Rend. Istit. Mat. Univ. Trieste
Suppl. 2 Vol. XXXII, 9–15 (2001)

Continuous Order-Preserving Functions on a Preordered Completely Regular Topological Space

Gianni Bosi and Romano Isler (*)

Summary. - *A necessary and sufficient condition is presented for the existence of a real continuous order-preserving function f on a topological preordered space $(X, \tau, \preceq)$, under a resonable continuity assumption concerning the preorder $\preceq$, called "quasi IC-continuity". Under the same continuity hypotheses, a sufficient condition is provided for the existence of a real continuous order-preserving function i n case that (X, τ) is a completely regular space.*

1. Introduction

In recent years there has been an increasing interest in mathematical economics and in mathematical psychology for the existence of continuous order-preserving functions on topological preordered spaces

(*) Authors' addresses: Gianni Bosi, Dipartimento di Matematica Applicata "Bruno de Finetti", Università di Trieste, Piazzale Europa 1, 34127 Trieste, Italy, email: giannibo@econ.univ.trieste.it
Romano Isler, Dipartimento di Matematica Applicata "Bruno de Finetti", Università di Trieste, Piazzale Europa 1, 34127 Trieste, Italy.
AMS (2000) Subject Classifications: 06A06, 54F05
Key Words: Topological preordered space, decreasing scale, order-preserving function.
Thanks are given to an anonymous referee for helpful suggestions for improvement.

(eventually endowed with an algebraic structure). Many impressive contributions in this sense were given by different authors (see, e.g., Candeal and Induráin [3], Herden [5], [6], [7] and Mehta [11], [12]). In this paper, based on resonable continuity hypotheses concerning the preorder $\preceq$, we present a necessary and sufficient condition for the existence of a real continuous order-preserving function f on a *topological preordered space* $(X, \tau, \preceq)$. Under the same continuity hypotheses, a sufficient condition for the existence of a real continuous order-preserving function is provided in case that (X, τ) is a *completely regular space*. Needless to say, the assumption of complete regularity is particularly interesting in decision theory since, for example, it is well known that each topological group is a completely regular space (see e.g. Engelking [4, Example 8.1.17]). Indeed, in the context of mathematical psychology, it is often assumed that a set X is endowed with a binary relation $\preceq$ (which is tipically a *preorder* on X), and a *concatenation operation* $+$, such that the pair $(X, +)$ is a (commutative) group (see e.g. Krantz, Luce, Suppes and Tversky [8]).

2. Definitions

Given a *preorder* $\preceq$ on an arbitrary set X (i.e., a binary relation on X which is *reflexive* and *transitive*), denote by $\prec$ and $\sim$ the *asymmetric part* and respectively the *symmetric part* of $\preceq$. A preorder $\preceq$ on X is *complete* if, for every $x, y \in X$, either $x \preceq y$ or $y \preceq x$.

If $(X, \preceq)$ is a preordered set, and τ is a topology on X, then the triplet $(X, \tau, \preceq)$ will be referred to as a *topological preordered space*. Observe that in the definition of a topological preordered space we have not assumed *a priori* that the preorder $\preceq$ is *continuous* in some sense as regards the given topology τ on X.

A subset A of a set X endowed with a preorder $\preceq$ is said to be *decreasing* (*increasing*) if $[(x \in A) \wedge (y \preceq x) \Rightarrow y \in A]$ $([(x \in A) \wedge (x \preceq y) \Rightarrow y \in A])$.

If A is any subset of a set X endowed with a preorder $\preceq$, then denote by $d(A)$ $(i(A))$ the intersection of all the decreasing (increasing) subsets of X containing A.

Given a topological preordered space $(X, \tau, \preceq)$, we say that $\preceq$ is *(i)* *quasi continuous* if $d(x) = d(\{x\})$ is a closed set for every $x \in X$,

(ii) *quasi I-continuous* if $d(A)$ is an open set for every open subset A of X

(iii) *quasi C-continuous* if $d(A)$ is a closed set for every closed subset A of X,

(iv) *quasi IC-continuous* if $\preceq$ is both quasi I-continuous and quasi C-continuous.

A quasi continuous preorder is usually called a *lower semicontinuous* preorder. Definitions *(ii)* and *(iii)* were introduced by McCartan [10]. The terminology here is a slight modification of the corresponding terminology in Künzi [9].

We say that a topological preordered space $(X, \tau, \preceq)$ is

(i) *Nachbin separable* if there exists a countable family $\{A_n, B_n\}_{n\in\mathbf{N}}$ of pairs of closed subsets of X such that A_n is decreasing, B_n is increasing, $A_n \cap B_n = \emptyset$ for every $n \in \mathbf{N}$, and $\{(x, y) \in X \times X : x \prec y\} \subseteq \bigcup_{n\in\mathbf{N}} A_n \times B_n$,

(ii) *strongly Nachbin separable* if there exists a countable family $\{x_n, B_n\}_{n\in\mathbf{N}}$ of pairs such that, for every $n \in \mathbf{N}, x_n \in X \setminus B_n$, B_n is a closed increasing subset of X, and $\{(x, y) \in X \times X : x \prec y\} \subseteq \bigcup_{n\in\mathbf{N}} d(x_n) \times B_n$.

The definition of Nachbin separability is already found in Mehta [11], who used *Nachbin separation theorem* in *normally preordered topological spaces* (see Nachbin [13]) in order to guarantee the existence of a real continuous order-preserving function on such spaces whenever the condition of Nachbin separability is satisfied (see also Mehta [12]).

Given a topological preordered space $(X, \tau, \preceq)$, it is clear that if $(X, \tau, \preceq)$ is strongly Nachbin separable, (X, τ) is a T_1-space, and $\preceq$ is quasi C-continuous, then $(X, \tau, \preceq)$ is Nachbin separable.

From Burgess and Fitzpatrick [2], given a topological preordered space $(X, \tau, \preceq)$, a family $\mathcal{G} = \{G_r : r \in \mathcal{S}\}$ of open decreasing subsets of X is said to be a *decreasing scale* in $(X, \tau, \preceq)$ if the following conditions are satisfied:

(i) $\mathcal{S}$ is a dense subset of $[0,1]$ such that $1 \in \mathcal{S}$ and $G_1 = X$,

(ii) for every $r_1, r_2 \in \mathcal{S}$ with $r_1 < r_2$, it is $\overline{G}_{r_1} \subseteq G_{r_2}$.

We observe that the definition of a decreasing scale introduced by Burgess and Fitzpatrick [2] is different from the definition of a decreasing scale adopted in measurement theory by Krantz, Luce, Suppes and Tversky [8].

If $(X, \preceq)$ is a preordered set, then a real function f on X is said to be

(i) *increasing* if, for every $x, y \in X$, $[x \preceq y \Rightarrow f(x) \leq f(y)]$,

(ii) *order-preserving* if it is increasing and, for every $x, y \in X$, $[x \prec y \Rightarrow f(x) < f(y)]$.

Finally, we recall that, given a topological space (X, τ), a family $\mathcal{G} = \{G_r : r \in \mathcal{S}\}$ of open subsets of X is said to be a *scale* in (X, τ) if $\mathcal{G}$ is a (decreasing) scale in $(X, \tau, =)$.

3. Existence of Continuous Order-Preserving Functions and Complete Regularity

A full characterization of the existence of a real continuous order-preserving function f on a topological preordered space $(X, \tau, \preceq)$ was provided by Herden [5, Theorem 4.1] (see also Mehta [12]). In the following theorem we present a necessary and sufficient condition for the existence of a real continuous order-preserving function f on a topological preordered space $(X, \tau, \preceq)$ in the case when $\preceq$ is quasi IC-continuous. It should be noted that a decreasing scale $\mathcal{G}$ is a (*linear*) *separable system* in Herden's terminology (see Herden [6]). So theorem 1 below is similar to theorem 4.1 of Herden [5], whose proof is based upon Nachbin's separation theorem (see Nachbin [13]). Nevertheless, the assumption of IC-continuity of the given preorder allows us to provide a more direct construction of a real continuous order-preserving function.

THEOREM 3.1. *Let $(X, \tau, \preceq)$ be a topological preordered space, and assume that $\preceq$ is quasi IC-continuous. Then the following conditions are equivalent:*

(i) There exists a real continuous order-preserving function f on $(X, \tau, \precsim)$ with values in $[0,1]$;

(ii) There exists a countable family $\{\mathcal{G}'_n = \{G'_{nr} : r \in \mathcal{S}_n\}\}_{n \in \mathbf{N}}$ of scales in (X, τ) such that, for every $x, y \in X$ with $x \prec y$, there exists $n \in \mathbf{N}$ such that $(x, y) \in d(G'_{nr}) \times (X \setminus d(G'_{nr}))$ for every $r \in \mathcal{S}_n \setminus \{1\}$.

Proof. (i) $\Rightarrow$ (ii). Assume that there exists a real continuous order-preserving function f on $(X, \tau, \precsim)$ with values in $[0,1]$. For every pair (p_i, q_i) of rational numbers such that $0 \leq p_i < q_i \leq 1$, define $\mathcal{S}_{(p_i,q_i)} \equiv]0,1] \cap \mathcal{Q}$, $G'_{(p_i,q_i)r} = f^{-1}([0, (q_i - p_i)r + p_i[)$ $(r \in]0,1[\cap \mathcal{Q})$, $G'_{(p_i,q_i)1} = X$, and consider the (decreasing) scale $\mathcal{G}'_{(p_i,q_i)} = \{G'_{(p_i,q_i)r} : r \in \mathcal{S}_{(p_i,q_i)}\}$ in $(X, \tau, \precsim)$. Since $\{\mathcal{G}'_{(p_i,q_i)} : p_i, q_i \in [0,1] \cap \mathcal{Q},\ p_i < q_i\}$ is a countable set, define $\{\mathcal{G}'_n\}_{n \in \mathbf{N}} = \{\mathcal{G}'_{(p_i,q_i)} : p_i, q_i \in [0,1] \cap \mathcal{Q},\ p_i < q_i\}$. Then it is easily seen that $\{\mathcal{G}'_n\}_{n \in \mathbf{N}}$ is a countable family of scales in (X, τ) satisfying condition *(ii)*. Indeed, for every $x, y \in X$ such that $x \prec y$, there exists a pair (p_i, q_i) of rational numbers such that $f(x) < p_i < q_i < f(y)$.

(ii) $\Rightarrow$ (i). Let $\{\mathcal{G}'_n = \{G'_{nr} : r \in \mathcal{S}_n\}\}_{n \in \mathbf{N}}$ be a countable family of scales in (X, τ) satisfying condition (ii). Define, for every $n \in \mathbf{N}$, and for every $r \in \mathcal{S}_n$, $G_{nr} = d(G'_{nr})$. Since $\precsim$ is quasi IC-continuous, $\mathcal{G}_n = \{G_{nr} : r \in \mathcal{S}_n\}$ is a decreasing scale in $(X, \tau, \precsim)$ for every $n \in \mathbf{N}$ (see Burgess and Fitzpatrick [2, Lemma 6.1]). Define, for every $n \in \mathbf{N}$, a real function $f_n : X \to [0,1]$ by $f_n(x) = inf\{r \in \mathcal{S}_n : x \in G_{nr}\}$. Let us prove that f_n is continuous. In order to show that f_n is upper semicontinuous, consider any element $x \in X$, and any real number $\alpha \leq 1$, such that $f_n(x) < \alpha$. From the definition of f_n, there exists $\bar{r} \in \mathcal{S}_n$ such that $f_n(x) < \bar{r} < \alpha$, $x \in G_{n\bar{r}}$. Then $G_{n\bar{r}}$ is an open subset of X such that $f_n(z) < \alpha$ for every $z \in G_{n\bar{r}}$, since $z \in G_{n\bar{r}}$ entails $f_n(z) \leq \bar{r}$.

In order to show that f_n is lower semicontinuous, let us first observe that it is $f_n(x) = inf\{r \in \mathcal{S}_n : x \in \overline{G}_{nr}\}$ for every $x \in X$. Consider any element $x \in X$, and any real number $\alpha \geq 0$, such that $\alpha < f_n(x)$. Then let $\bar{r} \in S$ be such that $\alpha < \bar{r} < f_n(x)$. Then $X \setminus \overline{G}_{n\bar{r}}$ is an open set containing x, such that $f_n(z) > \alpha$ for every $z \in X \setminus \overline{G}_{n\bar{r}}$, since $inf\{r \in \mathcal{S}_n : z \in \overline{G}_{nr}\} = f_n(z) \leq \alpha$ entails $z \in \overline{G}_{n\bar{r}}$. So we have shown that f_n is continuous for every $n \in \mathbf{N}$.

Further, f_n is increasing, since for every $x, y \in X$ such that $x \preceq y$, $\{r \in \mathcal{S}_n : y \in G_{nr}\} \subseteq \{r \in \mathcal{S}_n : x \in G_{nr}\}$, and therefore $f_n(x) \leq f_n(y)$ from the definition of f_n.
From condition *(ii)*, for every $x, y \in X$ with $x \prec y$, there exists $n \in \mathbf{N}$ such that $f_n(x) = 0$ and $f_n(y) = 1$ (see Burgess and Fitzpatrick [2, Theorem 4.1]). Define $f = \sum_{n \in \mathbf{N}} 2^{-n} f_n$. It is immediate to observe that f is real continuous order-preserving function on $(X, \tau, \preceq)$. □

A sufficient condition for the existence of a real continuous order-preserving function f on a topological preordered space $(X, \tau, \preceq)$, with (X, τ) completely regular, and $\preceq$ IC-continuous, is presented in the following proposition.

PROPOSITION 3.2. *Let $(X, \tau, \preceq)$ be a topological preordered space, and assume that (X, τ) is completely regular, and that $\preceq$ is quasi IC-continuous. If $(X, \tau, \preceq)$ is strongly Nachbin separable, then there exists a real continuous order-preserving function f on $(X, \tau, \preceq)$ with values in* $[0, 1]$.

Proof. Consider a countable family $\{x_n, B_n\}_{n \in \mathbf{N}}$ satisfying the definition of strong Nachbin separability. Since (X, τ) is completely regular, for every $n \in \mathbf{N}$ there exists a real continuous function f'_n on X with values in $[0, 1]$ such that $f'_n(x_n) = 0$ and $f'_n = 1$ on B_n. Then there exists a scale $\mathcal{G}''_n = \{G''_{nr} : r \in \mathcal{S}_n\}$ in (X, τ) such that $x_n \in G''_{nr}$, $G''_{nr} \subseteq X \setminus B_n$ for every $r \in \mathcal{S}_n \setminus \{1\}$. Indeed, given $n \in \mathbf{N}$, we can define $\mathcal{S}_n = \mathcal{Q} \cap]0, 1]$, $G''_{nr} = f_n'^{-1}([0, r))$ $(r \in \mathcal{S}_n \setminus \{1\})$, $G''_{n1} = X$. Hence, using the fact that $X \setminus B_n$ is a decreasing set for every $n \in \mathbf{N}$, we have that $x_n \in d(G''_{nr})$, $B_n \subseteq X \setminus d(G''_{nr})$ for every $r \in \mathcal{S}_n \setminus \{1\}$.

Define, for every $n \in \mathbf{N}$, and for every $r \in \mathcal{S}_n$, $G'_{nr} = d(G''_{nr})$. Since $\preceq$ is quasi IC-continuous, it has been already observed in the proof of theorem 1 that $\mathcal{G}'_n = \{G'_{nr} : r \in \mathcal{S}_n\}$ is a decreasing scale in $(X, \tau, \preceq)$ for every $n \in \mathbf{N}$. Since, for every $x, y \in X$ such that $x \prec y$, there exists $n \in \mathbf{N}$ such that $(x, y) \in d(x_n) \times B_n$, it is easily seen that $\{\mathcal{G}'_n\}_{n \in \mathbf{N}}$ is a countable family of (decreasing) scales satisfying condition *(ii)* of theorem 1. Hence, there exists a real continuous order-preserving function f on $(X, \tau, \preceq)$ with values in $[0, 1]$. So the proof is complete. □

References

[1] G. Bosi and G.B. Mehta, *Semicontinuous order-preserving functions and separable systems*, Preprint.

[2] D.C.J. Burgess and M. Fitzpatrick, *On separation axioms for certain types of topological spaces*, Mathematical Proceedings of the Cambridge Philosophical Society **82** (1977), 59–65.

[3] J.C. Candeal and E. Induráin, *Utility representations from the concept of measure*, Mathematical Social Sciences **26** (1993), 51–62.

[4] R. Engelking, *General Topology*, Polish Scientific Publishers, 1977.

[5] G. Herden, *On the existence of utility functions*, Mathematical Social Sciences **17** (1989), 297–313.

[6] G. Herden, *On the existence of utility functions II*, Mathematical Social Sciences **18** (1989), 107–117.

[7] G. Herden, *On some equivalent approaches to mathematical utility theory*, Mathematical Social Sciences **29** (1995), 19–31.

[8] D.H. Krantz, R.D. Luce, P. Suppes, and A. Tversky, *Foundations of Measurement*, vol. I, Academic Press, New York, 1971.

[9] H.-P. A. Künzi, *Completely regular ordered spaces*, Order **7** (1990), 283–293.

[10] S.D. McCartan, *Bicontinuous preordered topological spaces*, Pacific Journal of Mathematics **38** (1971), no. 2, 523–529.

[11] G.B. Mehta, *Some general theorems on the existence of order-preserving functions*, Mathematical Social Sciences **15** (1988), 135–143.

[12] G.B. Mehta, *Preference and utility*, Handbook of Utility Theory (S. Barberà, P.J. Hammond, and C. Seidl, eds.), vol. I, Kluwer Academic Publishers, 1998.

[13] L. Nachbin, *Topology and order*, D. Van Nostrand Company, 1965.

Received February 1, 2000.