

# On quotient orbifolds of hyperbolic 3-manifolds of genus two

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**ABSTRACT.** *We analyse the orbifolds that can be obtained as quotients of genus two hyperbolic 3-manifolds by their orientation preserving isometry groups. The genus two hyperbolic 3-manifolds are exactly the hyperbolic 2-fold branched coverings of 3-bridge links. If the 3-bridge link is a knot, we prove that the underlying topological space of the quotient orbifold is either the 3-sphere or a lens space and we describe the combinatorial setting of the singular set for each possible isometry group. In the case of 3-bridge links with two or three components, the situation is more complicated and we show that the underlying topological space is the 3-sphere, a lens space or a prism manifold. Finally we present an infinite family of hyperbolic 3-manifolds that are simultaneously the 2-fold branched covering of three inequivalent knots, two with bridge number three and the third one with bridge number strictly greater than three.*

**Keywords:** Genus two 3-manifold, 3-bridge knot, 2-fold branched covering, quotient orbifold.

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## 1. Introduction

In this paper we consider quotient orbifolds obtained from the smooth action of finite groups on hyperbolic 3-manifolds admitting a Heegaard splitting of genus two.

A genus  $n$  Heegaard splitting of a closed orientable 3-manifold  $M$  is a decomposition of  $M$  into a union  $V_1 \cup V_2$  of two handlebodies of genus  $n$  intersecting in their common boundary (the Heegaard surface of the splitting). The genus of  $M$  is the lowest genus for which  $M$  admits a Heegaard splitting. The only 3-manifold of genus 0 is the 3-sphere while the genus one 3-manifolds are the lens spaces and  $S^1 \times S^2$ . We remark that two is the lowest possible genus for a hyperbolic manifold.

Quotient orbifolds of 3-manifolds admitting a Heegaard splitting of genus 2 were also studied by J. Kim by using different methods (see [9]). In his paper J. Kim considered only groups leaving invariant the Heegaard splitting

of genus 2. Here we do not make this assumption. On the other hand the results in [9] include also non hyperbolic 3-manifolds.

We recall that 3-bridge knots are strictly related to genus two 3-manifolds. An  $m$ -bridge presentation of a knot  $K$  in the 3-sphere  $S^3$  is a decomposition of the pair  $(S^3, K)$  into a union  $(B_1, a_1) \cup (B_2, a_2)$  where  $B_i$  for  $i = 1, 2$  is a 3-ball and  $a_i$  is a set of  $m$  arcs which are embedded in the standard way in  $B_i$ . We shall say that  $K$  is an  $m$ -bridge knot if  $m$  is the minimal number for which  $K$  admits an  $m$ -bridge presentation (see [17]).

Now a genus two closed orientable surface admits a *hyperelliptic involution*, (i.e. the quotient of the surface by the involution is  $S^2$ ). This involution extends in a standard way to a handlebody of genus two and has the property that, up to isotopy, any diffeomorphism of the surface commutes with it. So for any genus two Heegaard splitting of  $M$  there exists an orientation preserving involution of  $M$ , which we shall also call *hyperelliptic*, which leaves invariant the Heegaard splitting and induces the hyperelliptic involution on the Heegaard surface (in contrast with the two-dimensional case, a genus two 3-manifold admits, in general, more than one hyperelliptic involution, even up to isotopy).

The quotient of  $M$  by a hyperelliptic involution is topologically  $S^3$  and its singular set is a link  $L$ . In this case the Heegaard splitting of  $M$  naturally induces a 3-bridge presentation  $(B_1, a_1) \cup (B_2, a_2)$  of  $L$  where each  $(B_i, a_i)$  is the quotient of a handlebody of the Heegaard splitting.

Conversely, a sphere that induces an  $m$ -bridge presentation of  $L$  lifts to a Heegaard surface of genus  $m - 1$  of the 2-fold branched covering of  $L$ . In particular a 3-bridge presentation induces a genus two Heegaard splitting of the 2-fold branched covering and the covering involution is hyperelliptic.

We can conclude that the hyperbolic 3-manifolds of genus two are exactly the hyperbolic 3-manifolds that are the 2-fold branched covering of a 3-bridge link. This representation is not unique, in fact there exist examples of three inequivalent 3-bridge knots with the same hyperbolic 2-fold branched covering (see [10, Section 5]). In [16] it is proved that a hyperbolic 3-manifold of genus two is the 2-fold branched covering of at most three 3-bridge links. The representation of 3-manifolds as 2-fold branched coverings of knots and links have been extensively studied (see for example the survey by L. Paoluzzi [15] and the recent results by J.E. Greene [7]).

In this paper we prove the following theorem about the structure of the quotient orbifolds of hyperbolic 3-manifolds of genus two. We remark that by the Thurston orbifold geometrization theorem (see [4]), any periodic diffeomorphism of a hyperbolic 3-manifold  $M$  is conjugate to an isometry of  $M$ , so we can suppose that the covering transformation of a 3-bridge link is an isometry. We recall also that a prism manifold is a Seifert 3-manifold such that its base orbifold is a 2-sphere with three singular points, two of them of singularity index 2 (see [14]).

**THEOREM 1.1.** *Let  $L$  be a 3-bridge link and let  $M$  be the 3-manifold of genus 2 that is the 2-fold branched covering of  $L$ . Suppose that  $M$  is hyperbolic and denote by  $G$  a group of orientation preserving isometries of  $M$  containing the covering involution of  $L$ .*

1. *If  $L$  is a knot, then the underlying topological space of  $M/G$  is either  $S^3$  or a lens space and the combinatorial setting of the singular set of  $M/G$  is represented in Figure 1. If the underlying topological space is a lens space, then the covering transformation of  $L$  is central in  $G$ .*
2. *If  $L$  has two or three components, the underlying topological space of  $M/G$  is  $S^3$ , a lens space or a prism manifold. If the underlying topological space is a prism manifold, then*
  - *the symmetries of  $L$  that are projections of elements of  $G$  and fix setwise each component of  $L$  form a non-trivial cyclic group acting freely on  $S^3$ ;*
  - *the symmetries of  $L$  preserving the orientation of  $S^3$  induce a group acting faithfully on the set of the components of  $L$  that is isomorphic to the symmetry group  $\mathcal{S}_n$  where  $n$  is the number of the components.*

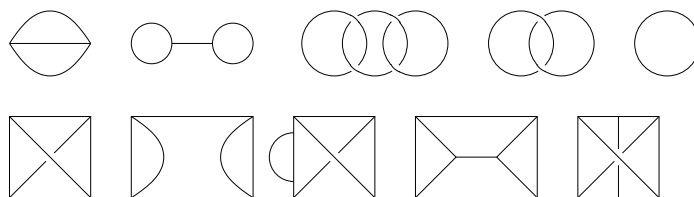


Figure 1: Admissible singular sets for  $M/G$  if  $L$  is a knot.

The proof of the theorem is based on the characterization of the isometry group of the hyperbolic 3-manifolds of genus two given in [10].

By Thurston orbifold geometrization theorem (see [4]) any finite group of diffeomorphisms of  $M$  is conjugate to a group of isometries, so the theorem holds also for finite groups of diffeomorphisms containing a hyperelliptic involution.

The underlying topological spaces are analysed both for knots and links with more than one component. In the knot case, if the underlying topological space is not  $S^3$ , then the hyperelliptic involution of  $L$  is central and each element of  $G$  projects to a symmetry of  $L$  (the 2-fold branched covering has no “hidden symmetries”). We remark also that the situation of quotient orbifolds with

underlying topological space that is neither the 3-sphere nor a lens space is very special.

More details about the quotients are contained in Section 3 and 4. In particular in Figure 20 the case of knots is summarized, distinguishing for each group, that can occur as  $G$ , the possible combinatorial settings of the singular set of  $M/G$ . In principle a similar analysis could be done when  $L$  is not a knot but in the link case the number of possible graphs for the singular set of  $M/G$  is very large and we obtain a very long and complicated list.

In this paper we define a hyperelliptic involution of a genus two 3-manifold as an extension of a hyperelliptic involution of the Heegaard surface of genus two. Often in literature an involution  $t$  acting on a 3-manifold  $M$  is defined to be hyperelliptic if it gives  $S^3$  as underlying topological of the quotient  $M/t$ . We know that a hyperelliptic involution in our sense is hyperelliptic in this broader sense. One might ask if for genus two 3-manifolds the two definitions coincide. The answer is no, in fact, in the last section of the paper, we present an infinite family of genus two 3-manifolds that are the 2-fold branched coverings of knots with bridge number strictly greater than three. Since the bridge number of the knots is not three, the covering involutions of these branched coverings are not hyperelliptic in our sense but they give  $S^3$  as underlying topological space of the quotient. Each of these manifolds is also the 2-fold branched covering of two inequivalent 3-bridge knots. This family gives also examples of 2-fold branched coverings where a sphere giving a minimal bridge presentation of the knot does not lift to a Heegaard surface of minimal genus. A different method to obtain examples of this phenomenon and some comments about it can be found in [8].

## 2. Preliminaries

In this section we present some preliminary results about finite group actions on 3-manifolds, and in particular on the 3-sphere.

**PROPOSITION 2.1.** *Let  $G$  be a finite group of orientation preserving diffeomorphisms of a closed orientable 3-manifold and let  $h$  be an element in  $G$  with nonempty connected fixed point set. Then the normalizer  $N_G h$  of the subgroup generated by  $h$  in  $G$  is isomorphic to a subgroup of a semidirect product  $\mathbb{Z}_2 \ltimes (\mathbb{Z}_a \times \mathbb{Z}_b)$ , for some positive integers  $a$  and  $b$ , where a generator of  $\mathbb{Z}_2$  (a  $h$ -reflection) acts on the normal subgroup  $\mathbb{Z}_a \times \mathbb{Z}_b$  of  $h$ -rotations by sending each element to its inverse.*

*Proof.* The fixed point set of  $h$  is a simple closed curve  $K$ , which is invariant under the action of  $N_G h$ . By a result of Newman (see [5, Theorem 9.5]), a periodic transformation of a manifold which is the identity on an open subset is the identity. Thus the action of an element of  $N_G h$  is determined by its

action on a regular neighborhood of  $K$  where it is a standard action on a solid torus. Every element of  $N_G h$  restricts to a reflection (strong inversion) or to a (possibly trivial) rotation on  $K$ . The subgroup of  $h$ -rotations has index one or two in  $N_G h$  and is abelian. It has a cyclic subgroup (the elements fixing  $K$  pointwise) with cyclic quotient group, so it is abelian of rank at most two.  $\square$

We consider now the finite subgroups of  $SO(4)$  and their action on the unit sphere  $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ . We recall that a non-trivial element of prime order in  $SO(4)$  either acts freely or fixes pointwise a simple closed curve in  $S^3$ . The finite subgroups of  $SO(4)$  are classified by Seifert and Threlfall ([19] and [20]). In Lemma 2.2 we collect some properties of these groups which we need in this paper. The results in Lemma 2.2 can be obtained from the classification and the results contained in [11], but here a direct proof seems to be more suitable. Point 2 of Lemma 2.2 is taken from [3]; the paper is no longer available, since it was a previous version of the paper [2], where this statement is not considered.

LEMMA 2.2. *Let  $G$  be a finite subgroup of  $SO(4)$ .*

1. *Suppose that  $G$  is abelian, then either it has rank at most two or it is an elementary 2-group of rank three. If  $G$  acts freely on  $S^3$ , it is cyclic. If  $G$  has rank at most two, then either at most two simple closed curves of  $S^3$  are fixed pointwise by some nontrivial element of  $G$  or  $G$  is an elementary 2-group of rank two and the whole  $G$  fixes two points (where the fixed-point sets of the three involutions meet).*
2. *If  $G$  is generalized dihedral (i.e.  $G$  is a semidirect product of an abelian subgroup of index two with a subgroup of order two whose generator acts dihedrally on the abelian subgroup of index two), then the underlying topological space of  $S^3/G$  is  $S^3$ .*
3. *If  $G$  has a cyclic normal subgroup  $H$  such that  $G/H$  is cyclic of odd order, then  $G$  is abelian.*

*Proof.* 1) Since  $G$  is abelian, the elements of the group can be simultaneously conjugate to block-diagonal matrices, i.e.  $G$  can be conjugate to a group such that each element has the following form:

$$\left( \begin{array}{c|c} A & 0_2 \\ \hline 0_2 & B \end{array} \right)$$

where  $A, B \in O(2)$  and  $0_2$  is the  $2 \times 2$  matrix whose entries are all zero. Then, by using standard arguments from linear algebra, Point 1 can be proved.

2) In this case the group  $G$  has an abelian subgroup  $R$  of rank at most two and index two. Let  $\phi$  and  $\psi$  be two elements in  $\text{SO}(4)$  that generate  $R$ . Since  $R$  is abelian, the action of  $R$  leaves setwise invariant a 2-dimensional plane  $P$  in  $\mathbb{R}^4$ , which corresponds to a simple closed curve in  $S^3$ . The group  $R$  leaves invariant also  $P^\perp$ , and  $P \oplus P^\perp$  is a  $R$ -invariant decomposition of  $\mathbb{R}^4$ . We will prove that also the action of  $G$  on  $\mathbb{R}^4$  leaves invariant setwise a 2-dimensional plane, that may be different from  $P$ . Let  $\sigma$  be an involution not in  $R$ . We have:  $\psi(\sigma(P)) = \sigma(\psi^{-1}(P)) = \sigma(P)$  and  $\phi(\sigma(P)) = \sigma(\phi^{-1}(P)) = \sigma(P)$ , this implies that the set  $Q = P \cap \sigma(P)$  is  $G$ -invariant.

If  $Q = P$ , then  $P$  is a 2-dimensional plane left setwise invariant by  $P$ .

If  $Q$  is a subspace of dimension 1, then we can construct explicitly another plane that is  $G$ -invariant. Let  $\{v, w\}$  be an orthonormal basis of  $P$  such that  $v \in Q$ . Since  $\psi(v) = \pm v$  and  $\phi(v) = \pm v$  and  $P$  is both  $\psi$ - and  $\phi$ -invariant, we must have that  $\psi(w) = \pm w$  and  $\phi(w) = \pm w$ . Therefore the plane spanned by the vectors  $v$  and  $w + \sigma(w)$  is  $G$ -invariant.

If  $Q = \{0\}$ , we fix again an orthonormal basis  $\{v, w\}$  of  $P$ . If  $\psi$  or  $\phi$  acts as a reflection on  $P$ , then a normal subgroup of  $G$  leaves pointwise invariant a 2-dimensional plane which is left setwise invariant by  $G$ . We can suppose that  $\psi$  and  $\phi$  act as rotations on the plane  $P$  and we will prove that the plane spanned by the couple of vectors  $v + \sigma(v)$  and  $w - \sigma(w)$  is a  $G$ -invariant plane. In fact we have that  $\psi(v + \sigma(v)) = \psi(v) + \psi(\sigma(v)) = \psi(v) + \sigma(\psi^{-1}(v))$ . Supposing  $\psi$  acting on the basis in the following way:  $\psi(v) = \alpha v + \beta w$ ,  $\psi(w) = -\beta v + \alpha w$ , we have that  $\psi^{-1}(v) = \alpha v - \beta w$  and  $\psi^{-1}(w) = \beta v + \alpha w$ . Then  $\psi(v + \sigma(v)) = \alpha v + \beta w + \sigma(\alpha v - \beta w) = \alpha(v + \sigma(v)) + \beta(w - \sigma(w))$  and  $\psi(w - \sigma(w)) = -\beta v + \alpha w - \sigma(\beta v + \alpha w) = -\beta(v + \sigma(v)) + \alpha(w - \sigma(w))$ . The same argument works with  $\phi$ , since it is a rotation on  $P$  too, moreover  $\sigma(v + \sigma(v)) = v + \sigma(v)$  and  $\sigma(w - \sigma(w)) = -(w - \sigma(w))$ . This completes the proof of the fact that  $G$  leaves invariant a 2- plane in  $\mathbb{R}^4$ .

At this point we can suppose that  $\sigma(x, y, z, w) = (x, -y, z, -w)$ , up to conjugacy. The whole isometry group  $G$  respects the Heegaard splitting  $S^3 = T_1 \cup T_2$ , where  $T_1 = \{(x, y, z, w) \in S^3 : x^2 + y^2 \geq 1/2\}$  and  $T_2 = \{(x, y, z, w) \in S^3 : x^2 + y^2 \leq 1/2\}$ . We obtain that  $G$  acts on the solid tori  $T_1$  and  $T_2$  in such a way that their quotients by  $G$  are two solid balls  $B_1$  and  $B_2$ ; then  $S^3/G$  is given by the gluing of a couple of solid balls, that is known to be a 3-sphere  $S^3$ .

3) Let  $\psi$  be a generator of  $H$  and  $\sigma$  be an element of  $G$  such that  $\sigma H$  is a generator of  $G/H$ . We denote by  $h$  the natural number smaller than the order of  $\psi$  such that  $\sigma \psi \sigma^{-1} = \psi^h$

If  $\psi$  (or any nontrivial element of  $H$ ) fixes pointwise a 2-dimensional plane  $P$  in  $\mathbb{R}^4$ , then  $\sigma$  fixes setwise the same 2-dimensional plane and  $P \oplus P^\perp$  is a  $G$ -invariant decomposition of  $\mathbb{R}^4$ . In this case we can reduce the problem to the analysis of the finite subgroups of  $O(2)$  and we are done.

We know that  $\psi$  can be conjugate by a matrix in  $SO(4)$  in the form:

$$\left( \begin{array}{c|c} A & 0_2 \\ \hline 0_2 & B \end{array} \right).$$

If  $A$  and  $B$  have different orders, then we obtain in  $H$  a nontrivial element fixing pointwise a 2-dimensional plane, hence we get the thesis.

We can suppose that  $A$  and  $B$  have the same order.

We consider  $\psi$  as a complex matrix; if  $v$  is an eigenvector of  $\phi$  corresponding to the eigenvalues  $\lambda$ , then  $\sigma(v)$  is an eigenvector for  $\phi$  corresponding to the eigenvalues  $\lambda^h$ .

Suppose first that  $\lambda^h = \lambda$  for an eigenvalue  $\lambda$ ; since the multiplicative order of  $\lambda$  equals the order of  $\psi$ , we obtain  $\sigma\psi\sigma^{-1} = \psi$  and we get the thesis.

Then we can suppose that  $\lambda^h \neq \lambda$  for each eigenvalue  $\lambda$ ;  $\sigma$  induces a bijection on the set of eigenvalues that does not fix any of them. If one of the eigenvalues is  $-1$ , then the order of  $\psi$  is two and the matrix is diagonal ( $A$  and  $B$  have the same order); in this case  $\psi$  is central in  $G$ . Therefore  $\psi$  has two or four different eigenvalues, in any case  $\sigma^4$  leaves invariant each eigenvalue, hence  $\sigma^4$  commutes with  $\psi$ . Since the order of  $\sigma H$  is odd,  $\sigma^4 H$  generates  $G/H$  and we obtained that  $G$  is an abelian group. □

### 3. The knot case

In this section we prove Theorem 1.1 in the knot case.

We recall that, since  $L$  is a knot,  $M$  is a  $\mathbb{Z}_2$ -homology sphere and, by Smith theory (see [12]), the fixed point set of an involution acting on  $M$  is either empty or a simple closed curve.

The method we use to investigate  $M/G$  is to pass through iterated quotients using a subnormal series of subgroups of  $G$ . This method can be applied thanks to the fact that, if  $G$  is a group acting on a manifold  $M$  and  $H$  is a normal subgroup of  $G$ , then the action of  $G$  induces an action of  $G/H$  on the quotient  $M/H$ .

In [10] it is proved that either there exists a hyperelliptic involution central in  $G$  or  $G$  is isomorphic to a subgroup of  $\mathbb{Z}_2 \times \mathbb{S}_4$ . We consider the two cases.

**Case 1:  $G$  contains a central hyperelliptic involution.**

Let  $t$  be a hyperelliptic involution contained in the centre of  $G$  and we suppose that  $L$  is the hyperbolic 3-bridge knot that is the projection of the fixed point set of  $t$  on  $M/\langle t \rangle \cong S^3$ .

The whole group  $G$  projects to  $M/\langle t \rangle$  and the quotient  $M/G$  can be factorized through  $(M/\langle t \rangle)/(G/\langle t \rangle)$ . The Thurston orbifold geometrization theorem

(see [4]) and the spherical space form conjecture for free actions on  $S^3$  proved by Perelman (see [13]) imply that every finite group of diffeomorphism of the 3-sphere is conjugate to a finite subgroup of  $SO(4)$ . The finite subgroups of  $SO(4)$  were classified by Seifert and Threlfall ([19] and [20], see also [11] for a more geometric approach in terms of quotient orbifolds). Thus we can suppose that  $G/\langle t \rangle$  is a group of isometries of  $S^3$  leaving setwise invariant the knot  $L$ . We remark that, since  $L$  is not a trivial knot, by the positive solution of Smith conjecture (see [12]),  $G/\langle t \rangle$  acts faithfully on  $L$ . In particular  $G/\langle t \rangle$  is cyclic or dihedral (see Proposition 2.1).

If the fixed point set (that may be empty) of a symmetry of  $L$  is disjoint from the knot, we call it a  $L$ -rotation. If the fixed point set of a symmetry of  $L$  intersects the knot in two points, we call it a  $L$ -reflection.

Suppose first that  $G/\langle t \rangle$  consists only of  $L$ -rotations. In this case  $G/\langle t \rangle$  is cyclic and there are at most two simple closed curves that are fixed pointwise by some nontrivial element of  $G/\langle t \rangle$  (see Lemma 2.2), thus the singular set of  $M/G$  is a link with at most three components. We recall that the quotient of a 3-sphere by an isometry with non-empty fixed point set is a 3-orbifold with the 3-sphere as underlying topological space and a trivial knot as singular set, while the quotient by an isometry acting freely is a lens space. Therefore if  $G/\langle t \rangle$  is generated by elements with nonempty fixed point set the underlying topological space of the orbifold  $M/G$  is a 3-sphere, otherwise it is a lens-space.

If the group  $G/\langle t \rangle$  contains a reflection of  $L$ , then  $G/\langle t \rangle$  is either dihedral or isomorphic to  $\mathbb{Z}_2$ . In any case, by Lemma 2.2, the underlying topological space of the quotient is a 3-sphere.

If  $G/\langle t \rangle \cong \mathbb{Z}_2$ , then the orbifold  $M/G$  can be obtained as the quotient of  $M/\langle t \rangle$  by a  $L$ -reflection; so the singular set of  $M/G$  is a theta-curve. For what concerns the dihedral case, first of all we recall that for  $G/\langle t \rangle$  to be dihedral means that it is generated, up to conjugacy, by a  $L$ -reflection  $s$  and by a  $L$ -rotation  $r$ . In Lemma 2.2 we defined two tori  $T_1$  and  $T_2$  such that  $T_1 \cup T_2 = S^3$  that are left invariant by  $G$ . Referring to the notation of the proof of Lemma 2.2, we define  $C_1 = \{(x, y, z, w) \in S^3 | x^2 + y^2 = 0\}$  and  $C_2 = \{(x, y, z, w) \in S^3 | x^2 + y^2 = 1\}$ ; these curves are the cores of the tori  $T_1$  and  $T_2$ . We can suppose by conjugacy that the fixed-point sets of the  $L$ -rotations are contained in  $C_1 \cup C_2$ .

We consider then the singular set of  $S^3/(G/\langle t \rangle)$  (where the knot  $L$  is not considered singular). The singular set of  $S^3/(G/\langle t \rangle)$  is contained in the union of the projection of  $C_1 \cup C_2$  with the projection of the fixed point sets of the  $L$ -reflections.

Let  $n$  be the order of the  $L$ -rotation  $r$ . We distinguish two cases:  $n$  odd or  $n$  even.

If  $n$  is odd, then all the  $L$ -reflections are conjugate. Therefore if we consider the fixed point sets of the  $L$ -reflections, the projections of these fixed point sets



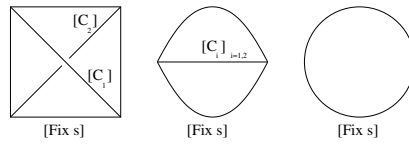


Figure 2: Possible singular sets of  $S^3/(G/\langle t \rangle)$  if  $G/\langle t \rangle$  is dihedral and  $n$  odd.

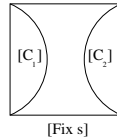


Figure 3: Another combinatorial setting build with a closed curve and two edges.

are all identified in a unique closed curve in the quotient  $S^3/\langle r \rangle$ . The involution  $s$  acts as a reflection also on the curves  $C_1$  and  $C_2$ . If we consider the action of the projection of  $s$  to  $S^3/\langle r \rangle$ , we can describe the possible combinatorial settings for the singular set. These are represented in Figure 2. Notice that the singular set of  $S^3/\langle r \rangle$  can also be empty or have only one component; the number of components of  $S^3/\langle r \rangle$  depends on the number of the simple closed curves, that are fixed pointwise by any  $L$ -rotation. We denote by  $[C_1]$ ,  $[C_2]$  and  $[\text{Fix } s]$  the projections to  $S^3/(G/\langle t \rangle)$  of  $C_1$ ,  $C_2$  and  $\text{Fix } s$ , respectively.

However here there is something to remark. The first graph is only one of the two combinatorial settings that can be built with a closed curve and two edges with different endpoints. The second possibility is the graph in Figure 3. By Lemma 2.2 we can choose up to conjugacy  $s : S^3 \subset \mathbb{R}^4 \rightarrow S^3$  as the map sending  $(x, y, z, w)$  to  $(x, -y, z, -w)$ , and it is easy to see that the fixed point set of  $r$  meets alternately  $C_1$  and  $C_2$ , so this graph does not occur.

To obtain the singular set of  $M/G$ , to each graph in Figure 2 we add  $[L]$ , where  $[L]$  is the projection of  $L$  to  $S^3/(G/\langle t \rangle)$ . Since  $s$  is a  $L$ -reflection,  $[L]$  is an edge with endpoints contained in  $[\text{Fix } s]$ . Figure 4 contains all the possibilities, up to knotting; all the edges, except  $[C_1]$  and  $[C_2]$ , must have singularity index two.

On the other hand if  $n$  is even, then we do not have a unique conjugacy class for all the  $L$ -reflections of  $G/\langle t \rangle$ . Since the fixed point sets of all the elements in the same conjugacy class project to a single curve in the quotient  $S^3/(G/\langle t \rangle)$ , we take into consideration from now on only  $\text{Fix } s$  and  $\text{Fix } rs$ , taking one representative element for each conjugacy class. In this case the

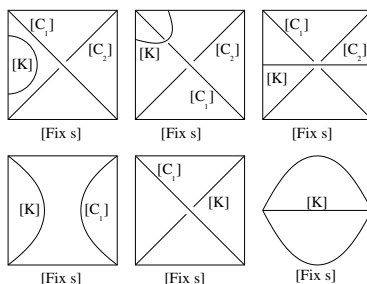


Figure 4: Possible singular sets of  $M/G$  when  $G/\langle t \rangle$  is dihedral of order  $2n$  with  $n$  odd.

fixed point sets of the  $L$ -reflections are not all identified in the quotient, but are collected into two different subsets of the singular set of  $S^3/(G/\langle t \rangle)$ , that we can denote simply by  $[Fix s]$  and  $[Fix rs]$ .

Notice also that if  $n$  is even,  $r^{n/2}$  is a central element in  $G/\langle t \rangle$ , hence we have a  $L$ -rotation fixing setwise  $Fix s$  and  $Fix rs$ .

The type of action of  $r^{n/2}$  on  $Fix s$  and  $Fix rs$  influences how these curves project to  $S^3/\langle r \rangle$ . In fact, according if this element has empty or non-empty fixed point set, different situations occur.

If  $Fix r^{n/2}$  is empty, then it acts on  $Fix s$  and  $Fix rs$  as a rotation, and the fixed point sets of  $s$  and  $rs$  project to two distinct closed curves in  $S^3/\langle r \rangle$ . We remark that  $S^3/\langle r \rangle$  is not a 3-sphere. The projection of  $s$  to  $S^3/\langle r \rangle$  is an involution which acts as a reflection on the projections of  $C_1$  and  $C_2$  and such that its fixed point set consists of the projections of  $Fix s$  and  $Fix rs$ . The possible combinatorial structures of the singular set of  $S^3/(G/\langle t \rangle)$  are presented in Figure 5.

We recall that  $L$  meets both  $Fix s$  and  $Fix sr$  and for the singular set of  $M/G$  we obtain one possibility for each of the graphs, as shown in Figure 6 (all the edges but  $[C_1]$  and  $[C_2]$  must have singularity index two).

If  $Fix r^{n/2}$  is non-empty, then clearly it coincides either with  $C_1$  or with  $C_2$ . In this case, since  $r^{n/2}$  commutes both with  $s$  and with  $rs$  and also with any other involution of  $G/\langle t \rangle$ , we obtain that  $r^{n/2}$  acts as a strong inversion on both of the closed curves  $Fix s$  and  $Fix rs$  (see Lemma 2.2). Therefore the projections of  $Fix s$  and  $Fix rs$  are two arcs in  $S^3/\langle r \rangle$  with both endpoints in common. Moreover the endpoints of  $[Fix s]$  and  $[Fix rs]$  in  $S^3/(G/\langle t \rangle)$  coincide with the endpoints of the arc given by the projection of  $Fix r^{n/2}$ . If  $C_1$  is  $Fix r^{n/2}$ , as before  $C_2$  links  $[Fix s]$  and  $[Fix rs]$ ; the roles of  $C_1$  and  $C_2$  can be exchanged. So the possible settings for the singular set of  $S^3/(G/\langle t \rangle)$  are the ones represented in Figure 7.

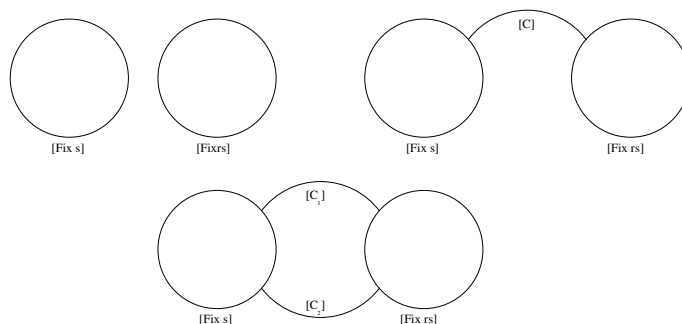


Figure 5: Possible singular sets of  $S^3/(G/\langle t \rangle)$  when  $G/\langle t \rangle$  is dihedral of order  $2n$  with  $n$  even and the central involution acts freely.

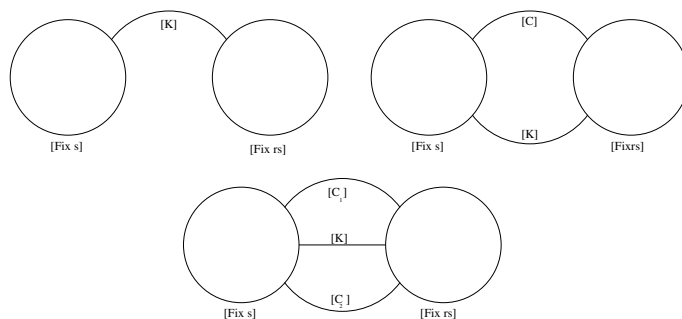


Figure 6: Possible singular sets of  $M/G$  when  $G/\langle t \rangle$  is dihedral of order  $2n$  with  $n$  even and the central involution acts freely.

From these we can build three different graphs that, up to knottings, are the possible singular sets of  $M/G$ , two from the first graph of Figure 7 and one from the second. The admissible results are shown in Figure 8 (again all the edges, except  $[C_1]$  and  $[C_2]$ , have singularity index two).

**Case 2:  $G$  is isomorphic to a subgroup of  $\mathbb{Z}_2 \times \mathbb{S}_4$ .**

By [16] the number of hyperelliptic involution is at most three. We recall that by [1] and [18] two hyperelliptic involutions commute and their fixed point sets meet in two points. Here we consider groups containing a non-central hyperelliptic involution. In this case  $G$  contains a conjugacy class of hyperelliptic involutions with two or three elements (the property to be hyperelliptic is invariant under conjugation). These groups are described in the proof of [10, Theorem 1] (case c) and d) - pages 7 and 8.)

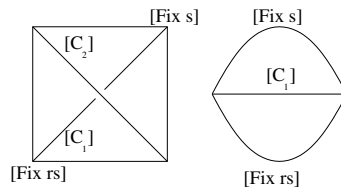


Figure 7: Possible singular sets of  $S^3/\langle G/\langle t \rangle \rangle$  when  $G/\langle t \rangle$  is dihedral of order  $2n$  with  $n$  even and the central involution does not act freely.

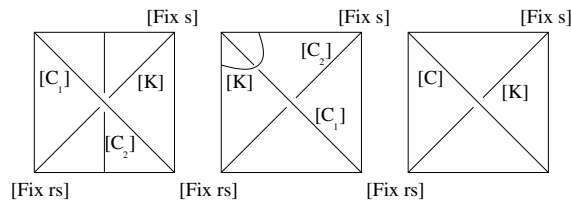


Figure 8: Possible singular sets of  $M/G$  when  $G/\langle t \rangle$  is dihedral of order  $2n$  with  $n$  even and the central involution does not act freely.

**Case 2.1:**  $G \cong \mathbb{D}_8$ .

This case occurs if we have a conjugacy class of hyperelliptic involutions with two elements which we denote by  $t_1$  and  $t_2$ . By the properties of hyperelliptic involutions,  $t_1$  and  $t_2$  generate an elementary subgroup of rank 2 in  $G$ , and we have a subnormal series,

$$\langle t_1 \rangle \triangleleft \langle t_1, t_2 \rangle \triangleleft G.$$

The orbifold  $M/\langle t_1 \rangle$  has  $S^3$  as underlying topological space and a knot as singular set. We consider now  $M/\langle t_1, t_2 \rangle$  which is diffeomorphic to the quotient of  $(M/\langle t_1 \rangle)$  by the projection of  $\langle t_1, t_2 \rangle$  to  $M/\langle t_1 \rangle$ . Since  $t_2$  has non empty fixed point set and is a  $\text{Fix } t_1$ -reflection, we obtain that the underlying topological space of  $M/\langle t_1, t_2 \rangle$  is  $S^3$  and its singular set is a knotted theta curve (a graph with two vertices and three edges; each of the three edges connects the two vertices).

Now we consider the action of  $G/\langle t_1, t_2 \rangle$  on  $M/\langle t_1, t_2 \rangle$ . For a period two action on a theta-curve  $\theta$  we have only three possibilities that are represented in Figure 9.

We can make some remarks on the actions represented. The first action fixes all the three edges and interchanges the vertices, therefore it acts as a rotation with period two around an axis that intersects all the three edges of the theta-curve and leaves the fixed point sets of  $t_1$  and  $t_2$  invariant; the second one acts

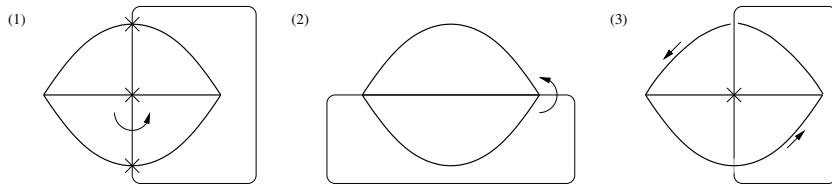


Figure 9: Possible actions with period two on a theta curve.

as a rotation of order two around an axis that contains one of the edges of the theta-curve, therefore it fixes one entire edge and the vertices and interchanges the remaining two edges; the third one acts as a rotation again of order two, but this time the axis intersects only one of the edges and in only one point, therefore it fixes only the intersection point of the theta-curve with the axis, leaves setwise invariant the edge that intersects the axis and interchanges the other two edges and the vertices. Since we already know that  $\mathbb{D}_8$  interchanges the fixed point sets of  $t_1$  and  $t_2$ , the first action is obviously not possible. Since the non-trivial element of  $G/\langle t_1, t_2 \rangle$  has non-empty fixed point set, the orbifold  $M/G$  has  $S^3$  as underlying topological space; the singular set is, up to knottings, one of the two graphs represented in Figure 10 (a theta curve and a "pince-nez" graph). If we obtain a theta curve, then one of the edge has singularity index four (in this case the elements of order four in  $G$  have nonempty fixed point set).

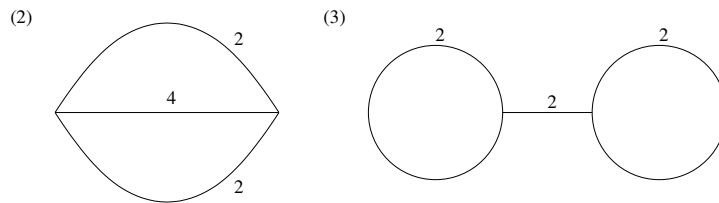


Figure 10: A theta-curve and a "pince-nez" graph.

**Case 2.2:**  $G \cong \mathbb{Z}_2 \times \mathbb{D}_8$ .

This is the second group occurring in the proof of [10, Theorem 1], when the existence of a conjugacy class of hyperelliptic involutions with two elements is assumed; we denote again the two hyperelliptic involutions by  $t_1$  and  $t_2$ .

The first two quotients we consider are the same of the preceding case and we obtain that  $M/\langle t_1, t_2 \rangle$  is known.

Let  $A$  be the subgroup of  $G$  obtained by extending  $\langle t_1, t_2 \rangle$  by a non-trivial

element of the centre of  $G$  ( $t_1$  and  $t_2$  cannot be in the centre of  $G$ ). This means  $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$  and  $A \triangleleft G$ . The subnormal series we consider in this case is the following:

$$\langle t_1 \rangle \triangleleft \langle t_1, t_2 \rangle \triangleleft A \triangleleft G$$

We consider now the projection of the action of  $A$  on  $M/\langle t_1, t_2 \rangle$ .  $A$  acts leaving both hyperelliptic involutions  $t_1$  and  $t_2$  fixed. This means that it does not interchange the fixed point sets of  $t_1$  and  $t_2$ . This time the only possible action of the three represented in Figure 9 is the first and the resulting singular set of  $M/A$  can be represented, up to knottings, as in Figure 11, by a tetrahedral graph. Since the action of  $A/\langle t_1, t_2 \rangle$  is not free, the underlying topological space of  $M/A$  is  $S^3$ .

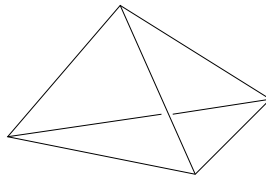


Figure 11: Singular set of  $M/A$ : a tetrahedral graph.

The last extension to take into consideration is  $A \triangleleft G$ , in particular we consider the action of  $G/A$  on  $M/A$ . We ask what actions of period two are combinatorially admissible on a tetrahedral graph. These are represented in Figure 12.

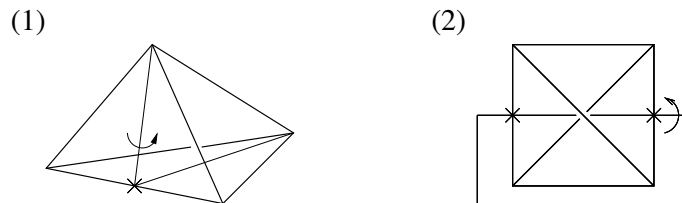


Figure 12: Possible actions of period two on a tetrahedral graph.

The actions represented are respectively a rotation around an axis containing one of the edges and meeting in a point the opposite one (1) and a rotation around an axis meeting a couple of non adjacent edges in one point (2). Therefore we obtain that  $M/G$  is an orbifold with underlying topological space a sphere  $S^3$  and with two possible singular sets, that are the graphs represented in Figure 13 (always up to possible knottings).

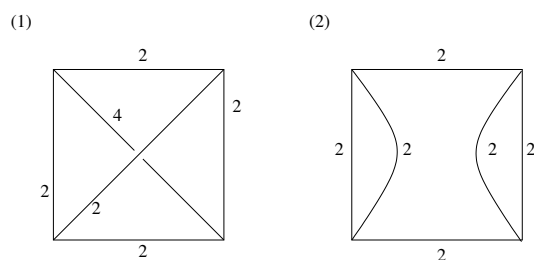


Figure 13: Graphs that can occur as singular sets of  $M/G$ , when  $G \cong \mathbb{Z}_2 \times \mathbb{D}_8$ .

**Case 2.3:**  $G \cong \mathbb{A}_4$ .

The case  $G \cong \mathbb{A}_4$  is the first we encounter in which  $M$  admits a conjugacy class of three hyperelliptic involutions. This condition is satisfied in all the remaining cases (see proof of [10, Theorem 1]) and we will denote the three hyperelliptic involutions by  $t_1, t_2$  and  $t_3$ . Just as before, we consider a subnormal series of subgroups of  $\mathbb{A}_4$ :

$$\langle t_1 \rangle \trianglelefteq \langle t_1, t_2 \rangle \trianglelefteq G.$$

The first two quotients we need to perform are the same encountered in the previous cases, therefore we begin analyzing the projection of the action of  $G$  on the last quotient  $M/\langle t_1, t_2 \rangle$ , that we recall is an orbifold with underlying topological space  $S^3$  and singular set a theta-curve. Noticing that the index of  $\langle t_1, t_2 \rangle$  in  $G$  is three, it follows that a non-trivial element  $G/\langle t_1, t_2 \rangle$  acts faithfully as a rotation with period three on  $M/\langle t_1, t_2 \rangle$ . There is only one action of this type, that is a rotation around an axis that passes through the vertices of the theta-curve; the rotation permutes the three edges cyclically.

The action of  $G/\langle t_1, t_2 \rangle$  on  $M/\langle t_1, t_2 \rangle$  is clearly not free, so the underlying topological space of  $M/G$  is necessarily  $S^3$ . Moreover we can notice that the result of this action is again a theta-curve, but with different singularity indices of the edges as shown in Figure 14.

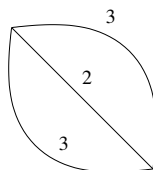


Figure 14: Singular set of  $M/G$ , when  $G \cong \mathbb{A}_4$ .

**Case 2.4:**  $G \cong \mathbb{Z}_2 \times \mathbb{A}_4$ .

The subnormal series of subgroups we use this time is the following:

$$\langle t_1 \rangle \triangleleft \langle t_1, t_2 \rangle \triangleleft A \triangleleft G,$$

where  $A$  is, as before, the normal subgroup of  $G$  isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$  obtained extending  $\langle t_1, t_2 \rangle$  by an element of the centre of the group  $G$ . In light of what we saw in Case 2.2, we already know that  $M/A$  is an orbifold with underlying topological space the 3-sphere  $S^3$  and singular set a tetrahedral graph. Therefore we can analyse directly the action of  $G/A$  on  $M/A$ . The group  $G/A$  has order three and there is only one admissible action of order three on a tetrahedral graph that is a rotation around an axis passing through one of the vertices of the graph that permutes cyclically the three edges containing the vertex fixed by the action, as well the three edges not containing the vertex. The singular set of  $M/G$  is shown, up to knottings, in Figure 15.

Notice that here too the singularity indices of the edges are different. Since the action of  $G/A$  is not free, the underlying topological space of  $M/G$  is  $S^3$ .

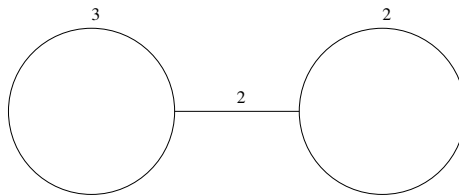


Figure 15: Singular set of  $M/G$ , when  $G \cong \mathbb{Z}_2 \times \mathbb{A}_4$ .

**Case 2.5:**  $G \cong \mathcal{S}_4$ .

Since  $t_1, t_2$  and  $t_3$  are conjugate and commute, the subgroup  $H$  of  $G$  isomorphic to  $\mathbb{A}_4$  contains the three hyperelliptic involutions. To study  $M/G$  we consider the following subnormal series of subgroups:

$$\langle t_1 \rangle \triangleleft \langle t_1, t_2 \rangle \triangleleft H \triangleleft G$$

We already know that the underlying topological space of  $M/H$  is the 3-sphere and that the singular set is the theta-curve represented in Figure 14, up to knottings. Now the point is to understand how  $G/H$  acts on  $M/H$ . Since it is clear that two fixed point sets that have different singularity indices cannot be identified, the possible actions of  $G/H$  are the three represented in Figure 16.

Nevertheless in this case we can exclude some actions. For example the second action would produce as quotient a theta-curve that has one edge of singularity index four, and hence we would have an element  $\alpha \in \mathcal{S}_4$  of order four



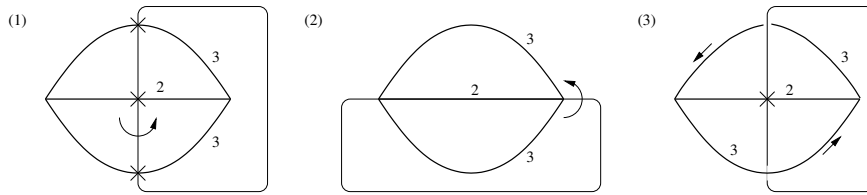


Figure 16: Possible actions of order two on the singular set of  $M/H$ .

and with non-empty fixed point set. This would mean that  $\alpha^2$  is a hyperelliptic involution of  $M$ , then  $\alpha$  projects to a symmetry that fixes pointwise a 3-bridge and by the positive solution of Smith Conjecture, this is impossible.

In this case we have a reason to reject also the third action. Notice that on  $M/\langle t_1, t_2 \rangle$  acts also the dihedral group  $\mathbb{D}_6 \cong G/\langle t_1, t_2 \rangle$ . The dihedral group with six elements is generated by a transformation of order two and by a transformation of order three. Since the transformation of order three acts on  $M/\langle t_1, t_2 \rangle$ , it must act also on its singular set, which is a theta-curve. Therefore, as we have already seen, the fixed point set of this transformation must be non-empty. Since the relation between the transformation of order two and the rotation of order three is dihedral in  $\mathbb{D}_6$ , the fixed point set of the involution is non-empty too and the involution acts as a strong inversion on the fixed point set of the rotation of order three (see Proposition 2.1).

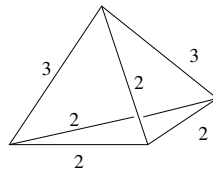


Figure 17: Singular set of  $M/G$ , when  $G \cong \mathcal{S}_4$ .

This means that the two fixed point sets intersect and this implies that the result of the action of  $G/\langle t_1, t_2 \rangle$  on  $S^3$  produces as singular set of the quotient a theta curve with two edges of singularity index two and one edge of singularity index three. This theta curve must be contained in the singular graph of  $M/G$ , but this does not happen for the "pince-nez" graph that we would obtain as singular set of the third action (while the tetrahedral graph resulting from the first case contains such a graph). Therefore both the second and the third actions are not admissible.

Finally the only possible combinatorial setting of the singular set of the orbifold  $M/G$  is the tetrahedral graph shown in Figure 17.

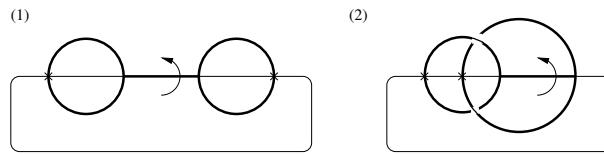


Figure 18: Reflections on the "pince-nez" graphs.

**Case 2.6:**  $G \cong \mathbb{Z}_2 \times \mathcal{S}_4$ .

We consider the following subnormal series of subgroups of  $G$

$$\langle t_1 \rangle \triangleleft \langle t_1, t_2 \rangle \triangleleft A \triangleleft J \triangleleft G,$$

where  $A$ , as before, is the subgroup of  $G$  isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$  obtained extending  $\langle t_1, t_2 \rangle$  by an element that belongs to the centre of the group  $G$  and  $J$  is the normal subgroup of  $G$  isomorphic to  $\mathbb{Z}_2 \times \mathbb{A}_4$  and containing  $A$  as normal subgroup. We already know that the underlying topological space of  $M/J$  is  $S^3$  and that its singular set is the "pince-nez" graph represented in Figure 15 (always up to knottings).

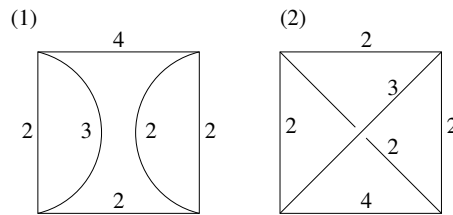


Figure 19: Admissible singular sets for  $M/G$ , when  $G \cong \mathbb{Z}_2 \times \mathcal{S}_4$ .

It is clear that the action of  $G/J$  on  $M/J$  is a transformation with period two, since  $\mathbb{Z}_2 \times \mathbb{A}_4$  has index two in  $\mathbb{Z}_2 \times \mathcal{S}_4$ . The peculiarity of this case is that the singular set of the quotient is different according to the knotting of the "pince-nez" graph. In fact the action on the graph is combinatorially unique, but the result depends on the order of intersections of the two loops with the axis of the involution. The action with period two on these graphs is always a rotation that fixes pointwise the middle edge and leaves invariant the two loops, operating a reflection on each of them, but we have to distinguish between the two cases represented by the graphs of Figure 18. In the first case one of the two arcs, in which is divided the axis by the intersection points between the axis and the first loop, does not contain any intersection point

of the second loop and the axis; in the second case both the arcs contain an intersection point of the second loop and the axis.

We obtain two possible singular sets for the orbifold  $M/G$ , which are shown, up to knottings, in Figure 19. In any case the underlying topological space is  $S^3$ .

In the table in Figure 20 we summarize the situation.

#### 4. The link case

In this section our aim is to generalize the work done in Chapter 3 on 2-fold branched coverings of 3-bridge knots extending our considerations to 2-fold branched coverings of 3-bridge links. In light of the definition of bridge number we can deduce that 3-bridge links can have a maximum of three components. Moreover, in contrast with the case of hyperbolic knots, the constituent knots of a hyperbolic link can also be all trivial.

We denote by  $t$  the hyperelliptic involution that is the covering transformation of  $L$ . In the last part of the proof of [16, Theorem 1] it is proved that, if  $L$  has more than one component, then  $t$  is central in  $G$ . Therefore we have that  $M/G \cong (M/\langle t \rangle)/(G/\langle t \rangle)$  and each element of  $G$  projects to a symmetry of  $L$ .

of  $S^3$  lifts to a finite group acting on  $M$  and containing  $t$  in its centre; by Thurston orbifold geometrization Theorem we can suppose that it is contained up to conjugacy in  $G$ .

In this case, since a long list of graphs would be produced (with respect to the one of the previous chapter), we don't consider the singular set of the quotients. Our only aim this time is to analyse what the underlying topological space of this quotient is.

If  $L$  is a link, the symmetry group of  $L$  is not as simple as when  $L$  is a knot: it is no more true that it is a subgroup of a dihedral group. Let  $G_0$  be the normal subgroup of  $G$  which consists of the elements fixing setwise each component of  $\text{Fix } t$ . We denote by  $\bar{G}$  (resp.  $\bar{G}_0$ ) the quotient  $G/\langle t \rangle$  (resp.  $G_0/\langle t \rangle$ ); the group  $\bar{G}_0$  fixes setwise each component of  $L$ .

Clearly we have that the quotient group  $\bar{G}/\bar{G}_0$  is a subgroup of the symmetry group  $\mathcal{S}_n$ , where  $n$  is the number of components of the link  $L$ , hence in our case  $\bar{G}/\bar{G}_0$  is either a subgroup of  $\mathcal{S}_3$  or a subgroup of  $\mathcal{S}_2$ . By Proposition 2.1,  $\bar{G}_0 \leq \mathbb{Z}_2(\mathbb{Z}_m \times \mathbb{Z}_n)$  for some  $m, n \in \mathbb{N}$ , i.e.  $\bar{G}_0$  is isomorphic to a subgroup of a generalized dihedral group. Therefore  $\bar{G}_0$  has an abelian subgroup of rank at most two of index at most two. We separately analyse the different cases.

**Case 1:**  $\bar{G}_0 \cong \mathbb{Z}_2(\mathbb{Z}_m \times \mathbb{Z}_n)$ , i.e.  $\bar{G}_0$  is generalized dihedral.

We recall that the underlying topological space of  $M/\langle t \rangle$  is  $S^3$ . By Lemma 2.2, we obtain that the underlying topological space of the orbifold  $(M/\langle t \rangle)/\bar{G}_0 \cong M/G_0$  is  $S^3$ . What remains to study now is the action of  $G/G_0$  on  $M/G_0$ , but  $G/G_0$  is either a subgroup of  $S_3$  or a subgroup of  $S_2$ , and hence it is cyclic or


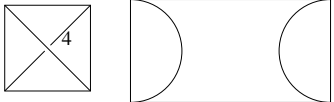
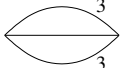
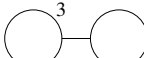

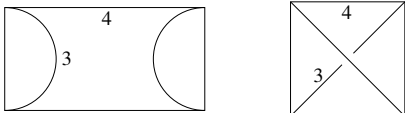
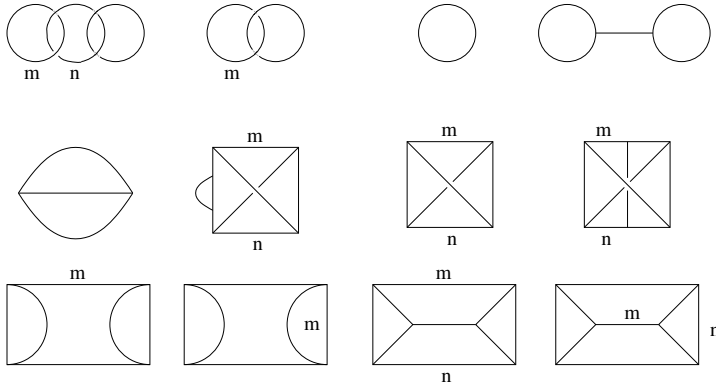
Iso <sup>+</sup> M	Possible singular sets of the quotient of M by Iso M <sup>+</sup>
<b>D<sub>8</sub></b>	
<b>Z<sub>2</sub> × D<sub>8</sub></b>	
<b>A<sub>4</sub></b>	
<b>Z<sub>2</sub> × A<sub>4</sub></b>	
<b>S<sub>4</sub></b>	
<b>Z<sub>2</sub> × S<sub>4</sub></b>	
The set of lifts of the isometries of Iso <sup>+</sup> O(K)	

Figure 20: Admissible singular set for  $M/G$ . For the edges of singularity index 2 we omit the label.

dihedral. By Lemma 2.2, we obtain that the underlying topological space of  $M/G \cong (M/G_0)/(G/G_0)$  is either  $S^3$  or a lens space.

**Case 2:**  $\bar{G}_0 \leq (\mathbb{Z}_m \times \mathbb{Z}_n)$ , i.e.  $\bar{G}_0$  is abelian.

In this case what is missing with respect to the previous one is the action of a strong inversion on a component of  $L$ . Again two cases can occur: rank  $\bar{G}_0 = 1$  or rank  $\bar{G}_0 = 2$ .

**Case 2.1: rank  $\bar{G}_0 = 2$ .**

In this case  $\bar{G}_0$  admits a subgroup either of type  $\mathbb{Z}_p \times \mathbb{Z}_p$  with  $p$  an odd prime or of type  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

We begin showing that the first case cannot occur. Suppose that there exists a subgroup  $D$  of  $\bar{G}_0$  such that  $D \cong \mathbb{Z}_p \times \mathbb{Z}_p$  for some prime  $p > 2$ . Let  $L_i$  be a connected component of the link  $L$  and let  $X_i$  be the subgroup of  $D$  made of the isometries that fix pointwise the component  $L_i$ . The group  $D/X_i$  acts faithfully on the component  $L_i$ , hence the group  $D/X_i$  must be either cyclic or dihedral. Clearly it cannot be dihedral, being a quotient of an abelian group. This means that  $D/X_i$  is cyclic, in particular isomorphic to  $\mathbb{Z}_p$ . Therefore, by Lemma 2.2 the group  $X_i$  is one of the two subgroups of  $D$  isomorphic to  $\mathbb{Z}_p$  that admit nonempty fixed-point set. This argument holds true for all the components of  $L$ : the components of  $L$  are two and, since  $D$  can be simultaneously conjugate to block-diagonal matrices (see proof of Lemma 2.2),  $L$  is the Hopf Link. Since the Hopf link is a well known 2-bridge link, this leads to a contradiction.

Suppose now that  $\bar{G}_0$  contains a subgroup  $D$  isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , in this case we prove that the underlying topological space of  $M/G_0$  is  $S^3$ . Again we consider the quotient  $D/X_i$ , where  $X_i$  is the normal subgroup of  $D$  consisting of the isometries that fix pointwise the component  $L_i$  of  $L$ . In this case  $D/X_i$  can be either cyclic or the whole group  $D$ .

We suppose that  $X_i$  is trivial for some component  $L_i$ , thus  $D$  contains an  $L_i$ -reflection which we denote by  $\alpha$ . Suppose that exists an element  $\beta \in \bar{G}_0$  of order different from two. This element must act as a rotation on the  $i$ -th component of  $L$ , but then, since  $\alpha$  is a  $L_i$ -reflection, we have that  $\alpha\beta\alpha^{-1} = \beta^{-1}$ . This implies that  $\bar{G}_0$  admits a dihedral subgroup, that leads to a contradiction, being  $\bar{G}_0$  abelian. Therefore we obtain that  $\bar{G}_0$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . In this case we are done. In fact, since  $\mathbb{Z}_2 \times \mathbb{Z}_2$  is always generated by a couple of involutions with non empty fixed point set, then the underlying topological space of  $M/G_0 \cong (M/\langle t \rangle)/(G_0/\langle t \rangle)$  is  $S^3$ .

On the other hand, if  $X_i$  is non-trivial for each component  $L_i$  of the link  $L$ , then again we obtain a contradiction by Lemma 2.2. In fact, if  $D$  contains two involutions with non empty fixed point set, then  $L$  should be the Hopf link. If  $D$  contains three involutions with non empty fixed point set, the components of  $L$  should intersect in two points and this is impossible.

Summarizing we obtained that if  $\bar{G}_0$  has rank two, then  $M/G_0$  has always

underlying topological space  $S^3$ . As in the previous case, the group  $G/G_0$  is either cyclic or isomorphic to  $\mathcal{S}_3$ , and this implies that the underlying topological space of  $M/G$  is either  $S^3$  or a lens space.

**Case 2.2: rank  $\bar{G}_0 = 1$ , i.e.  $\bar{G}_0$  is cyclic.**

The quotient of  $S^3$  by a cyclic group of isometries is an orbifold with underlying topological space either  $S^3$  or a lens space.

We distinguish two cases:  $\bar{G}_0$  admits at least one element acting with non empty fixed point set or  $\bar{G}_0$  acts freely.

If  $\bar{G}_0$  does not act freely, then each element in the normalizer of  $G_0$  fixes setwise each curve fixed by a nontrivial element of  $G_0$ . In fact the different curves are fixed pointwise by elements of different order. This means that  $G$  fixes setwise at least a closed curve, therefore, thanks to Proposition 2.1, we can say that  $G$  must be a subgroup of a generalized dihedral group and we are done by Lemma 2.2.

On the other hand if  $\bar{G}_0$  acts freely, then the analysis of the quotient  $M/G$  is more complicated. If  $\bar{G} = \bar{G}_0$ , then the underlying topological space is a lens space.

Otherwise the quotient  $G/G_0$  is isomorphic to  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$  or  $\mathcal{S}_3$ .

If  $\bar{G}/\bar{G}_0 \cong \mathbb{Z}_3$ , then by Lemma 2.2 the group  $\bar{G}$  is abelian and the underlying topological space of  $M/G$  is either  $S^3$  or a lens space. If the components are three and  $\bar{G}/\bar{G}_0 \cong \mathbb{Z}_2$ , then one of the component of  $L$  is fixed setwise by  $\bar{G}$  and we are done.

In the remaining cases we can suppose that  $\bar{G}$  has an abelian subgroup of index two. Up to now we were able to prove that the underlying topological space of  $M/G$  is either  $S^3$  or a lens space, unfortunately in the remaining cases some groups can admit as underlying topological space of the quotient a prism manifold. By analyzing the remaining groups case by case, we can deduce more information about the situations in which a prism manifold can occur, but at this point we prefer to give a shorter argument that simply exclude tetrahedral, octahedral and icosahedral manifolds as underlying topological space of the quotient. Since  $\bar{G}$  has an abelian subgroup of index two, the group leaves invariant a fibration of  $S^3$  (see [11]). The quotient orbifold  $S^3/\bar{G}$  admits a Seifert fibration induced by the fibration of  $S^3$  left invariant by  $G$ . By [11, Lemma 2], the base 2-orbifold  $B$  of  $S^3/\bar{G}$  is the quotient of  $S^2$  by the action of  $\bar{G}$  (which is possibly non-faithfully.) Since  $G$  has an abelian subgroup of index 2, either  $B$  has a disk as underlying topological space or it is a 2-sphere with at most one singular point of index strictly greater than 2 (i.e. the cases with base 2-orbifold  $S^2(2, 3, 3)$ ,  $S^2(2, 3, 4)$  and  $S^2(2, 3, 5)$  in [11, Table 4] are excluded).

If  $B$  has underlying topological space the 2-disk, then, by [6, Proposition 2.11], the underlying topological space of  $S^3/\bar{G}$  (and hence of  $M/G$ ) is either  $S^3$  or a lens space.

On the other hand, if the base 2-orbifold has no boundary component, then, by forgetting the orbifold singularity of the fibers, we obtain, from the Seifert fibration of  $S^3/\bar{G}$ , a Seifert fibration of the underlying topological space of  $S^3/\bar{G}$ . The base 2-orbifold of the underlying topological space of  $S^3/\bar{G}$  can be obtained from  $B$  by dividing the index of the singular points by the singularity index of the corresponding fibers. The Euler number of the fibration is not affected by the singularity forgetting process. Since  $S^2(2, 3, 3)$ ,  $S^2(2, 3, 4)$  and  $S^2(2, 3, 5)$  are excluded as base 2-orbifolds, it turns out that the underlying topological space of  $S^3/\bar{G}$  is  $S^3$ , a lens space or a prism manifold (see [14] and [11, Table 2,3,4])

### 5. An example

In this section we describe an infinite family of hyperbolic 3-manifolds such that each of them is the 2-fold branched covering of three inequivalent knots, two of them with bridge number equal to three and the third one with bridge number strictly greater than three.

For any triple of nonzero integers  $(i, j, k)$  we can define the 3-bridge knot  $K_{ijk}$  presented in Figure 21.

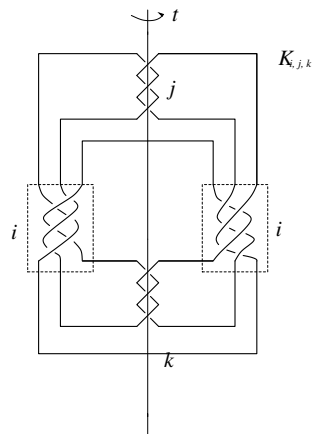


Figure 21: The knot  $K_{ijk}$ .

In Figure 21 we have also drawn the axis of a strong inversion  $t$  of  $K_{ijk}$ . Let  $\mathcal{O}(K_{ijk})$  be the orbifold with underlying topological space  $S^3$  and  $K_{ijk}$  as singular set of index 2. We consider the quotient orbifold  $\mathcal{O}(\theta_{ijk}) := \mathcal{O}(K_{ijk})/\langle t \rangle$  which has  $S^3$  as underlying topological space. The singular set is a theta-curve  $\theta_{ijk}$  with edges  $e_1, e_2$  and  $e_3$  and constituent knots  $A_1 = e_2 \cup e_3, A_2 = e_1 \cup e_3$

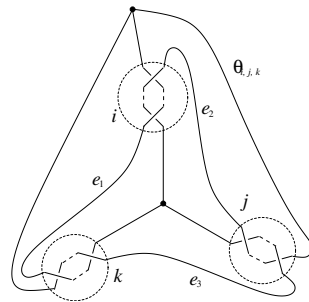


Figure 22: The theta curve  $\theta_{i,j,k}$ .

and  $A_3 = e_1 \cup e_2$ . The theta curve  $\theta_{i,j,k}$  is represented in Figure 22 (how to obtain this planar diagram of  $\theta_{i,j,k}$  is explained in [21]).

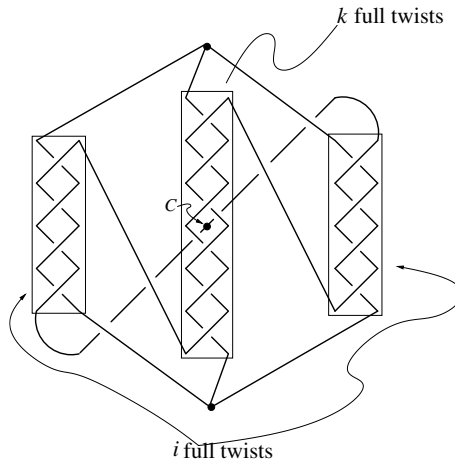


Figure 23: Symmetry of  $\theta_{i,i,k}$ .

The three constituent knots are trivial and the preimage of  $e_1, e_2$  respectively  $e_3$  in the 2-fold cyclic branched covering of  $S^3$  along  $A_1, A_2$  respectively  $A_3$  is  $K_1 = K_{i,j,k}, K_2 = K_{j,k,i}$  respectively  $K_3 = K_{k,i,j}$ . Finally, if we take the two fold branched covering of  $K_1, K_2$  or  $K_3$ , we get the same manifold  $M$ : the manifold  $M$  is the  $\mathbb{D}_4$  covering of  $\mathcal{O}(\theta_{i,j,k})$ . In [21] it is proved that  $M$  is hyperbolic for  $|i|, |j|, |k|$  sufficiently large. The isometry group of  $M$  was studied in [10]. Here we consider the case where two of the indices are equal, while the third one is different. If  $\{i, j, k\}$  is not of the form  $\{lm, (l-1)m\}$ , with  $m$



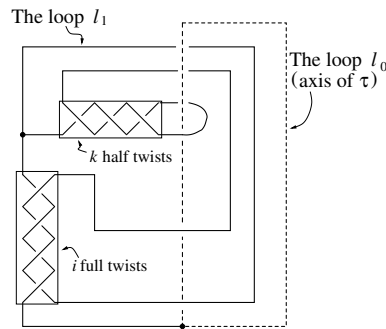


Figure 24: The singular set of  $M/G$ .

and  $l$  integers and  $l$  even, then the isometry group of  $M$  is isomorphic to  $\mathbb{D}_8$  and it does not contain any orientation-reversing isometry (see [10, page 8]). We denote by  $G$  the isometry group of  $M$  and we suppose  $i = j$ . In  $G$  there are three conjugacy classes of involutions, two of them consist of hyperelliptic involutions and correspond to the 3-bridge knots. One of the hyperelliptic involutions is central in  $G$ . The involutions in the third conjugacy class are not hyperelliptic. In this section we prove that the quotient orbifold of  $M$  by one of these involutions has underlying topological space  $S^3$ ; the singular set of this orbifold is a knot that has bridge number different from three. If  $i = j$ , the theta curve  $\theta_{iik}$  has a symmetry of order 2 exchanging the two vertices and leaving setwise invariant only one of the edges. In the diagram of  $\theta_{iik}$  presented in Figure 23 the symmetry is evident, it consists of a  $\pi$ -rotation around the point  $C$ . We denote this involution by  $\tau$ .

The quotient of  $\mathcal{O}(\theta_{iik})$  by  $\tau$  is an orbifold with  $S^3$  as underlying topological space and with the knotted pince-nez graph represented in Figure 24 as singular set. This orbifold is  $M/G$  (this situation corresponds to Case 2.1. in Section 3). We denote by  $l_0$  and  $l_1$  the loops of the pince-nez graph. In particular let  $l_1$  be the projection of the axis of  $\tau$  (the dotted line in Figure 24).

Now we consider the orbifold obtained by taking the 2-fold covering of  $M/G$  branched over the loop  $l_1$ . We remark that the loop  $l_1$  is a trivial knot. This gives an orbifold  $\mathcal{O}(\Gamma_{ik})$  with  $S^3$  as underlying topological space and the theta curve  $\Gamma_{ik}$  represented in Figure 25 as singular set (of singularity index 2).

To draw explicitly  $\Gamma_{ik}$  we use a planar diagram of the singular set of  $M/G$  where  $l_1$  has a trivial projection. In Figure 26 how to obtain such a diagram is explained. From this representation of the graph it is easy to reconstruct a diagram of  $\Gamma_{ik}$ . To help the reader, in Figure 25 we represent explicitly the axis of the involution acting on  $\mathcal{O}(\Gamma_{ik})$  that gives  $M/G$  as quotient.

The orbifold  $\mathcal{O}(\Gamma_{ik})$  is the quotient of  $M$  by the group generated by the

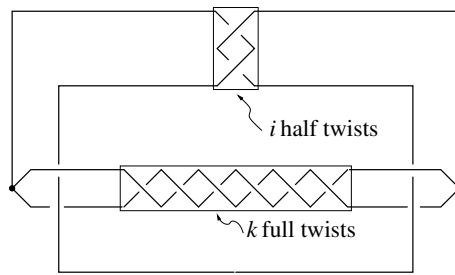


Figure 25:  $\Gamma_{ik}$

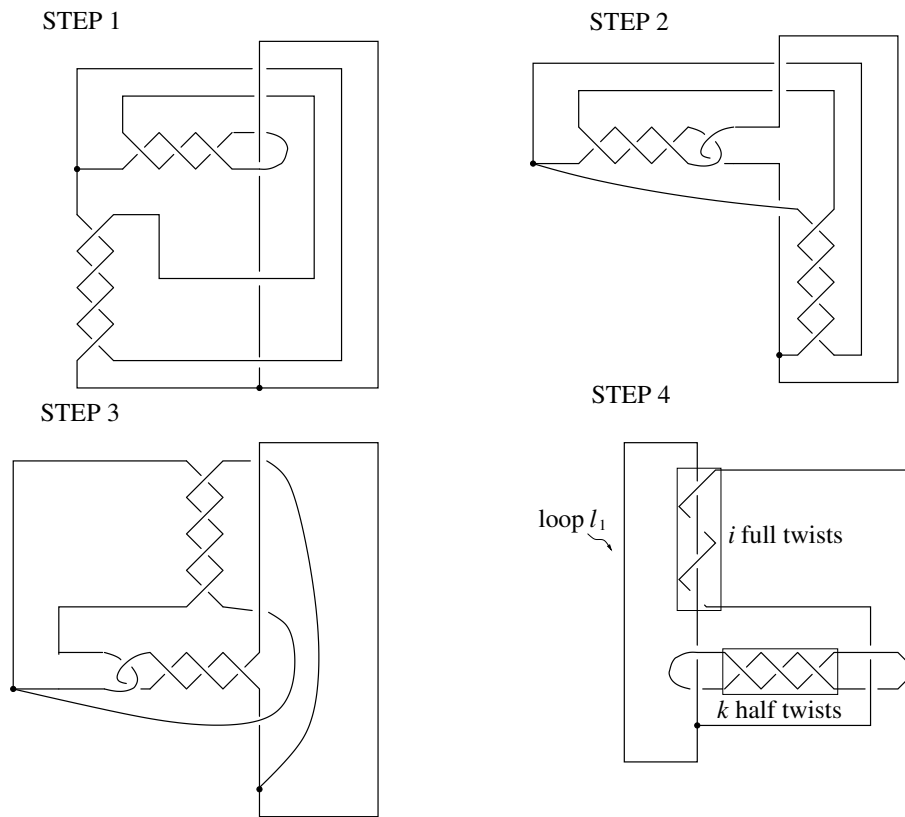


Figure 26: Equivalent diagrams of the singular set of  $M/G$

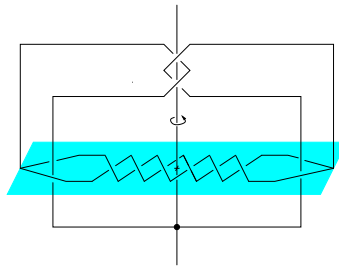


Figure 27:  $\Gamma_{ik}$

central hyperelliptic involution and by one involution that is not hyperelliptic (we call  $u$  such an involution). The orbifold  $M/\langle u \rangle$  is the two fold covering of  $\mathcal{O}(\Gamma_{ik})$  branched over the appropriate constituent knot. We remark that all the three constituent knots of  $\Gamma_{ik}$  are trivial. This implies that the underlying topological space of  $M/\langle u \rangle$  is  $S^3$  and that  $M$  is the 2-fold covering of  $S^3$  branched over the knot that is the singular set of  $M/\langle u \rangle$ . We denote this knot by  $L_{ik}$ . Since  $u$  is not hyperelliptic, the bridge number of  $L_{ik}$  is different from 3. Since the only knot with bridge number one is the unknot and the 2-fold branched coverings of the 2-bridge knots are the lens spaces, these knots have bridge number strictly greater than three. We give a diagram of  $L_{ik}$  in Figure 28 and Figure 29 shows a procedure to get it. Indeed an explicit diagram of  $L_{ik}$  is not necessary to get the properties we need. We remark that the Heegaard splitting induced by a minimal bridge presentation of  $L_{ik}$  is not of minimal genus.

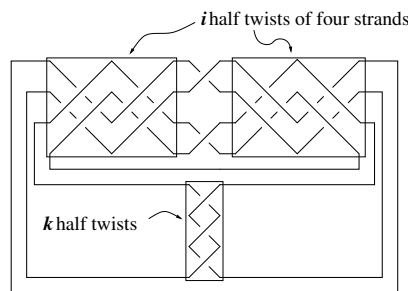
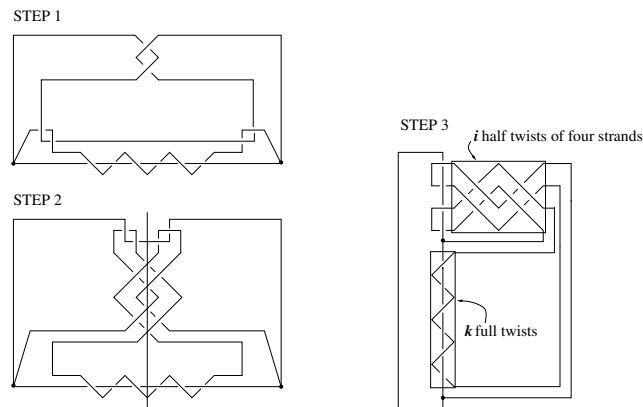


Figure 28: the knot  $L_{ik}$

Figure 29: Equivalent diagrams of  $\Gamma_{ik}$ 

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