can be straightforwardly proven considering that, in view of Theorem 3.3.1 and taking in account inequalities (A.5) and (A.9), the optimal finite horizon cost function satisfies the condition

$$\begin{aligned}&V(x_{t+1}) - V(x_t) \leq -\alpha_3(|x_t|) + \sigma(\eta_u(|v_t|) + \eta_x(|q_t|) + |w_t|) \\ &\leq -\alpha_3(|x_t|) + \sigma(3\eta_u(|v_t|)) + \sigma(3\eta_x(|q_t|)) + \sigma(3|w_t|) \\ &= -\alpha_3(|x_t|) + \sigma_v(|v_t|) + \sigma_q(|q_t|) + \sigma_w(|w_t|),\end{aligned} \tag{A.12}$$

where $\sigma_v(s) \triangleq \sigma(3\eta_u(s))$, $\sigma_q(s) \triangleq \sigma(3\eta_x(s))$, $\sigma_w(|w_t|) \triangleq \sigma(3s)$, $s \in \mathbb{R}_{\geq 0}$. Hence, in view of Theorem 2.2.1, the closed loop system is regional-ISS in Ξ w.r.t. the bounded approximation-induced perturbations $v_t \in V$, $q_t \in Q$ and $w_t \in W$.

Proof of Proposition 3.5.1

Let's denote with $\hat{f}_{(1)}(x_{(1)}, u)$ and $\hat{f}_{(2)}(x_{(2)}, u)$ respectively the two components of the state transition function (3.19). Consider the first one: given two points $x_{(1)}$ and $x'_{(1)} = x_{(1)} + \delta x_{(1)}$, we have

$$
\begin{aligned}
&\left|\hat{f}_{(1)}(x'_{(1)}, u) - \hat{f}_{(1)}(x_{(1)}, u)\right| = \\
&\qquad = \left|(x_{(1)} + \delta x_{(1)})\left(p_1 + \text{sign}(x_{(1)} + \delta x_{(1)})u\right) - x_{(1)}\left(p_1 + \text{sign}(x_{(1)})u\right)\right| \\
&\qquad = \left|x_{(1)}\left(\text{sign}(x_{(1)} + \delta x_{(1)}) - \text{sign}(x_{(1)})\right)u + \delta x_{(1)}\left(p_1 + \text{sign}(x_{(1)} + \delta x_{(1)})u\right)\right| \\
&\qquad \leq \left|x_{(1)}\right| \left|\text{sign}(x_{(1)} + \delta x_{(1)}) - \text{sign}(x_{(1)})\right| R + \left|\delta x_{(1)}\right| (p_1 + R).
\end{aligned}
$$

Note that, if $\left|x_{(1)}\right| > \left|\delta x_{(1)}\right|$, then the first term in the right-hand side of the last inequality becomes null and hence

$$
\left|\hat{f}_{(1)}(x'_{(1)}, u) - \hat{f}_{(1)}(x_{(1)}, u)\right| \leq \left|\delta x_{(1)}\right| (p_1 + R).
$$

Conversely, when $\left|x_{(1)}\right| \leq \left|\delta x_{(1)}\right|$, we have that

$$
\left|\hat{f}_{(1)}(x'_{(1)}, u) - \hat{f}_{(1)}(x_{(1)}, u)\right| \leq \left|\delta x_{(1)}\right| (p_1 + 3R).
$$

Finally, an upper bound for the Lipschitz constant of the transition function $\hat{f}$ can be established as

$$
L_{f_x} \leq \max\left\{p_1 + 3R, e^{-2p_3}(p_2 + (1 - p_2)p_1/p_3)\right\}.
$$

Proof of Claim 3.5.1

The proof will be carried out by contradiction. Assume that there exists a bounded state feedback control law $u_t = \kappa(x_t)$, $\kappa : \mathcal{B}^2(\gamma) \to \mathbb{R}$, $\gamma > 0$,

$|\kappa(x)| \leq R$, $\forall x \in \mathbb{R}^2$, for some finite $R \in \mathbb{R}_{>0}$, continuous in its arguments, capable to locally stabilize the 0-equilibrium in the $\sigma - \epsilon$ sense. Then, for all $\epsilon \geq 0$ there should exist $\delta > 0$ such that the solution of the closed-loop system driven by $\kappa(x_{(1)_t}, x_{(2)_t})$ verifies

$$|(x_{(1)_t}, x_{(2)_t})| \leq \epsilon, \;\; \forall t \geq 0 \tag{A.13}$$

whenever $|x_0| \leq \delta$. Now take, for the sake of simplicity, an initial condition $x_0 = (x_{(1)_0}, x_{(2)_0}) = (\bar{x}_{(1)}, 0)$, for which the solution verifies $x_{(2)_t} \equiv 0$, $\forall t \geq 0$. By this position we can consider only the first equation in (3.19) to prove the non-stabilizability of the system by continuous state-feedback.

From the (uniform) continuity of $\kappa(\cdot)$, it holds that $\forall \eta > 0$, $\exists \mu_\eta > 0$ such that $|\kappa(x) - \kappa(x')| < \eta, \forall |x - x'| \leq \mu_\eta, \forall x \in \mathcal{B}^2(\gamma)$ and $\forall x' \in \mathcal{B}^2(\gamma)$. Let us pick an arbitrary η such that $0 < \eta < 2(p_1 - 1)$, then there exists a finite scalar μ_η such that

$$|\kappa(x) - \kappa(x')| < 2(p_1 - 1), \forall |x - x'| \leq \mu_\eta, \forall x \in \mathcal{B}^2(\gamma). \tag{A.14}$$

In the following, we will show that the continuity of the control function is not compatible with the stability of the 0-equilibrium. Now let us choose an arbitrary $\epsilon < \min\{\mu_\eta/2, \gamma\}$; in order to verify (A.13), whatever be the initial condition $|x_{(1)_0}| \leq \delta \leq \epsilon$, the following inequalities must hold for at least one $x_{(1)+} \in [\delta, \epsilon]$ and one $x_{(1)-} \in [-\epsilon, -\delta]$:

$$\begin{aligned} x_{(1)+} \left(p_1 + \text{sign}(x_{(1)+})\kappa(x_{(1)+})\right) &\leq x_{(1)+}, \\ x_{(1)-} \left(p_1 + \text{sign}(x_{(1)-})\kappa(x_{(1)-})\right) &\geq x_{(1)-}. \end{aligned} \tag{A.15}$$

Indeed, if, for instance, the first condition is not satisfied for at least one $x_{(1)+} \in [\delta, \epsilon]$, then it holds that $\hat{f}_{(1)}(x_{(1)}) > x_{(1)}$, $\forall x_{(1)} \in [\delta, \epsilon]$, which, by the continuity of $\kappa(x_1)$ and $\hat{f}_{(1)}(x_1)$ in $[\delta, \epsilon]$ implies that, given the initial condition $x_{(1)_0} = \delta$, there will exist a finite time instant $T \in \mathbb{Z}_{>0}$ such that $x_{(1)_T} > \epsilon$, which is not compatible with the assumption of $\sigma - \epsilon$ stability of the origin. The same argument can be used to prove that the existence of a suitable $x_{(1)-} \in [-\epsilon, -\delta]$, satisfying the second inequality of (A.15), is necessary for the $\sigma-\epsilon$ stability of the 0-equilibrium. The necessary conditions (A.15) are equivalent to

$$\kappa(x_{(1)+}) \leq -p_1 + 1\,, \quad \kappa(x_{(1)-}) \geq p_1 - 1\,,$$

that together imply

$$\left|\kappa(x_{(1)+}) - \kappa(x_{(1)-})\right| \geq 2(p_1 - 1). \tag{A.16}$$

Since $|x_{(1)+} - x_{(1)-}| \leq 2\epsilon < \mu_\eta$, by the continuity of κ we also have that (A.14) holds, which is incompatible with (A.16). Hence, the 0-equilibrium is not stabilizable in the $\sigma-\epsilon$ sense by a (uniformly) continuous time-invariant state-feedback under the specified parametrized class of transition maps.

A.2 Proofs of Chapter 4

Proof of Lemma 4.2.2

Given the state measurement $x_{t-\tau_c(t)}$, available at time t at the controller node, let us consider the combined sequence of controls, $\mathbf{c}^*$, formed by:
$i)$ the subsequence used for estimating $\hat{x}_{t|t-\tau_c(t)}$ (i.e., the true overall correction sequence $\mathbf{c}_{t-\tau_c(t),t-1}$ applied from $t-\tau_c(t)$ to $t-1$)
$ii)$ a feasible sequence $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$.

The resulting sequence will be then

$$\mathbf{c}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)} \triangleq \mathrm{col}[\mathbf{c}_{t-\tau_c(t),t-1}, \overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}]. \tag{A.17}$$

Thanks to Lemma 4.2.1, the prediction error

$$\hat{e}_{t-\tau_c(t)+j|t-\tau_c(t)} \triangleq x_{t-\tau_c(t)+j} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)},$$

with $j \in \{1, \ldots, N_c + \tau_c(t)\}$ and $x_{t-\tau_c(t)+j}$ obtained by applying $\mathbf{c}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ in open loop to the uncertain pre-compensated system (4.2), verifies the inclusion

$$\hat{e}_{t-\tau_c(t)+j|t-\tau_c(t)} \in T_{\mathbf{K}j}(D), \ \ \forall j \in \{1, \ldots, N_c + \tau_c(t)\}$$

Being $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$ feasible, it holds that $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_{\mathbf{K}j}(D)$, $\forall j \in \{\tau_c(t)+1, \ldots, N_c + \tau_c(t)\}$, then it follows immediately that

$$x_{t-\tau_c(t)+j} = \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \hat{e}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X.$$

With the same arguments as above, considering that the deviation of any possible perturbed trajectory from the predicted one does not exceed the uncertainty envelope, it is possible to prove that also the input constraint $u_t \in U$ is satisfied.

Proof of Theorem 4.2.1

The proof consists in showing that if, at time t, the input sequence computed by the controller $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$ is feasible in the sense of Definition 4.2.6, and if the perturbed system evolves under the action of the MPC–NDC scheme, there will exist a feasible control sequence at time instant $t+1$. Finally, the recursive feasibility follows by induction. Now, the proof will be carried out in four steps.

i) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$:

Let us consider the sequence $\mathbf{c}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ defined in (A.17). It is straightforward to prove that the two trajectories $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)}$ (initiated by $x_{t-\tau_c(t)}$ and $x_{t+1-\tau_c(t+1)}$), respectively obtained by applying to the nominal model the sequence $\mathbf{c}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and its subsequence $\mathbf{c}^*_{t+1-\tau_c(t+1),t-\tau_c(t)+j-1|t-\tau_c(t)}$, verify the following inclusion $\forall j \in \{i, \ldots, N_c + \tau_c(t)\}$:

$$\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \oplus \mathbf{A}_{\mathbf{K}}^{j-i} T_{\mathbf{K}i}(D), \qquad \text{(A.18)}$$

where we have posed $i = \tau_c(t) - \tau_c(t+1) + 1$.

Now, consider the case $j = N_c + \tau_c(t)$; then (A.18) yields to

$$\hat{x}_{t+N_c|t+1-\tau_c(t+1)} \in \hat{x}_{t+N_c|t-\tau_c(t)} \oplus \mathbf{A}_{\mathbf{K}}^{N_c+\tau_c(t+1)-1} T_{\mathbf{K}\tau_c(t)-\tau_c(t+1)+1}(D). \quad \text{(A.19)}$$

Then, by posing $r = \tau_c(t+1) - 1$ and $s \triangleq \tau_c(t)$, in view of Point 1) of the statement of the theorem, it holds that

$$\hat{x}_{t+N_c|t+1-\tau_c(t+1)} \in \mathcal{C}_1(X_f|\mathbf{A}_{\mathbf{K}}, \mathbf{B}, U \backsim \mathbf{K}(X_f \oplus T_{\mathbf{K}\tau_c(t+1)+N_c-1}(\gamma S))),$$

whatever be the values of $\tau_c(t)$ and $\tau_c(t+1)$. Hence, from the definition of $\mathcal{C}_1$, there exists a feasible control move (for the RHOCP at time $t+1$) $\overline{c}_{t+N_c|t+1-\tau_c(t+1)}$ which can steer the state vector from $\hat{x}_{t+N_c|t+1-\tau_c(t+1)}$ to $\hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$.

ii) $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_{\mathbf{K}j}(D) \Rightarrow \hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{j-i}(\overline{d})$, with $i = \tau_c(t) - \tau_c(t+1) + 1$ and $\forall j \in \{\tau_c(t)+1, \ldots, N_c + \tau_c(t)\}$:

Consider the predictions $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i}$(initiated by $x_{t-\tau_c(t)}$ and $x_{t-\tau_c(t)+i}$), respectively obtained with the sequence $\mathbf{c}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and its subsequence $\mathbf{c}^*_{t-\tau_c(t)+i,t-\tau_c(t)+j-1|t-\tau_c(t)}$. Assuming that $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X \backsim T_{\mathbf{K}j}(D)$, let us introduce $\eta \in T_{\mathbf{K}j-i}(D)$. Let $\xi \triangleq \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \eta$, then, in view of (A.18), it follows that

$$\xi \in T_{\mathbf{K}j}(D)\,. \quad \text{(A.20)}$$

Hence $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \xi \in X$, $\forall \eta \in T_{\mathbf{K}j-i}(D)$, yielding to $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{\mathbf{K}j-\tau_c(t)+\tau_c(t+1)-1}(D)$.

iii) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_{\mathbf{K}N_c+\tau_c(t+1)}(D)$;

Thanks to Point i), there exists a feasible control sequence at time $t+1$ which yields to $\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_f$. In view to Point 2) in the statement of the Theorem, it follows that

$$\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_f \subseteq X_{N_c+\tau_c(t+1)}, \ \forall \tau_c(t+1) \leq \overline{\tau}_c.$$

iv) Posing

$$\overline{\mathbf{c}}^c_{t+1,t+N_c+1|t+1-\tau_c(t+1)} = \text{col}[\mathbf{c}^*_{t+1,t+N_c-1|t-\tau_c(t)}, \overline{c}_{t+N_c|t+1-\tau_c(t+1)}],$$

we have that (A.18) yields to

$$\overline{c}^c_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in U_{\mathbf{K}j}(D),$$

with $j \in \{\tau_c(t), \dots, \tau_c(t) + N_c - 1\}$. Indeed, by posing

$$i = \tau_c(t) - \tau_c(t+1) + 1$$

we have that

$$\begin{aligned}\overline{c}^c_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} &+ \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in \\ &c^*_{t-\tau_c(t)+j|t-\tau_c(t)} + \mathbf{K}(\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \oplus \mathbf{A}_{\mathbf{K}}^{j-i} T_{\mathbf{K}i}(D)) \subseteq \\ &\qquad U_{\mathbf{K}j}(D) \oplus \mathbf{K}\mathbf{A}_{\mathbf{K}}^{j-i} T_{\mathbf{K}i}(D).\end{aligned}$$

In view of the fact that $U_{\mathbf{K}j}(D) = U \backsim \mathbf{K}T_{\mathbf{K}j}(D)$, and that $\mathbf{A}_{\mathbf{K}}^{j-i} T_{\mathbf{K}i}(D) = T_{\mathbf{K}j}(D) \backsim T_{\mathbf{K}j-i}(D)$, by posing $k = \tau_c(t+1) - \tau_c(t) + j - 1 = j - i$ we have that

$$\begin{aligned}\overline{c}^c_{t+1-\tau_c(t+1)+k|t+1-\tau_c(t+1)} &+ \mathbf{K}\hat{x}_{t+1-\tau_c(t+1)+k|t+1-\tau_c(t+1)} \subseteq \\ &U \backsim \mathbf{K}T_{\mathbf{K}k}(D) = U_{\mathbf{K}k}(D),\end{aligned}$$

which is verified $\forall k \in \{0, \dots, N_c - 2\}$.

Then, the sub-sequence $\overline{\mathbf{c}}^c_{t+\overline{\tau}_a,t+N_c+1|t+1-\tau_c(t+1)}$ is feasible at time $t+1$ with respect to the restricted input constraints of the RHOCP.

Then, under the assumptions posed in the statement of Theorem 4.2.1, given $\overline{x}_0 \in X_{MPC}$, and being $\tau_c(0) = 0$ (i.e. at the first time instant the buffers of the local controllers are initiated with a feasible sequence) in view of Points i)–iii) it holds that at any time $t \in \mathbb{Z}_{>0}$ a feasible control sequence exists and can be chosen as $\overline{\mathbf{c}}^c_{t+1,t+N_c+1|t+1-\tau_c(t+1)}$ specified in Point iv) above. Therefore the recursive feasibility of the scheme is ensured.

Proof of Lemma 4.3.1

Given the state measurement $x_{t-\tau_c(t)}$, available at time t at the controller node, let us consider the combined sequence of controls, $\mathbf{c}^*$, formed by:
i) the subsequence used for estimating $\hat{x}_{t|t-\tau_c(t)}$ (that is the true overall correction sequence $\mathbf{c}_{t-\tau_c(t),t-1}$ applied from $t-\tau_c(t)$ to $t-1$);
ii) a feasible sequence $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$,

$$\mathbf{c}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)} \triangleq \text{col}[\mathbf{c}_{t-\tau_c(t),t-1}, \overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}]. \tag{A.21}$$

Then, the prediction error $\hat{e}_{t-\tau_c(t)+j|t-\tau_c(t)} \triangleq x_{t-\tau_c(t)+j} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$, with $j \in \{1, \ldots, N_c + \tau_c(t)\}$ and $x_{t-\tau_c(t)+j}$ obtained by applying $\mathbf{c}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ in open loop to system (4.22) is upper bounded by ([62]):

$$|\hat{e}_{t-\tau_c(t)+j|t-\tau_c(t)}| \leq \frac{L_g^j - 1}{L_g - 1}\overline{d}, \quad \forall j \in \{1, \ldots, N_c + \tau_c(t)\},$$

where $\overline{d}$ is the quantity defined in Assumption 4. This inequality can be easily proved by induction considering that $\hat{e}_{t-\tau_c(t)|t-\tau_c(t)} = 0$ and

$$\begin{aligned}|\hat{e}_{t-\tau_c(t)+j|t-\tau_c(t)}| =& |g(x_{t-\tau_c+j-1}, \overline{c}^c_{t-\tau_c(t)+j-1|t-\tau_c(t)}) + \overline{d} \\ & - g(\hat{x}_{t-\tau_c+j-1}, \overline{c}^c_{t-\tau_c(t)+j-1|t-\tau_c(t)})| \\ & \leq L_g|\hat{e}_{t-\tau_c(t)+j-1|t-\tau_c(t)}| + \overline{d} = \overline{d}\sum_{k=0}^{j-1}(L_g^k + 1) = \frac{L_g^j - 1}{L_g - 1}\overline{d}.\end{aligned}$$

Being $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$ feasible, it holds that $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_j(\overline{d})$,
$\forall j \in \{\tau_c(t)+1, \ldots, N_c + \tau_c(t)\}$, then it follows immediately that

$$x_{t-\tau_c(t)+j} = \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \hat{e}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X.$$

With the same arguments as above, considering that the deviation of any possible perturbed trajectory from the predicted one does not exceed the uncertainty envelope, it is possible to prove that also the input constraint

$u_t \in U$ is satisfied.

Proof of Theorem 4.3.1

Similarly to the one of Theorem 4.2.1, the proof will be carried out in four steps and consists in showing that if, at time t, the input sequence computed by the controller $\overline{\mathbf{c}}^c_{t,t+N_c-1|t-\tau_c(t)}$ is feasible in the sense of Definition 4.3.3, and if the perturbed system evolves under the action of the MPC-NDC scheme, then there will exist a feasible control sequence at time instant $t+1$. Again, the recursive feasibility will follow by induction.

i) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$:

Let us consider the sequence $\mathbf{c}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ defined in (A.21). It is straightforward to prove that the norm difference between the two trajectories $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)}$ (initiated by $x_{t-\tau_c(t)}$ and $x_{t+1-\tau_c(t+1)}$), respectively obtained by applying to the nominal model the sequence $\mathbf{c}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and $\mathbf{c}^*_{t+1-\tau_c(t+1),t-\tau_c(t)+j-1|t-\tau_c(t)}$ (its subsequence), can be upper bounded noting:

$$|\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}| \leq L_g^{j-i} \sum_{l=0}^{i-1} L_g^l \overline{d} \qquad \text{(A.22)}$$

where we have posed $i = \tau_c(t) - \tau_c(t+1) + 1$ and with $j \in \{i, \ldots, N_c + \tau_c(t)\}$. If now $j = N_c + \tau_c(t)$, (A.22) yields to

$$|\hat{x}_{t+N_c|t+1-\tau_c(t+1)} - \hat{x}_{t+N_c|t-\tau_c(t)}| \leq \frac{L_g^{N_c+\tau_c(t)} - L_g^{N_c+\tau_c(t)-i}}{L_g - 1} \overline{d}.$$

If the following inequality holds $\forall k \in \{1, \ldots, \overline{\tau}_c\}$

$$\overline{d} \leq \frac{L_g - 1}{L_g^{N_c+k} - L_g^{N_c-1}} \mathrm{dist}(\mathbb{R}^n \setminus \mathcal{C}_1(X_f), X_f),$$

then $\hat{x}_{t+N_c|t+1-\tau_c(t+1)} \in \mathcal{C}_1(X_f)$, whatever be the values of $\tau_c(t)$ and

$\tau_c(t+1)$. Hence, from the definition of $\mathcal{C}_1$, there exists a feasible control move (for the RHOCP at time $t+1$) $\overline{c}_{t+N_c|t+1-\tau_c(t+1)}$ which can steer the state vector from $\hat{x}_{t+N_c|t+1-\tau_c(t+1)}$ to $\hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$.

ii) $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_j(\overline{d}) \Rightarrow \hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{j-i}(\overline{d})$, with $i = \tau_c(t) - \tau_c(t+1) + 1$ and $\forall j \in \{\tau_c(t)+1, \ldots, N_c + \tau_c(t)\}$:

Consider the predictions $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i}$ (initiated by $x_{t-\tau_c(t)}$ and $x_{t-\tau_c(t)+i}$), respectively obtained with the sequence $\mathbf{c}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and its subsequence $\mathbf{c}^*_{t-\tau_c(t)+i,t-\tau_c(t)+j-1|t-\tau_c(t)}$. Assuming that

$$\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X \backsim \mathcal{B}^n(\frac{L_g^j - 1}{L_g - 1}\overline{d}),$$

let us introduce

$$\eta \in \mathcal{B}^n(\frac{L_g^{j-i} - 1}{L_g - 1}\overline{d}).$$

Let

$$\xi \triangleq \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \eta;$$

then, in view of (A.22), it follows that

$$\begin{aligned} |\xi| &\leq |\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_{rt}(t)+j|t-\tau_c(t)}| + |\eta| \\ &\leq \frac{L_g^j - 1}{L_g - 1}\overline{d} \end{aligned}$$

and hence $\xi \in \mathcal{B}^n((L_g^j - 1)/(L_g - 1)\overline{d})$. Since $\hat{x}_{t-\tau_c(t)+j|t} \in X_j(\overline{d})$, it follows that

$$\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \xi = \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} + \eta \in X, \forall \eta \in \mathcal{B}^n\left(\frac{L_g^{j-i} - 1}{L_g - 1}\overline{d}\right),$$

yielding to $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{j-\tau_c(t)+\tau_c(t+1)-1}(\overline{d})$.

iii) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_{N_c+\tau_c(t+1)}(\overline{d})$:

Thanks to Point i), there exists a feasible control sequence at time $t+1$ yielding to $\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_f$. If $\overline{d}$ satisfies

$$\overline{d} \leq \min_{j\in\{N_c,\dots,N_c+\overline{\tau}_c\}} \left\{ \frac{L_g - 1}{L_g^j - 1} \mathrm{dist}(\mathbb{R}^n \setminus X_j(\overline{d}), X_f) \right\},$$

it follows that $\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_{N_c+\tau_c(t+1)}(\overline{d})$, $\forall \tau_c(t+1) \leq \overline{\tau}_c$.

iv) Posing $\overline{\mathbf{c}}^c_{t+1,t+N_c+1|t+1-\tau_c(t+1)} = \mathrm{col}[\mathbf{c}^*_{t+1,t+N_c-1|t-\tau_c(t)}, \overline{c}_{t+N_c|t+1-\tau_c(t+1)}]$ and $i = \tau_c(t) - \tau_c(t+1) + 1$, thanks to (A.22) we have that

$$\begin{aligned}
&\overline{c}^c_{t-\tau_c(t)+j|t-\tau_c(t)+i} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} \\
&\qquad \in c^*_{t-\tau_c(t)+j|t-\tau_c(t)} + \mathbf{K}(\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \mathcal{B}_{j-i}) \\
&\qquad = (c^*_{t-\tau_c(t)+j|t-\tau_c(t)} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}) + \mathbf{K}\mathcal{B}_{j-i},
\end{aligned}$$

where we indicated with $\mathcal{B}_{j-i}$ the set $\mathcal{B}^n(L_g^{j-i}\sum_{l=0}^{i-1} L_g^l \overline{d})$. Since the first term is contained in the set $U_j(\overline{d})$ while the second one lies in $\mathbf{K}\mathcal{B}_{j-i}$, it turns out that

$$\begin{aligned}
&\overline{c}^c_{t-\tau_c(t)+j|t-\tau_c(t)+i} + \mathbf{K}\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} \\
&\qquad \in U \setminus \mathbf{K}\mathcal{B}^n\left(\left(\frac{L_g^j - 1}{L_g - 1} - L_g^{j-i}\frac{L_g^i - 1}{L_g - 1}\right)\overline{d}\right) \\
&\qquad = U \setminus \mathbf{K}\mathcal{B}^n\left(\left(\frac{L_g^{j-i} - 1}{L_g - 1}\right)\overline{d}\right) = U_{j-i}.
\end{aligned}$$

Posing $k = \tau_c(t+1) - \tau_c(t) + j - 1 = j - i$, this finally implies that $\overline{c}^c_{t+1-\tau_c(t+1)+k|t+1-\tau_c(t+1)} + \mathbf{K}\hat{x}_{t+1-\tau_c(t+1)+k|t+1-\tau_c(t+1)} \in U_k(\overline{d})$, which is verified $\forall k \in \{0,\dots,N_c-2\}$. Then, the sub-sequence $\overline{\mathbf{c}}^c_{t+\overline{\tau}_{rt},t+N_c+1|t+1-\tau_c(t+1)}$ is feasible at time $t+1$ with respect to the restricted input constraints of the RHOCP.

Then, under the assumptions posed in the statement of Theorem 4.3.1, given $\overline{x}_0 \in X_{MPC}$, and being $\tau_c(0) = 0$ (i.e. at the first time instant the buffers of the local controllers are initiated with a feasible sequence) in view of Points i)-iii) it holds that at any time $t \in \mathbb{Z}_{>0}$ a feasible control sequence exists and can be chosen as $\overline{\mathbf{c}}^c_{t+1,t+N_c+1|t+1-\tau_c(t+1)}$, specified in Point iv) above. Therefore the recursive feasibility of the scheme is ensured.