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Non-vanishing Theorems for Rank Two Vector Bundles on Threefolds[1]

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Abstract. *The paper investigates the non-vanishing of $H^1(\mathcal{E}(n))$, where $\mathcal{E}$ is a (normalized) rank two vector bundle over any smooth irreducible threefold X with $\mathrm{Pic}(X) \cong \mathbb{Z}$. If ϵ is defined by the equality $\omega_X = \mathcal{O}_X(\epsilon)$, and α is the least integer t such that $H^0(\mathcal{E}(t)) \neq 0$, then, for a non-stable $\mathcal{E}$, $H^1(\mathcal{E}(n))$ does not vanish at least between $\frac{\epsilon - c_1}{2}$ and $-\alpha - c_1 - 1$. The paper also shows that there are other non-vanishing intervals, whose endpoints depend on α and on the second Chern class of $\mathcal{E}$. If $\mathcal{E}$ is stable $H^1(\mathcal{E}(n))$ does not vanish at least between $\frac{\epsilon - c_1}{2}$ and $\alpha - 2$. The paper considers also the case of a threefold X with $\mathrm{Pic}(X) \neq \mathbb{Z}$ but $\mathrm{Num}(X) \cong \mathbb{Z}$ and gives similar non-vanishing results.*

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1. Introduction

In 1942 G. Gherardelli ([5]) proved that, if C is a smooth irreducible curve in $\mathbb{P}^3$ whose canonical divisors are cut out by the surfaces of some degree e and moreover all linear series cut out by the surfaces in $\mathbb{P}^3$ are complete, then C is the complete intersection of two surfaces. Shortly and in the language of modern algebraic geometry: every e-subcanonical smooth curve C in $\mathbb{P}^3$ such that $h^1(\mathcal{I}_C(n)) = 0$ for all n is the complete intersection of two surfaces.

Thanks to the Serre correspondence between curves and vector bundles (see [7, 8, 9]) the above statement is equivalent to the following one: if $\mathcal{E}$ is a rank two vector bundle on $\mathbb{P}^3$ such that $h^1(\mathcal{E}(n)) = 0$ for all n, then $\mathcal{E}$ splits.

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There are many improvements of the above result with a variety of different approaches (see for instance [2, 3, 4, 13, 15]): it comes out that a rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ is forced to split if $h^1(\mathcal{E}(n))$ vanishes for just one strategic n, and such a value n can be chosen arbitrarily within a suitable interval, whose endpoints depend on the Chern classes and the least number α such that $h^0(\mathcal{E}(\alpha)) \neq 0$.

When rank two vector bundles on a smooth threefold X of degree d in $\mathbb{P}^4$ are concerned, similar results can be obtained, with some interesting difference.

In 1998 Madonna ([11]) proved that on a smooth threefold X of degree d in $\mathbb{P}^4$ there are ACM rank two vector bundles (i.e. whose 1-cohomology vanishes for all twists) that do not split. And this can happen, for a normalized vector bundle $\mathcal{E}$ ($c_1 \in \{0, -1\}$), only when $1 - \frac{d+c_1}{2} < \alpha < \frac{d-c_1}{2}$, while an ACM rank two vector bundle on X whose α lies outside of the interval is forced to split.

The following non-vanishing results for a normalized non-split rank two vector bundle on a smooth irreducible thereefold of degree d in $\mathbb{P}^4$ are proved in [11]:

- if $\alpha \leq 1 - \frac{d+c_1}{2}$, then $h^1(\mathcal{E}(\frac{d-3-c_1}{2})) \neq 0$ if $d + c_1$ is odd, $h^1(\mathcal{E}(\frac{d-4-c_1}{2})) \neq 0, h^1(\mathcal{E}(\frac{d-2-c_1}{2})) \neq 0$ if $d + c_1$ is even, while $h^1(\mathcal{E}(\frac{d-c_1}{2})) \neq 0$ if $d + c_1$ is even and moreover $\alpha \leq -\frac{d+c_1}{2}$;
- if $\alpha \geq \frac{d-c_1}{2}$, then $h^1(\mathcal{E}(\frac{d-3-c_1}{2})) \neq 0$ if $d + c_1$ is odd, while $h^1(\mathcal{E}(\frac{d-4-c_1}{2})) \neq 0$ if $d + c_1$ is even.

In [11] it is also claimed that the same techniques work to obtain similar non-vanishing results on any smooth threefold X with $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $h^1(\mathcal{O}_X(n)) = 0$, for every n.

The present paper investigates the non-vanishing of $H^1(\mathcal{E}(n))$, where $\mathcal{E}$ is a rank two vector bundle over any smooth irreducible threefold X such that $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $H^1(\mathcal{O}_X(n)) = 0$, for all n. Actually we can prove that for such an $\mathcal{E}$ there is a wider range of non-vanishing for $h^1(\mathcal{E}(n))$, so improving the above results.

More precisely, when $\mathcal{E}$ is (normalized and) non-stable ($\alpha \leq 0$) the first cohomology module does not vanish at least between the endpoints $\frac{\epsilon-c_1}{2}$ and $-\alpha - c_1 - 1$, where ϵ is defined by the equality $\omega(X) = \mathcal{O}_X(\epsilon)$ (and is $d - 5$ if $X \subset \mathbb{P}^4$, where $d = \deg(X)$). But we can show that there are other non-vanishing intervals, whose endpoints depend on α and also on the second Chern class c_2 of $\mathcal{E}$.

If on the contrary $\mathcal{E}$ is stable the first cohomology module does not vanish at least between the endpoints $\frac{\epsilon-c_1}{2}$ and $\alpha - 2$, but other ranges of non-vanishing can be produced.

We give a few examples obtained by pull-back from vector bundles on $\mathbb{P}^3$.

We must remark that most of our non-vanishing results do not exclude the range for α between the endpoints $1 - \frac{d+c_1}{2}$ and $\frac{d-c_1}{2}$ (for a general threefold it becomes $-\frac{\epsilon+3+c_1}{2} < \alpha < \frac{\epsilon+5-c_1}{2}$). Actually [11] produces some examples of non-split ACM rank two vector bundles on smooth hypersurfaces in $\mathbb{P}^4$, but it can be seen that they do not conflict with our theorems.

As to threefolds with $\mathrm{Pic}(X) \neq \mathbb{Z}$, we need to observe that a key point is a good definition of the integer α. We are able to prove, by using a boundedness argument, that α exists when $\mathrm{Pic}(X) \neq \mathbb{Z}$ but $\mathrm{Num}(X) \cong \mathbb{Z}$. In this event the correspondence between rank two vector bundles and two-codimensional subschemes can be proved to hold. In order to obtain non-vanishing results that are similar to the results proved when $\mathrm{Pic}(X) \cong \mathbb{Z}$, we need also use the Kodaira vanishing theorem, which holds in characteristic 0. We can extend the results to characteristic $p > 0$ if we assume a Kodaira-type vanishing condition.

In this paper we investigate non-vanishing theorems for rank two vector bundles on *any* threefold. The problem looks quite different if the threefold is general of belongs to some family (for the case of ACM bundles see for instance [14] and [1]).

Moreover we observe that our examples of section 6 are sharp but the threefolds (except one) are quadric hypersurfaces, so that one can guess that some stronger statement holds when the degree d is large enough.

2. Notation

We work over an algebraically closed field $\mathbf{k}$ of any characteristic.

Let X be a non-singular irreducible projective algebraic variety of dimension 3, for short a smooth threefold. We fix an ample divisor H on X, so we consider the polarized threefold (X, H). We denote with $\mathcal{O}_X(n)$, instead of $\mathcal{O}_X(nH)$, the invertible sheaf corresponding to the divisor nH, for each $n \in \mathbb{Z}$.

For every cycle Z on X of codimension i it is defined its degree with respect to H, i.e. $\deg(Z; H) := Z \cdot H^{3-i}$, having identified a codimension 3 cycle on X, i.e. a 0-dimensional cycle, with its degree, which is an integer.

From now on (with the exception of section 7) we consider a smooth polarized threefold $(X, \mathcal{O}_X(1)) = (X, H)$ that satifies the following conditions:

(**C1**) $\mathrm{Pic}(X) \cong \mathbb{Z}$ generated by $[H]$,

(**C2**) $H^1(X, \mathcal{O}_X(n)) = 0$ for every $n \in \mathbb{Z}$,

(**C3**) $H^0(X, \mathcal{O}_X(1)) \neq 0$.

By condition (**C1**) every divisor on X is linearly equivalent to aH for some integer $a \in \mathbb{Z}$, i.e. every invertible sheaf on X is (up to an isomorphism) of type $\mathcal{O}_X(a)$ for some $a \in \mathbb{Z}$, in particular we have for the canonical divisor $K_X \sim \epsilon H$, or equivalently $\omega_X \simeq \mathcal{O}_X(\epsilon)$, for a suitable integer ϵ. Furthermore, by Serre duality condition (**C2**) implies that $H^2(X, \mathcal{O}_X(n)) = 0$ for all $n \in \mathbb{Z}$.

Since by assumption $A^1(X) = \mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$ through the map $[H] \mapsto 1$, where $[H] = c_1(\mathcal{O}_X(1))$, we identify the first Chern class $c_1(\mathcal{F})$ of a coherent sheaf with a whole number c_1, where $c_1(\mathcal{F}) = c_1 H$.

The second Chern class $c_2(\mathcal{F})$ gives the integer $c_2 = c_2(\mathcal{F}) \cdot H$ and we will call this integer the second Chern number or the second Chern class of $\mathcal{F}$.

We set

$$d := \deg(X; H) = H^3,$$

so d is the "degree" of the threefold X with respect to the ample divisor H.

Let $c_1(X)$ and $c_2(X)$ be the first and second Chern classes of X, that is of its tangent bundle TX (which is a locally free sheaf of rank 3); then we have

$$c_1(X) = [-K_X] = -\epsilon[H],$$

so we identify the first Chern class of X with the integer $-\epsilon$. Moreover we set

$$\tau := \deg(c_2(X); H) = c_2(X) \cdot H,$$

i.e. τ is the degree of the second Chern class of the threefold X.

In the following we will call the triple of integers (d, ϵ, τ) the *characteristic numbers* of the polarized threefold $(X, \mathcal{O}_X(1))$.

We recall the well-known Riemann-Roch formula on the threefold X (e.g. see [18], Proposition 4).

Theorem 2.1 (Riemann-Roch). *Let $\mathcal{F}$ be a rank r coherent sheaf on X with Chern classes $c_1(\mathcal{F})$, $c_2(\mathcal{F})$ and $c_3(\mathcal{F})$. Then the Euler-Poincaré characteristic of $\mathcal{F}$ is*

$$\begin{aligned}\chi(\mathcal{F}) =& \frac{1}{6}\Big(c_1(\mathcal{F})^3 - 3c_1(\mathcal{F}) \cdot c_2(\mathcal{F}) + 3c_3(\mathcal{F})\Big) + \frac{1}{4}\Big(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})\Big) \cdot c_1(X) \\ &+ \frac{1}{12}c_1(\mathcal{F}) \cdot \Big(c_1(X)^2 + c_2(X)\Big) + \frac{r}{24}c_1(X) \cdot c_2(X)\end{aligned}$$

where $c_1(X)$ and $c_2(X)$ are the Chern classes of X, that is the Chern classes of the tangent bundle TX of X.

So applying the Riemann-Roch Theorem to the invertible sheaf $\mathcal{O}_X(n)$, for each $n \in \mathbb{Z}$, we get the Hilbert polynomial of the sheaf $\mathcal{O}_X(1)$

$$\chi(\mathcal{O}_X(n)) = \frac{d}{6}\left(n - \frac{\epsilon}{2}\right)\left[\left(n - \frac{\epsilon}{2}\right)^2 + \frac{\tau}{2d} - \frac{\epsilon^2}{4}\right]. \tag{1}$$

Let $\mathcal{E}$ be a rank 2 vector bundle on the threefold X with Chern classes $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$, i.e. with Chern numbers c_1 and c_2. We assume that $\mathcal{E}$ is normalized, i.e. that $c_1 \in \{0, -1\}$. It is defined the integer α, the so called *first relevant* *level*, such that $h^0(\mathcal{E}(\alpha)) \neq 0, h^0(\mathcal{E}(\alpha-1)) = 0$. If $\alpha > 0$, $\mathcal{E}$ is called stable, non-stable otherwise. We set

$$\vartheta = \frac{3c_2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}, \qquad \zeta_0 = \frac{\epsilon - c_1}{2}, \quad \text{and} \quad w_0 = [\zeta_0] + 1,$$

where $[\zeta_0]$ = integer part of ζ_0, so the Hilbert polynomial of $\mathcal{E}$ can be written as

$$\chi(\mathcal{E}(n)) = \frac{d}{3}\big(n - \zeta_0\big)\Big[\big(n - \zeta_0\big)^2 - \vartheta\Big]. \tag{2}$$

If $\vartheta \geq 0$ we set

$$\zeta = \zeta_0 + \sqrt{\vartheta}$$

so in this case the Hilbert polynomial of $\mathcal{E}$ has the three real roots $\zeta' \leq \zeta_0 \leq \zeta$ where $\zeta' = \zeta_0 - \sqrt{\vartheta}$. We also define $\bar{\alpha} = [\zeta] + 1$.

The polinomial $\chi(\mathcal{E}(n))$, as a rational polynomial, has three real roots if and only if $\vartheta \geq 0$, and it has only one real root if and only if $\vartheta < 0$.

If $\mathcal{E}$ is normalized, we set

$$\delta = c_2 + c_1 d\alpha + d\alpha^2.$$

Proposition 2.2. *It holds $\delta = 0$ if and only if $\mathcal{E}$ splits.*

Proof. (see also [17], Lemma 3.13) In fact, if $\mathcal{E} = \mathcal{O}_X(a) \otimes \mathcal{O}_X(-a + c_1)$, for some $a \geq 0$, then a direct computation shows that $\delta = 0$. Conversely, if $\mathcal{E}$ is a non-split bundle, then $\mathcal{E}(\alpha)$ has a non-vanishing section that gives rise to a two-codimensional scheme, whose degree, by [6], Appendix A, 3, C6, is δ, which cannot be 0. □

Unless stated otherwise, we work over the smooth polarized threefold X and $\mathcal{E}$ *is a normalized non-split rank two vector bundle on* X.

3. About the Characteristic Numbers ϵ and τ

In this section we want to recall some essentially known properties of the characteristic numbers of the threefold X (see also [16] for more general statements). We start with the following remark.

Remark 3.1. *For the fixed ample invertible sheaf $\mathcal{O}_X(1)$ we have:*

$$h^0(\mathcal{O}_X(n)) = 0 \text{ for } n < 0, \quad h^0(\mathcal{O}_X) = 1, \quad h^0(\mathcal{O}_X(n)) \neq 0 \text{ for } n > 0,$$

and also $h^0(\mathcal{O}_X(m)) - h^0(\mathcal{O}_X(n)) > 0$ for all $n, m \in \mathbb{Z}$ with $m > n \geq 0$.

Moreover it holds

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^3(\mathcal{O}_X) = 1 - h^0(\mathcal{O}_X(\epsilon)),$$

so we have:

$$\chi(\mathcal{O}_X) = 1 \iff \epsilon < 0, \quad \chi(\mathcal{O}_X) = 0 \iff \epsilon = 0, \quad \chi(\mathcal{O}_X) < 0 \iff \epsilon > 0.$$

Proposition 3.2. *Let $(X, \mathcal{O}_X(1))$ be a smooth polarized threefold with characteristic numbers (d, ϵ, τ). Then it holds:*

1) $\epsilon \geq -4$,
2) $\epsilon = -4$ *if and only if* $X = \mathbb{P}^3$, *i.e.* $(d, \epsilon, \tau) = (1, -4, 6)$ *and so* $\frac{\tau}{2d} - \frac{\epsilon^2}{4} = -1$,
3) *if* $\epsilon = -3$, *then* $(d, \epsilon, \tau) = (2, -3, 8)$ *and* $\frac{\tau}{2d} - \frac{\epsilon^2}{4} = -\frac{1}{4}$,
4) $\epsilon\tau$ *is a multiple of* 24, *in particular if* $\epsilon < 0$ *then* $\epsilon\tau = -24$ *and moreover the only possibilities for* (ϵ, τ) *are the following:*
$$(\epsilon, \tau) \in \{(-4, 6), (-3, 8), (-2, 12), (-1, 24)\},$$
5) *if* $\epsilon \neq 0$, *then* $\tau > 0$,
6) *if* $\epsilon = 0$, *then* $\tau > -2d$,
7) τ *is always even,*
8) *if* ϵ *is even, then* $\frac{\tau}{2d} - \frac{\epsilon^2}{4} \geq -1$,
9) *if* ϵ *is odd, then* $\frac{\tau}{2d} - \frac{\epsilon^2}{4} \geq -\frac{1}{4}$.

Proof. For statements 1), 2), 3) see [16].

4) Observe that $\chi(\mathcal{O}_X) = -\frac{1}{24}\epsilon\tau$ is an integer, and moreover, if $\epsilon < 0$, then $\chi(\mathcal{O}_X) = 1$. If $\epsilon < 0$, then by 1) we have $\epsilon \in \{-4, -3, -2, -1\}$ and so we obtain the thesis.
5) By Remark 3.1 we have: if $\epsilon > 0$ then $-\frac{1}{24}\epsilon\tau < 0$, while if $\epsilon < 0$ then $-\frac{1}{24}\epsilon\tau > 0$. In both cases we deduce $\tau > 0$.
6) If $\epsilon = 0$, then we have
$$\chi(\mathcal{O}_X(n)) = \frac{d}{6}n\left(n^2 + \frac{\tau}{2d}\right),$$
and also
$$\chi(\mathcal{O}_X(n)) = h^0(\mathcal{O}_X(n)) > 0 \quad \forall n > 0,$$
therefore we must have $\frac{2d+\tau}{12} > 0$, so $\tau > -2d$.
7) Assume that ϵ is even, then we have
$$d\left(1 - \frac{\epsilon}{2}\right)\left(1 + \frac{\epsilon}{2}\right) + \frac{\tau}{2} = d\left(1 - \frac{\epsilon^2}{4} + \frac{\tau}{2d}\right) = 6\,\chi\left(\mathcal{O}_X\left(\frac{\epsilon}{2} + 1\right)\right) \in \mathbb{Z}$$
and moreover $d\left(1 - \frac{\epsilon}{2}\right)\left(1 + \frac{\epsilon}{2}\right) \in \mathbb{Z}$, so τ must be even.
If ϵ is odd, the proof is quite similar.

8) Let ϵ be even. If it holds

$$h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2}+1\right)\right)-h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2}-1\right)\right)=\chi\left(\mathcal{O}_X\left(\frac{\epsilon}{2}+1\right)\right)<0,$$

then we must have $h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2}-1\right)\right)\neq 0$, which implies

$$h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2}+1\right)\right)-h^0\left(\mathcal{O}_X\left(\frac{\epsilon}{2}-1\right)\right)\geq 0,$$

a contradiction. So we must have

$$\chi\left(\mathcal{O}_X\left(\frac{\epsilon}{2}+1\right)\right)=\frac{d}{6}\left(1+\frac{\tau}{2d}-\frac{\epsilon^2}{4}\right)\geq 0,$$

therefore

$$\frac{\tau}{2d}-\frac{\epsilon^2}{4}\geq -1.$$

9) The proof is quite similar to the proof of 8). □

4. Non-stable Vector Bundles ($\alpha \leq 0$)

We make the following assumption:

$\mathcal{E}$ is a normalized non-split rank two vector bundle with $\alpha \leq 0$.

Lemma 4.1. *For every integer n it holds:*

$$\chi(\mathcal{O}_X(n-\alpha))-\chi(\mathcal{O}_X(\epsilon-n-\alpha-c_1))-\chi(\mathcal{E}(n))=(n-\zeta_0)\delta.$$

Proof. It is a straightforward computation using formulas (1) and (2) for the Hilbert polynomial of $\mathcal{O}_X(1)$ and $\mathcal{E}$, respectively. □

Proposition 4.2. *Assume that $\zeta_0 < -\alpha - c_1 - 1$. Then it holds:*

$$h^1(\mathcal{E}(n))-h^2(\mathcal{E}(n))=(n-\zeta_0)\delta$$

for every integer n such that $\zeta_0 < n \leq -\alpha - c_1 - 1$.

Proof. For each n such that $\zeta_0 < n \leq -\alpha - c_1 - 1$ it holds: $\epsilon - n + \alpha < -1$ and $n + \alpha + c_1 \leq -1$, so we have

$$h^3(\mathcal{O}_X(n-\alpha))=h^0(\mathcal{O}_X(\epsilon-n+\alpha))=0$$
$$h^3(\mathcal{O}_X(\epsilon-n-\alpha-c_1))=h^0(\mathcal{O}_X(n+\alpha+c_1))=0,$$

therefore we obtain:

$$\begin{aligned} h^0(\mathcal{E}(n)) &= h^0(\mathcal{O}_X(n-\alpha)) = \chi(\mathcal{O}_X(n-\alpha)) \\ h^3(\mathcal{E}(n)) &= h^0(\mathcal{E}(\epsilon-n-c_1)) = h^0(\mathcal{O}_X(\epsilon-n-\alpha-c_1)) \\ &= \chi(\mathcal{O}_X(\epsilon-n-\alpha-c_1)). \end{aligned}$$

Hence

$$\begin{aligned} h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) &= h^0(\mathcal{E}(n)) - h^3(\mathcal{E}(n)) - \chi(\mathcal{E}(n)) = \\ &= \chi(\mathcal{O}_X(n-\alpha)) - \chi(\mathcal{O}_X(\epsilon-n-\alpha-c_1)) - \chi(\mathcal{E}(n)), \end{aligned}$$

so using Lemma 4.1 we obtain tha claim. □

THEOREM 4.3. *Let us assume that $\zeta_0 < -\alpha - c_1 - 1$ and let n be such that $\zeta_0 < n \leq -\alpha - 1 - c_1$. Then $h^1(\mathcal{E}(n)) \geq (n-\zeta_0)\delta$. In particular $h^1(\mathcal{E}(n)) \neq 0$.*

Proof. It is enough to observe that $h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) = (n-\zeta_0)\delta$, by Proposition 4.2, and that the right side of this equality is strictly positive for a non-split vector bundle. □

REMARK 4.4. *Observe that the above theorem describes a non-empty set of integers if and only if $-\alpha - c_1 - 1 > \zeta_0$; this means $\alpha < -\frac{\epsilon+2+c_1}{2}$, i.e. $\alpha \leq -\frac{\epsilon+3+c_1}{2}$. So our assumption on α agrees with the bound of [11].*
Observe that the inequality on α implies that $\alpha \leq -2$ if $\epsilon \geq 1$.

The non-vanishing result above can be improved, if other invariants both of the threefold and the bundle are considered.

Now we set $\lambda = \frac{\tau}{2d} - \frac{\epsilon^2}{4}$ and consider the following degree 3 polynomial:

$$F(X) = X^3 + \left(\lambda - \frac{6\delta}{d}\right) X + \frac{6\delta}{d}\left(\alpha + \frac{c_1}{2}\right).$$

It is easy to see that, if $\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} \leq 0$, then $F(X)$ is strictly increasing and so it has only one real root X_0.

THEOREM 4.5. *Assume that $\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} \leq 0$. Let n be such that $\epsilon - \alpha - c_1 + 1 \leq n < -\alpha + X_0 + \zeta_0$, where X_0 = unique real root of $F(X)$. Then $h^1(\mathcal{E}(n)) \geq -\frac{d}{6}F\left(n + \alpha - \zeta_0 + \frac{c_1}{2}\right) > -\frac{d}{6}F(X_0) = 0$. In particular $h^1(\mathcal{E}(n)) \neq 0$.*

Proof. For each n such that $\epsilon - \alpha - c_1 + 1 \leq n < -\alpha + X_0 + \zeta_0$ it holds: $\epsilon - n + \alpha \leq -1$ and $\epsilon - n - c_1 \leq \alpha - 1$, so we have

$$\begin{aligned} h^3(\mathcal{O}_X(n-\alpha)) &= h^0(\mathcal{O}_X(\epsilon-n+\alpha)) = 0 \\ h^3(\mathcal{E}(n)) &= h^0(\mathcal{E}(\epsilon-n-c_1)) = 0. \end{aligned}$$

Moreover, taking into account the exact sequence

$$0 \to \mathcal{O}_X(n-\alpha) \to \mathcal{E}(n) \to \mathcal{I}_Z(n+\alpha) \to 0$$

which arises from the Serre correspondence (see [18], Theorem 4), and where Z is the zero-locus of a non-zero section of $\mathcal{E}(\alpha)$, we obtain:

$$h^0(\mathcal{E}(n)) \geq h^0(\mathcal{O}_X(n-\alpha)) = \chi(\mathcal{O}_X(n-\alpha)).$$

Hence

$$\begin{aligned} h^1(\mathcal{E}(n)) &= h^0(\mathcal{E}(n)) + h^2(\mathcal{E}(n)) - h^3(\mathcal{E}(n)) - \chi(\mathcal{E}(n)) \\ &\geq \chi(\mathcal{O}_X(n-\alpha)) - \chi(\mathcal{E}(n)) = \text{ (by Lemma 4.1)} \\ &= (n-\zeta_0)\delta + \chi(\mathcal{O}_X(\epsilon - n - \alpha - c_1)) \\ &= (n-\zeta_0)\delta - \frac{d}{6}\left(n+\alpha-\zeta_0+\frac{c_1}{2}\right)\left[\left(n+\alpha-\zeta_0+\frac{c_1}{2}\right)^2+\lambda\right], \end{aligned}$$

so, if we put $X = n+\alpha-\zeta_0+\frac{c_1}{2}$, then we obtain: $h^1(\mathcal{E}(n)) \geq -\frac{d}{6}F(X) > -\frac{d}{6}F(X_0) = 0$, because of the hypothesis $n < -\alpha + X_0 + \zeta_0$ and the fact that F is strictly increasing. □

The proofs of the above theorems work perfectly without any restriction on ϵ, while for the proof of the following theorem a few more words are required if $\epsilon \leq 0$.

THEOREM 4.6. *Assume that* $\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4} \geq 0$. *Let* $n > \zeta_0$ *be such that* $\epsilon - \alpha - c_1 + 1 \leq n < \zeta_0 + \sqrt{\frac{6\delta}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}}$ *and put*

$$S(n) = \frac{d}{6}\left(n - \frac{\epsilon - c_1}{2}\right)\left[\left(n - \frac{\epsilon - c_1}{2}\right)^2 - 6\frac{c_2 + d\alpha^2 + c_1 d\alpha}{d} + \frac{\tau}{2d} - \frac{\epsilon^2}{4} + \frac{3c_1^2}{4}\right].$$

Then $h^1(\mathcal{E}(n)) \geq -S(n) > 0$. *In particular* $h^1(\mathcal{E}(n)) \neq 0$.

Proof. **Case 1: $\epsilon \geq 1$.** Assume $c_1 = 0$. Under our hypothesis $h^0(\mathcal{E}(\epsilon - n)) = 0$ and so $h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) \geq h^0(\mathcal{O}_X(n-\alpha)) - \chi(\mathcal{E}(n))$. Observe that $h^0(\mathcal{O}_X(n-\alpha)) - \chi(\mathcal{E}(n)) + S(n) = -\frac{1}{2}nd\alpha(-\epsilon+n+\alpha)) - \frac{1}{12}d\alpha(-3\epsilon\alpha + 2\alpha^2 + \epsilon^2 + \frac{\tau}{d}) \geq 0$ (by direct computation). Therefore we have: $h^1(\mathcal{E}(n)) \geq h^2(\mathcal{E}(n)) - S(n)$. Hence $h^1(\mathcal{E}(n))$ may possibly vanish when

$$\left(n - \frac{\epsilon}{2}\right)^2 - 6\frac{c_2 + d\alpha^2}{d} + \frac{\tau}{2d} - \frac{\epsilon^2}{4} \geq 0.$$

When $S(n) < 0$, so $-S(n) > 0$, $h^1(\mathcal{E}(n)) \geq -S(n) > 0$ and in particular it cannot vanish.

If $c_1 = -1$ the proof is quite similar.

Case 2: $\epsilon \leq 0$.

A. $\epsilon \leq -2$.

We need to know that

$$\frac{1}{2}nd\alpha(-\epsilon + n + \alpha) + \frac{1}{12}d\alpha(\epsilon^2 + \frac{\tau}{d} - 3\epsilon\alpha + 2\alpha^2) \leq 0.$$

The first term of the sum is for sure negative; as for

$$\frac{1}{12}d\alpha\left(\epsilon^2 + \frac{\tau}{d}\right) + \frac{1}{12}d\alpha^2(-3\epsilon + 2\alpha)$$

we observe that the quantity in brackets has discriminant

$$\Delta = \epsilon^2 - 8\frac{\tau}{d} = 4\left(\frac{\epsilon^2}{4} - \frac{\tau}{2d} + \frac{\tau}{2d} - 8\frac{\tau}{d}\right) \leq 4(1 - 15) < 0.$$

Therefore it is positive for all $\alpha \leq 0$ and the product is negative.

B. $\epsilon = -1$.

We need to know that

$$\frac{1}{2}nd\alpha(1 + n + \alpha) + \frac{1}{12}d\alpha\left(1 + \frac{\tau}{d}\right) + \frac{1}{12}d\alpha^2(3 + 2\alpha) \leq 0.$$

If $\alpha \leq -2$, then it is enough to observe that $\frac{\tau}{d} + 3\alpha + 2\alpha^2 \geq 0$. If $\alpha = -1$ we have to consider $-\frac{1}{2}n^2d + \frac{1}{12}d\frac{\tau}{d}$ and then we observe that $6n^2 + \frac{\tau}{d} > 0$. If $\alpha = 0$ obviously the quantity is 0.

C. $\epsilon = 0$.

In theorem 4.5 we need to know that

$$\frac{1}{2}nd\alpha(n + \alpha) + \frac{1}{12}d\alpha\left(\frac{\tau}{d}\right) + \frac{1}{12}d\alpha^2(2\alpha) \leq 0.$$

It is enough to observe that $n + \alpha \geq 1$ and that $2\alpha^2 + \frac{\tau}{d} > 0$ (by Proposition 3.2(6)), if $\alpha < 0$; otherwise we have a 0 quantity. □

Remark 4.7. *Observe that in Theorems 4.5 and 4.6 α can be zero.*

Remark 4.8. *Observe that the case $\alpha = 0$ in Theorem 4.3 can occur only if $\epsilon \leq -c_1 - 3$.*

Remark 4.9. *In theorem 4.6 we do not use the hypothesis* $-\frac{\epsilon+3}{2} \geq \alpha$*, but we assume that* $6\frac{c_2+d\alpha^2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - 1 \geq 0$*. In theorem 4.5 we do not use the hypothesis* $-\frac{\epsilon+3}{2} \geq \alpha$*, but we assume that* $6\frac{c_2+d\alpha^2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} < 0$*. Moreover in both theorems there is a range for* n*, the left endpoint being* $\epsilon - \alpha - c_1 + 1$ *and the right endpoint being either* $\zeta_0 + \sqrt{6\frac{c_2+d\alpha^2}{d} - \frac{\tau}{2d} + \frac{\epsilon^2}{4} - 1}$ *(4.6) or* $\zeta_0 - \alpha + X_0$ *(4.5).*

In [11] there are examples of ACM non-split vector bundles on smooth threefolds in $\mathbb{P}^4$*, with* $-\frac{\epsilon+3+c_1}{2} < \alpha < \frac{\epsilon+5-c_1}{2}$*. We want to emphasize that our theorems do not conflict with the examples of [11]: if* C *is any curve described in [11] and lying on a smooth threefold of degree* d*, then our numerical constraints cannot be satisfied (we have checked it directly in many but not all cases).*

Remark 4.10. *Let us consider a smooth degree* d *threefold* $X \subset \mathbb{P}^4$*.*

We have:

$$\epsilon = d - 5, \quad \tau = d(10 - 5d + d^2), \quad \vartheta = \frac{3c_2}{d} - \frac{d^2 - 5 + 3c_1^2}{4}$$

(see [18]). As to the characteristic function of $\mathcal{O}_X$ *and* $\mathcal{E}$*, it holds:*

$$\chi(\mathcal{O}_X(n)) = \frac{d}{6}\left(n - \frac{d-5}{2}\right)\left[\left(n - \frac{d-5}{2}\right)^2 + \frac{d^2-5}{4}\right],$$

$$\chi(\mathcal{E}(n)) = \frac{d}{3}\left(n - \frac{d-5-c_1}{2}\right)\left[\left(n - \frac{d-5-c_1}{2}\right)^2 + \frac{d^2}{4} - \frac{5}{4} + \frac{3c_1^2}{4} - \frac{3c_2}{d}\right].$$

Then it is easy to see that the hypothesis of Theorem 4.6, i.e. $6\frac{\delta}{d} - \frac{d^2-5+3c_1^2}{4} \geq 0$ *is for sure fulfilled if* $c_2 \geq 0, \alpha \leq -\frac{d-2+c_1}{2}$*. In fact we have (for the sake of simplicity when* $c_1 = 0$*):* $-6\frac{6c_2+d\alpha^2}{d} + \frac{d^2-5}{4} \leq \frac{d^2-5}{4} - 6\frac{d^2-2d+1}{4} = -\frac{5d^2-12d+11}{4} < 0$*.*

Remark 4.11. *Condition* (**C2**) *holds for sure if* X *is a smooth hypersurface of* $\mathbb{P}^4$*. In general, for a characteristic* 0 *base field, only the Kodaira vanishing holds ([6], remark 7.15) and so, unless we work over a threefold* X *having some stronger vanishing, we need assume, in Theorems 4.3, 4.5, 4.6 that* $n - \alpha \notin \{0, \ldots, \epsilon\}$ *(which implies, by duality, that also* $\epsilon - n + \alpha \notin \{0, \ldots, \epsilon\}$*).*

Observe that the first assumption ($n - \alpha \notin \{0, \ldots, \epsilon\}$) *in the case of Theorem 4.3 is automatically fulfilled because of the hypothesis* $\zeta_0 < -\alpha - c_1 - 1$*, and in Theorems 4.5 and 4.6 because of the hypothesis* $\epsilon - \alpha - c_1 + 1 \leq n$*. In fact* $n - \alpha$ *is greater than* ϵ*. But this implies that* $\epsilon - n + \alpha < 0$ *and so also the second condition is fulfilled, at least when* $\epsilon \geq 0$*. For the case* $\epsilon < 0$ *in positive characteristic see [16].*

Observe that, if $\epsilon < 0$*, Kodaira, and so* (**C2**)*, holds for every* n*.*

For a general discussion, also in characteristic $p > 0$*, of this question, see section 7, Remark 7.8.*

Remark 4.12. *In the above theorems we assume that $\mathcal{E}$ is a non-split bundle. If $\mathcal{E}$ splits, then (see section 2) $\delta = 0$. In Theorem 4.3 this implies $h^1(\mathcal{E}(n)) - h^2(\mathcal{E}(n)) = 0$ and so nothing can be said on the non-vanishing.*

Let us now consider Theorem 4.6. If $\delta = 0$, then we must have: $\zeta_0 < n < \zeta_0 + \sqrt{-\frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}} \le \zeta_0 + 1$ (the last inequality depending on Proposition 3.2(8) and (9)). As a consequence ζ_0 cannot be a whole number. Moreover, since we have $2\zeta_0 - \alpha + 1 \le n < \zeta_0 + \sqrt{-\frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}}$, we obtain that $\zeta_0 < \alpha \le 0$, hence $\epsilon - c_1 \le -1$. If $c_1 = 0$, $\epsilon \in \{-1, -3\}$. If $\epsilon = -3$, then n must satisfy the following inequalities: $-\frac{3}{2} < n < -1$ (see Proposition 3.2(8)), which is a contradiction. If $\epsilon = -1$, then, by Proposition 3.2(8), we have $-1 + \alpha + 1 < -\frac{1}{2} + \frac{1}{2} = 0$, which implies $\alpha > 0$, a contradiction. If $c_1 = -1$, then $\epsilon \in \{-2, -4\}$. If $\epsilon = -4$, we have $\sqrt{-\frac{\tau}{2d} + \frac{\epsilon^2}{4} - \frac{3c_1^2}{4}} = \frac{1}{2}$, and so we must have: $-\frac{3}{2} < n < -1$, which is impossible. If $\epsilon = -2$, then $\zeta_0 = -\frac{1}{2}$ and so $-2 - \alpha + 2 < -\frac{1}{2} + \sqrt{1 - \frac{3}{4}}$, which implies $-\alpha < 0$, hence $\alpha > 0$, a contradiction with the non-stability of $\mathcal{E}$.

Then we consider Theorem 4.5. The vanishing of δ on the one hand implies $\lambda > 0$ and $X_0 = 0$. But on the other hand from our hypothesis on the range of n we see that $\zeta_0 \le -2$, hence $\epsilon = -4, c_1 = 0$. But this contradicts Proposition 3.2(2).

5. Stable Vector Bundles

We start with the following lemma which holds both in the stable and in the non-stable case but is useful only in the present section.

Lemma 5.1. *If $h^1(\mathcal{E}(m)) = 0$ for some integer $m \le \alpha - 2$, then $h^1(\mathcal{E}(n)) = 0$ for all $n \le m$.*

Proof. First of all observe that, by our condition (**C3**), from the restriction exact sequence we can obtain in cohomology the exact sequence

$$0 \to H^0(\mathcal{E}(m)) \to H^0(\mathcal{E}(m+1)) \to H^0(\mathcal{E}_H(m+1)) \to 0.$$

Since $m+1 \le \alpha - 1$ we obtain that $h^0(\mathcal{E}_H(m+1)) = 0$, and so $h^0(\mathcal{E}_H(t)) = 0$ for every $t \le m+1$. This implies that $h^1(\mathcal{E}(t-1)) \le h^1(\mathcal{E}(t))$ for each $t \le m+1$, and so we prove the claim. (Our proof is quite similar to the one given in [17] for $\mathbb{P}^3$, where condition (**C3**) is automatically fulfilled). □

In the present section we assume that $\alpha \ge \frac{\epsilon - c_1 + 5}{2}$, or equivalently that $c_1 + 2\alpha \ge \epsilon + 5$. This means that $\alpha \ge 1$ in any event, so $\mathcal{E}$ is stable.

Theorem 5.2. *Let $\mathcal{E}$ be a rank 2 vector bundle on the threefold X with first relevant level α. If $\alpha \ge \frac{\epsilon + 5 - c_1}{2}$, then $h^1(\mathcal{E}(n)) \ne 0$ for $w_0 \le n \le \alpha - 2$.*

Proof. By the hypothesis it holds $w_0 \leq \alpha - 2$, so we have $h^0(\mathcal{E}(n)) = 0$ for all $n \leq w_0 + 1$. Assume $h^1(\mathcal{E}(w_0)) = 0$, then by Lemma 5.1 it holds $h^1(\mathcal{E}(n)) = 0$ for every $n \leq w_0$. Therefore we have

$$\chi(\mathcal{E}(w_0)) = h^0(\mathcal{E}(w_0)) + h^1(\mathcal{E}(-w_0 + \epsilon - c_1)) - h^0(\mathcal{E}(-w_0 + \epsilon - c_1)) = 0.$$

Now observe that the characteristic function has at most three real roots, that are symmetric with respect to ζ_0. Therefore, if w_0 is a root, then $w_0 = \zeta_0 + \sqrt{\vartheta}$ and the other roots are ζ_0 and $\zeta_0 - \sqrt{\vartheta}$. This implies that $\chi(\mathcal{E}(w_0 + 1)) > 0$. On the other hand

$$\chi(\mathcal{E}(w_0 + 1)) = -h^1(\mathcal{E}(w_0 + 1)) \leq 0,$$

a contradiction. So we must have $h^1(\mathcal{E}(w_0)) \neq 0$, then by Lemma 5.1 we obtain the thesis. □

Remark 5.3. *If $\mathcal{E}$ is* ACM, *then $\alpha < \frac{\epsilon+5-c_1}{2}$.*

Theorem 5.4. *Let $\mathcal{E}$ be a normalized rank 2 vector bundle on the threefold X with $\vartheta \geq 0$ and $w_0 < \zeta$. Then the following hold:*

1) *$h^1(\mathcal{E}(n)) \neq 0$ for $\zeta_0 < n < \zeta$, i.e. for $w_0 \leq n \leq \bar{\alpha} - 2$, and also for $n = \bar{\alpha} - 1$ if $\zeta \notin \mathbb{Z}$.*
2) *If $\zeta \in \mathbb{Z}$ and $\alpha < \bar{\alpha}$, then $h^1(\mathcal{E}(\bar{\alpha} - 1)) \neq 0$.*

Proof.

1) The Hilbert polynomial of the bundle $\mathcal{E}$ is strictly negative for each integer such that $w_0 \leq n < \zeta$, but for such an integer n we have $h^2(\mathcal{E}(n)) \geq 0$ and $h^0(\mathcal{E}(n)) - h^0(\mathcal{E}(-n + \epsilon - c_1)) \geq 0$ since $n \geq -n + \epsilon - c_1$ for every $n \geq w_0$, therefore we must have $h^1(\mathcal{E}(n)) \neq 0$. The other statements hold because $\bar{\alpha}$ is, by definition, the integral part of $\zeta + 1$.
2) If $\zeta \in \mathbb{Z}$, then $\zeta = \bar{\alpha} - 1$, so we have $\chi(\mathcal{E}(\bar{\alpha} - 1)) = \chi(\mathcal{E}(\zeta)) = 0$. Moreover $h^0(\mathcal{E}(\bar{\alpha} - 1)) \neq 0$ since $\alpha < \bar{\alpha}$, therefore $h^0(\mathcal{E}(\bar{\alpha} - 1)) - h^3(\mathcal{E}(\bar{\alpha} - 1)) > 0$, and $h^1(\mathcal{E}(n)) = 0$ implies $h^1(\mathcal{E}(m))$, for all $m \leq n$; hence we must have $h^1(\mathcal{E}(\bar{\alpha} - 1)) \neq 0$ to obtain the vanishing of $\chi(\mathcal{E}(\bar{\alpha} - 1))$. □

Remark 5.5. *Observe that in this section we assume $\alpha \geq \frac{\epsilon-c_1+5}{2}$, in order to have $w_0 \leq \alpha - 2$ and so to have a non-empty range for n in Theorem 5.2.*

Remark 5.6. *Observe that in the stable case we need not assume any vanishing of $h^1(\mathcal{O}_X(n))$.*

Remark 5.7. *Observe that split bundles are excluded in this section because they cannot be stable.*

6. Examples

We need the following

REMARK 6.1. *Let $X \subset \mathbb{P}^4$ be a smooth threefold of degree d and let f be the projection onto $\mathbb{P}^3$ from a general point of $\mathbb{P}^4$ not on X, and consider a normalized rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ which gives rise to the pull-back $\mathcal{F} = f^*(\mathcal{E})$. We want to check that $f_*(\mathcal{O}_X) \cong \oplus_{i=0}^{d-1}\mathcal{O}_{\mathbb{P}^3}(-i)$.*

Since f is flat and $\deg(f) = d$, $f_*(\mathcal{O}_X)$ is a rank d vector bundle. The projection formula and the cohomology of the hypersurface X shows that $f_*(\mathcal{O}_X)$ is ACM. Thus there are integers $a_0 \geq \cdots \geq a_{d-1}$ such that $f_*(\mathcal{O}_X) \cong \oplus_{i=0}^{d-1}\mathcal{O}_{\mathbb{P}^3}(a_i)$. Since $h^0(X, \mathcal{O}_X) = 1$, the projection formula gives $a_0 = 0$ and $a_i < 0$ for all $i > 0$. Since $h^0(X, \mathcal{O}_X(1)) = 5 = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) + h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})$, the projection formula gives $a_1 = -1$ and $a_i \leq -2$ for all $i \geq 2$. Fix an integer $t \leq d-2$ and assume proved $a_i = -i$ for all $i \leq t$ and $a_i < -t$ for all $i > t$. Since $h^0(X, \mathcal{O}_X(t+1)) = \binom{t+5}{4} = \sum_{i=0}^{t}\binom{t+4-i}{3}$, we get $a_{t+1} = -t-1$ and, if $t+1 \leq d-2$, $a_i < -t-1$ for all $i > t+1$. Since $f_*(\mathcal{O}_X) \cong \oplus_{i=0}^{d-1}\mathcal{O}_{\mathbb{P}^3}(-i)$, the projection formula gives the following formula for the first cohomology module:

$$H^i(\mathcal{F}(n)) \cong H^i(\mathcal{E}(n)) \oplus H^i(\mathcal{E}(n-1)) \oplus \cdots \oplus H^i(\mathcal{E}(n-d+1))$$

all i. Observe that, as a consequence of the above equalitiy for $i = 0$, we obtain that $\mathcal{F}$ has the same α as $\mathcal{E}$. Moreover the pull-back $\mathcal{F} = f^*(\mathcal{E})$ and $\mathcal{E}$ have the same Chern class c_1, while $c_2(\mathcal{F}) = d\, c_2(\mathcal{E})$ and therefore $\delta(\mathcal{F}) = d\, \delta(\mathcal{E})$.

Examples:

1. (a stable vector bundle with $c_1 = 0$, $c_2 = 4$ on a quadric hypersurface X).

   Choose $d = 2$ and take the pull-back $\mathcal{F}$ of the stable vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ of [17], example 4.1. Then the numbers of $\mathcal{F}$ (see Notation) are: $c_1 = 0$, $c_2 = 4$, $\alpha = 1$, $\bar{\alpha} = 2$, $\zeta_0 = -\frac{3}{2}$, $w_0 = -1$, $\vartheta = \frac{25}{4}$, $\zeta = -\frac{3}{2} + \sqrt{\frac{25}{4}} = 1 \in \mathbb{Z}$. From [17], example 4.1, we know that $h^1(\mathcal{E}) \neq 0$. Since $H^1(\mathcal{F}(1)) \cong H^1(\mathcal{E}(1)) \oplus H^1(\mathcal{E})$, we have: $h^1(\mathcal{F}(1)) \neq 0$, one shift higher than it is stated in Theorem 5.4(2).

2. (a non-stable vector bundle with $c_1 = 0$, $c_2 = 45$ on a hypersurface of degree 5).

   Choose $d = 5$ and take the pull-back $\mathcal{F}$ of the stable vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ of [17], example 4.5. Then the numbers of $\mathcal{F}$ (see Notation) are: $c_1 = 0$, $c_2 = 45$, $\alpha = -3$, $\delta = 90$, $\zeta_0 = 0$. From [17], Theorem 3.8, we know that $h^1(\mathcal{E}(12)) \neq 0$. Since $H^1(\mathcal{F}(16)) \cong H^1(\mathcal{E}(16)) \oplus \cdots \oplus H^1(\mathcal{E}(12))$, we have: $h^1(\mathcal{F}(16)) \neq 0$ (Theorem 4.5 states that $h^1(\mathcal{F}(10)) \neq 0$.

3. (a stable vector bundle with $c_1 = -1$, $c_2 = 2$ on a quadric hypersurface).

   Let $\mathcal{E}$ be the rank two vector bundle corresponding to the union of two skew lines on a smooth quadric hypersurface $Q \subset \mathbb{P}^4$. Then its numbers are : $c_1 = -1$, $c_2 = 2$, $\alpha = 1$ and it is known that $h^1(\mathcal{E}(n)) \neq 0$ if and only if $n = 0$.

   Observe that in this case $\vartheta = \frac{5}{2} \geq 0$, $\zeta_0 = -1$, $\bar{\alpha} = 1$. Therefore Theorem 5.4 states exactly that $h^1(\mathcal{E}) \neq 0$, hence this example is sharp.

4. (a non-stable vector bundle with $c_1 = 0$, $c_2 = 8$ on a quadric hypersurface).

   Choose $d = 2$ and take the pull-back $\mathcal{F}$ of the non-stable vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ of [17], example 4.10. Then the numbers of $\mathcal{F}$ (see Notation) are: $c_1 = 0$, $c_2 = 8$, $\alpha = 0$, $\zeta_0 = -\frac{3}{2}$, $\delta = 8$. We know (see [17], example 4.10) that $h^1(\mathcal{E}(2)) \neq 0, h^1(\mathcal{E}(3)) = 0$. Since $H^1(\mathcal{F}(3)) \cong H^1(\mathcal{E}(3)) \oplus H^1(\mathcal{E}(2))$, we have: $h^1(\mathcal{F}(3)) \neq 0$, exactly the bound of Theorem 4.6.

REMARK 6.2. *The bounds for a degree d threefold in $\mathbb{P}^4$ agree with [17], where $\mathbb{P}^3$ is considered.*

7. Threefolds with $\mathrm{Pic}(X) \neq \mathbb{Z}$

Let X be a smooth and connected projective threefold defined over an algebraically closed field $\mathbf{k}$. Let $\mathrm{Num}(X)$ denote the quotient of $\mathrm{Pic}(X)$ by numerical equivalence. Numerical classes are denoted by square brackets [ ]. We assume $\mathrm{Num}(X) \cong \mathbb{Z}$ and take the unique isomorphism $\eta \colon \mathrm{Num}(X) \to \mathbb{Z}$ such that 1 is the image of a fixed ample line bundle. Notice that $M \in \mathrm{Pic}(X)$ is ample if and only if $\eta([M]) > 0$.

REMARK 7.1. *Let $\eta \colon \mathrm{Num}(X) \to \mathbb{Z}$ be as before. Notice that every effective divisor on X is ample and hence its η is strictly positive. For any $t \in \mathbb{Z}$ set $\mathrm{Pic}_t(X) := \{L \in \mathrm{Pic}(X) \mid \eta([L]) = t\}$. Hence $\mathrm{Pic}_0(X)$ is the set of all isomorphism classes of numerically trivial line bundles on X. The set $\mathrm{Pic}_0(X)$ is parametrized by a scheme of finite type ([10], Proposition 1.4.37). Hence for each $t \in \mathbb{Z}$ the set $\mathrm{Pic}_t(X)$ is bounded. Let now $\mathcal{E}$ be a rank 2 vector bundle on X. Since $\mathrm{Pic}_1(X)$ is bounded there is a minimal integer t such that there is $B \in \mathrm{Pic}_t(X)$ and $h^0(\mathcal{E} \otimes B) > 0$. Call it $\alpha(\mathcal{E})$ or just α. By the definition of α there is $B \in \mathrm{Pic}_\alpha(X)$ such that $h^0(X, \mathcal{E} \otimes B) > 0$. Hence there is a non-zero map $j \colon B^* \to \mathcal{E}$. Since B^* is a line bundle and $j \neq 0$, j is injective. The definition of α gives the non-existence of a non-zero effective divisor D such that j factors through an inclusion $B^* \to B^*(D)$, because $\eta([D]) > 0$. Thus the inclusion j induces an exact sequence*

$$0 \to B^* \to \mathcal{E} \to \mathcal{I}_Z \otimes B \otimes \det\ (\mathcal{E}) \to 0 \tag{3}$$

in which Z is a closed subscheme of X of pure codimension 2.

Observe that $\eta([B]) = \alpha, \eta([B^*]) = -\alpha, \eta([B \otimes det(\mathcal{E})]) = \alpha + c_1$, *hence the exact sequence is quite similar to the usual exact sequence that holds true in the case* $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Notation. We set $\epsilon := \eta([\omega_X])$, $\alpha := \alpha\ (\mathcal{E})$ and $c_1 := \eta([\det\ (\mathcal{E})])$. So we can speak of a normalized vector bundle $\mathcal{E}$, with $c_1 \in \{0, -1\}$. Moreover we say that $\mathcal{E}$ is stable if $\alpha > 0$, non-stable if $\alpha \leq 0$. Furthermore ζ_0, ζ, w_0, $\bar{\alpha}$, ϑ are defined as in section 2.

Remark 7.2. *Fix any* $L \in \mathrm{Pic}_1(X)$ *and set:* $d = L^3 =$ *degree of* X. *The degree* d *does not depend on the numerical equivalence class. In fact, if* R *is numerically equivalent to* 0, *then* $(L+R)^3 = L^3 + R^3 + 3L^2R + 3LR^2 = L^3 + 0 + 0 + 0 = L^3$. *Then it is easy to see that the formulas for* $\chi(\mathcal{O}_X(n))$ *and* $\chi(\mathcal{E}(n))$ *given in section 2 still hold if we consider* $\mathcal{O}_X \otimes L^{\otimes n}$ *and* $\mathcal{E} \otimes L^{\otimes n}$ *(see [18]).*

Remark 7.3.

(a) Assume the existence of $L \in \mathrm{Pic}(X)$ *such that* $\eta([L]) = 1$ *and* $h^0(X, L) > 0$. *Then for every integer* $t > \alpha$ *there is* $M \in \mathrm{Pic}(X)$ *such that* $\eta([M]) = t$ *and* $h^0(X, \mathcal{E} \otimes M) > 0$.

(b) Assume $h^0(X, L) > 0$ *for every* $L \in \mathrm{Pic}(X)$ *such that* $\eta([L]) = 1$. *Then* $h^0(X, \mathcal{E} \otimes M) > 0$ *for every* $M \in \mathrm{Pic}(X)$ *such that* $\eta([M]) > \alpha$.

Proposition 7.4. *Let* $\mathcal{E}$ *be a normalized rank two vector bundle and assume the existence of a spanned* $R \in \mathrm{Pic}(X)$ *such that* $\eta([R]) = 1$. *If* $\mathrm{char}(\mathbf{k}) > 0$, *assume that* $|R|$ *induces an embedding of* X *outside finitely many points. Assume*

$$2\alpha \leq -\epsilon - 3 - c_1 \tag{4}$$

and $h^1(X, \mathcal{E} \otimes N) = 0$ *for every* $N \in \mathrm{Pic}(X)$ *such that* $\eta([N]) \in \{-\alpha - c_1 - 1, \alpha + 2 + e\}$. *If* $h^1(X, B) = 0$ *for every* $B \in \mathrm{Pic}(X)$ *such that* $\eta([B]) = -2\alpha - c_1$, *then* $\mathcal{E}$ *splits.*

If moreover $h^1(X, M) = 0$ *for every* $M \in \mathrm{Pic}(X)$ *then it is enough to assume that* $h^1(X, \mathcal{E} \otimes N) = 0$ *for every* $N \in \mathrm{Pic}(X)$ *such that* $\eta([N]) = -\alpha - c_1 - 1$.

Proof. By assumption there is $M \in \mathrm{Pic}(X)$ such that $\eta([M]) = \alpha$ and $h^0(X, \mathcal{E} \otimes M) > 0$. Set $A := M^*$. We have seen in remark 7.1 that $\mathcal{E}$ fits into an extension of the following type:

$$0 \to A \to \mathcal{E} \to \mathcal{I}_C \otimes \det(\mathcal{E}) \otimes A^* \to 0 \tag{5}$$

with C a locally complete intersection closed subscheme of pure dimension 1. Let H be a general element of $|R|$ and T the intersection of H with another general element of $|R|$. Observe that T, under our assumptions, is generically reduced by Bertini's Theorem (see [6], Theorem II, 8.18 and Remark II, 8.18.1). Since R is spanned, T is a locally complete intersection curve and $C \cap T = \emptyset$. Hence $\mathcal{E}|_T$ is an extension of $\det(\mathcal{E}) \otimes A^*|_T$ by $A|_T$. Since T is generically reduced and locally a complete intersection, it is reduced. Hence $h^0(T, M^*) = 0$ for every ample line bundle M on T. Since $\omega_T \cong (\omega_X \otimes R^{\otimes 2})|_T$, we have $\dim(\mathrm{Ext}^1_T(\det(\mathcal{E}) \otimes A^*, A)) = h^0(T, (\det(\mathcal{E}) \otimes (A^*)^{\otimes 2} \otimes \omega_X \otimes R^{\otimes 2})|_T) = 0$ (indeed $\eta([\det(\mathcal{E}) \otimes (A^*)^{\otimes 2} \otimes \omega_X \otimes R^{\otimes 2}]) = 2\alpha + c_1 + e + 2 < 0$). Hence $\mathcal{E}|_T \cong A|_T \oplus (\det(\mathcal{E}) \otimes A^*)|_T$. Let σ be the non-zero section of $(\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^*)|_T$ coming from the projection onto the second factor of the decomposition just given. The vector bundle $\mathcal{E}|_H$ is an extension of $(\det(\mathcal{E}) \otimes A^*)|_H$ by $A|_H$ if and only if $C \cap H = \emptyset$. Since R is ample, $C \cap H = \emptyset$ if and only if $C = \emptyset$. Hence we get simultaneously $C \cap H = \emptyset$ and $\mathcal{E}|_H \cong A|_H \oplus (\det(\mathcal{E}) \otimes A^*)|_H$ if we prove the existence of $\tau \in H^0(H, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^*)|_H)$ such that $\tau|_T = \sigma$. To get τ it is sufficient to have $H^1(H, (E \otimes (A \otimes \det(\mathcal{E})^* \otimes R^*)|_H) = 0$. A standard exact sequence shows that $H^1(H, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^*)|_H) = 0$ if $h^1(X, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^*)) = 0$ and $h^2(X, (\mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^* \otimes R^*)) = 0$. Since $\mathcal{E}^* \cong \mathcal{E} \otimes \det(\mathcal{E})^*$, Serre duality gives $h^2(X, \mathcal{E} \otimes (A \otimes \det(\mathcal{E})^* \otimes R^* \otimes R^*)) = h^1(X, \mathcal{E} \otimes A \otimes R^{\otimes 2} \otimes \omega_X)$. Since $\eta([A \otimes \det(\mathcal{E})^* \otimes R^*]) = -\alpha - c_1 - 1$ and $\eta([A \otimes R^{\otimes 2} \otimes \omega_X]) = \alpha + e + 2$, we get that $C = \emptyset$. The last sentence follows because $\eta([A^{\otimes 2} \otimes \det(\mathcal{E})^*]) = -2\alpha - c_1$. □

Remark 7.5. *Fix integers $t < z \leq \alpha - 2$. Assume the existence of $L \in \mathrm{Pic}(X)$ such that $\eta([L]) = z$ and $h^1(X, \mathcal{E} \otimes L) = 0$. If there is $R \in \mathrm{Pic}(X)$ such that $\eta([R]) = 1$ and $h^0(X, R) > 0$, then there exists $M \in \mathrm{Pic}(X)$ such that $\eta([M]) = t$ and $h^1(X, \mathcal{E} \otimes M) = 0$. If $h^0(X, R) > 0$ for every $R \in \mathrm{Pic}(X)$ such that $\eta([R]) = 1$, then $h^1(X, \mathcal{E} \otimes M) = 0$ for every $M \in \mathrm{Pic}(X)$ such that $\eta([M]) = t$.*

The proof can follow the lines of Lemma 5.1. In fact consider a line bundle R with $\eta([R]) = 1$ and let H be the zero-locus of a non-zero section of R; then we have the following exact sequence:

$$0 \to \mathcal{E} \otimes L \to \mathcal{E} \otimes L \otimes R \to (\mathcal{E} \otimes L \otimes R)|_H \to 0.$$

Now observe that the vanishing of $h^1(X, \mathcal{E} \otimes L)$ implies that $h^0((\mathcal{E} \otimes L \otimes R)|_H) = 0$. And now we can argue as in Lemma 5.1 (see also [17]).

Remark 7.6.

(a) *Assume the existence of $L \in \mathrm{Pic}(X)$ such that $\eta([L]) = 1$ and $h^0(X, L) > 0$. Then for every integer $t > \alpha$ there is $M \in \mathrm{Pic}(X)$ such that $\eta([M]) = t$ and $h^0(X, \mathcal{E} \otimes M) > 0$.*

(b) *Assume $h^0(X, L) > 0$ for every $L \in \mathrm{Pic}(X)$ such that $\eta([L]) = 1$. Then $h^0(X, \mathcal{E} \otimes M) > 0$ for every $M \in \mathrm{Pic}(X)$ such that $\eta([M]) > \alpha$.*

Remark 7.7. *In all our results of sections 4 and 5 we use the vanishing of* $h^1(\mathcal{O}_X(n))$ *for all* n *(and by Serre duality of* $h^2(\mathcal{O}_X(n))$*) (or, at least,* $\forall n \notin \{0, \cdots, \epsilon\}$*), see Remark 4.11.*

From now on we need to use similar vanishing conditions and so we introduce the following condition:

(**C4**) $h^1(X, L) = 0$ *for all* $L \in \mathrm{Pic}(X)$ *such that either* $\eta([L]) < 0$ *or* $\eta([L]) > \epsilon$.

Observe that (**C4**) *is always satisfied in characteristic 0 (by the Kodaira vanishing theorem). In positive characteristic it is often satisfied. This is always the case if* X *is an abelian variety ([12] page 150).*

Observe also that, if $\epsilon \leq -1$*, the Kodaira vanishing and our condition put no restriction on* n *(see also Remark 4.12).*

Example. If (4) holds, then $-2\alpha - c_1 > \epsilon$. Hence we may apply Proposition 7.4 to X. In particular observe that, in the case of an abelian variety with $\mathrm{Num}(X) \cong \mathbb{Z}$ or in the case of a Calabi-Yau threefold with $\mathrm{Num}(X) \cong \mathbb{Z}$, we have $\epsilon = 0$. Notice that Proposition 7.4 also applies to any threefold X whose ω_X has finite order.

With the assumption of condition (**C4**) the proofs of Theorems 4.3, 4.5, 4.6 can be easily modified in order to obtain the statements below ($\mathcal{E}$ is normalized, i.e. $\eta([\det(\mathcal{E})]) \in \{-1, 0\}$), where, by the sake of simplicity, we assume $\epsilon \geq 0$ (if $\epsilon < 0$, (**C4**), which holds by [16], implies that all the vanishing of h^1 and h^2 for all $L \in \mathrm{Pic}(X)$ hold).

Theorem 7.8. *Assume* (**C4**)*,* $\alpha \leq 0$*, the existence of* $R \in \mathrm{Pic}(X)$ *such that* $\eta([R]) = 1$ *and* $\zeta_0 < -\alpha - c_1 - 1$*. Fix an integer* n *such that* $\zeta_0 < n \leq -\alpha - 1 - c_1$*. Fix* $L \in \mathrm{Pic}(X)$ *such that* $\eta([L]) = n$*. Then* $h^1(\mathcal{E} \otimes L) \geq (n - \zeta_0)\delta > 0$.

Remark 7.9. *Observe that we should require the following conditions:* $n - \alpha \notin \{0, \ldots, \epsilon\}, \epsilon - n + \alpha \notin \{0, \ldots, \epsilon\}$*. But they are automatically fulfiled under the assumption that* $\zeta_0 < -\alpha - c_1 - 1$.

Theorem 7.10. *Assume* (**C4**)*,* $\alpha \leq 0$*, the existence of* $R \in \mathrm{Pic}(X)$ *such that* $\eta([R]) = 1$ *and the same hypotheses of Theorem 4.6. Fix* $L \in \mathrm{Pic}(X)$ *such that* $\eta([L]) = n$*. Then* $h^1(\mathcal{E} \otimes L) \geq -S(n) > 0$ *(*$S(n)$ *being defined as in Theorem 4.6).*

Theorem 7.11. *Assumption as in Theorem 4.5. Moreover assume* (**C4**) *and* $n - \alpha \notin \{0, \ldots, \epsilon\}$*. Fix* $L \in \mathrm{Pic}(X)$ *such that* $\eta([L]) = n$*. Then* $h^1(\mathcal{E} \otimes L) \geq -\frac{d}{6}F(n + \alpha - \zeta_0 + \frac{c_1}{2}) > 0$ *(*F *being defined as in Theorem 4.5).*

Remark 7.12. *Observe that in Theorems 7.10 and 7.11 we should require* $n - \alpha \notin \{0, \ldots, \epsilon\}$*, but the assumption* $\epsilon - \alpha - c_1 + 1 \leq n$ *implies that it is automatically fulfilled.*

The proofs of the above theorems are based on the existence of the exact sequence (3) and on the properties of α. They follow the lines of the proofs given in the case $\mathrm{Pic}(X) \cong \mathbb{Z}$. Here and in section 4 we actually need only the Kodaira vanishing (true in characteristic 0 and assumed in characteristic $p > 0$) and no further vanishing of the first cohomology.

Also the stable case can be extended to a smooth threefold with $\mathrm{Num}(X) \cong \mathbb{Z}$. Observe that the proofs can follow the lines of the proofs given in the case $\mathrm{Pic}(X) \cong \mathbb{Z}$ and make use of Remark 7.6 (which extends Theorem 5.1).

More precisely we have:

Theorem 7.13. *Assumptions as in Theorem 5.2 and fix $L \in \mathrm{Pic}(X)$ such that $\eta([L]) = n$. Then, if $\alpha \geq \frac{\epsilon+5-c_1}{2}$, then $h^1(\mathcal{E} \otimes L) \neq 0$ for $w_0 \leq n \leq \alpha - 2$.*

Theorem 7.14. *Assumptions as in Theorem 5.4 and fix $L \in \mathrm{Pic}(X)$ such that $\eta([L]) = n$. Then the following hold:*

1) *$h^1(\mathcal{E} \otimes L) \neq 0$ for $\zeta_0 < n < \zeta$, i.e. for $w_0 \leq n \leq \bar{\alpha} - 2$, and also for $n = \bar{\alpha} - 1$ if $\zeta \notin \mathbb{Z}$.*

2) *If $\zeta \in \mathbb{Z}$ and $\alpha < \bar{\alpha}$, then $h^1(\mathcal{E} \otimes N) \neq 0$, for every N such that $\eta([N]) = \bar{\alpha} - 1$.*

Remark 7.15. *The above theorems can be applied to any X such that* $\mathrm{Num}(X) \cong \mathbb{Z}$*, $\epsilon = 0$ and $h^1(X, L) = 0$ for all $L \in \mathrm{Pic}(X)$ such that $\eta([L]) \neq 0$, for instance to $X =$ an abelian threefold with* $\mathrm{Num}(X) \cong \mathbb{Z}$*.*

Remark 7.16. *If X is any threefold (in characteristic 0 or positive) such that $h^1(X, L) = 0$, for all $L \in \mathrm{Pic}(X)$, then we can avoid the restriction $n - \alpha \notin \{0, \ldots, \epsilon\}$. Not many threefolds, beside any $X \subset \mathbb{P}^4$, fulfill these conditions.*

Remark 7.17. *Observe that in Theorems 7.13 and 7.14 we do not assume* (**C4**) *(see also Remark 5.6).*

Remark 7.18. *Observe that also in the present case* ($\mathrm{Num}(X) \cong \mathbb{Z}$)*, we have: $\delta = 0$ if and only if $\mathcal{E}$ splits. Therefore Remarks 4.12 and 5.7 apply here.*

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