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Uniform Continuity in Fuzzy Metric Spaces

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Summary. - *With the help of appropriate fuzzy notions of equinormality and Lebesgue property, several characterizations of those fuzzy metric spaces, in the sense of George and Veeramani, for which every real valued continuous function is uniformly continuous are obtained.*

1. Introduction

There exist many attempts of obtaining a notion of fuzzy metric space that permitted to extend the classical theory of metric and metrizable spaces to the setting of fuzzy topology. In this direction, and by modifying a definition of fuzzy metric space given by Kramosil and Michalek [8], George and Veeramani ([4], [1]) have introduced and studied a new and interesting notion of a fuzzy metric space.

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According to [9] a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $*$ satisfies the following conditions:
(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for every $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

According to [4] and [1], a fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm on X and M is a fuzzy set of $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:
(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) $M(x, y, \cdot) : (0, +\infty) \to [0,1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a fuzzy metric on X.

Let (X, d) be a metric space. Define $a * b = ab$ for every $a, b \in [0,1]$, and let M_d be the function defined on $X \times X \times (0, +\infty)$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then, $(X, M_d, *)$ is a fuzzy metric space, and $(M_d, *)$ is called the fuzzy metric induced by d (see [4]). In the case that d is the Euclidean metric on $\mathbb{R}$, the induced fuzzy metric will be called *the Euclidean fuzzy metric on* $\mathbb{R}$.

George and Veeramani proved in [4] that every fuzzy metric $(M, *)$ on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, r, t) : x \in X,\ 0 < r < 1,\ t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for every $r \in (0,1)$ and $t > 0$.

They proved that (X, τ_M) is a Hausdorff first countable topological space. Moreover, if (X, d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the fuzzy metric $(M_d, *)$.

We say that a topological space (X, τ) admits a fuzzy metric if there is a fuzzy metric space $(X, M, *)$ such that $\tau = \tau_M$. Thus, by results of George and Veeramani mentioned above, every metrizable topological space admits a fuzzy metric. In [7] it was proved that if $(X, M, *)$ is a fuzzy metric space, then $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}_M$ compatible with τ_M, where $U_n = \{(x, y) : M(x, y, 1/n) > 1 - 1/n\}$ for all $n \in \mathbb{N}$. Hence if $(X, M, *)$ is a fuzzy metric space, then the topological space (X, τ_M) is metrizable (see [7] Theorem 1).

Several properties of completeness, compactness and precompactness of fuzzy metric spaces have been obtained in [4], [1] and [7]. In [5] the notion of uniform continuity of mappings on fuzzy metric spaces was introduced, and it was observed that, similarly to the classical metric case, every real valued continuous function on a compact fuzzy metric space is uniformly continuous. It then seems interesting to characterize those fuzzy metric spaces for which every real valued continuous function is uniformly continuous. With the help of appropriate fuzzy notions of equinormality and Lebesgue property we shall present some solutions to this problem. Such solutions provide satisfactory extensions of important and well-known theorems on metric and topological metrizable spaces, respectively, to the fuzzy setting.

Let us recall that a uniformity $\mathcal{U}$ on a set X has the Lebesgue property provided that for each open cover $\mathcal{G}$ of X there is $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{G}$, and $\mathcal{U}$ is said to be equinormal if for each pair of disjoint nonempty closed subsets A and B of X there is $U \in \mathcal{U}$ such that $U(A) \cap B = \emptyset$. A metric d on X has the Lebesgue property provided that the uniformity $\mathcal{U}_d$, induced by d, has the Lebesgue property and d is equinormal provided that $\mathcal{U}_d$ so is (see, for instance, [3]).

In the following the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and positive integer numbers, respectively.

2. The results

Definition 2.1. *[5] A mapping f from a fuzzy metric space $(X, M, *)$ to a fuzzy metric space $(Y, N, \star)$ is called uniformly continuous if for* *each* $\varepsilon \in (0,1)$ *and each* $t > 0$, *there exist* $r \in (0,1)$ *and* $s > 0$ *such that* $N(f(x), f(y), t) > 1 - \varepsilon$ *whenever* $M(x, y, s) > 1 - r$.

It is clear that every uniformly continuous mapping from the fuzzy metric space $(X, M, *)$ to the fuzzy metric space $(Y, N, \star)$ is continuous from (X, τ_M) to (Y, τ_N).

Definition 2.2. *We say that a real valued function* f *on the fuzzy metric space* $(X, M, *)$ *is* $\mathbb{R}$*-uniformly continuous provided that for each* $\varepsilon > 0$ *there exist* $r \in (0,1)$ *and* $s > 0$ *such that* $| f(x) - f(y) | < \varepsilon$ *whenever* $M(x, y, s) > 1 - r$.

Here by a compact fuzzy metric space we mean a fuzzy metric space $(X, M, *)$ such that (X, τ_M) is a compact topological space.

Note that if f is a continuous mapping from the compact fuzzy metric space $(X, M, *)$ to the fuzzy metric space $(Y, N, \star)$, and we fix $\varepsilon \in (0,1)$ and $t > 0$, then there exists $\delta > 0$ such that $(1 - \delta) \star (1 - \delta) > 1 - \varepsilon$, by the continuity of $\star$. So, for each $x \in X$ there exist $r_x, r'_x \in (0,1)$ and $s_x > 0$ such that $f(B_M(x, r'_x, s_x)) \subseteq B_N(f(x), \delta, t/2)$ and $(1 - r_x) * (1 - r_x) > 1 - r'_x$. Now, there exists a finite subset A of X such that $X = \cup_{x \in A} B_M(x, r_x, s_x/2)$. Put $r = \min\{r_x : x \in A\}$ and $s = \max\{s_x/2 : x \in A\}$. It is routine to show that $N(f(x), f(y), t) > 1 - \varepsilon$ whenever $M(x, y, s) > 1 - r$. So f is uniformly continuous in the sense of Definition 1 (see also [5] Result 1.8).

In our main result we shall characterize those fuzzy metric spaces for which real valued continuous functions are uniformly continuous.

Definition 2.3. *A fuzzy metric* $(M, *)$ *on a set* X *is called equinormal if for each pair of disjoint nonempty closed subsets* A *and* B *of* (X, τ_M) *there is* $s > 0$ *such that* $\sup\{M(a, b, s) : a \in A,\ b \in B\} < 1$.

Definition 2.4. *We say that a fuzzy metric* $(M, *)$ *on a set* X *has the Lebesgue property if for each open cover* $\mathcal{G}$ *of* (X, τ_M) *there exist* $r \in (0,1)$ *and* $s > 0$ *such that* $\{B_M(x, r, s) : x \in X\}$ *refines* $\mathcal{G}$.

Remark 2.5. *Notice that if* (X, d) *is a metric space, then the fuzzy metric* $(M_d, *)$ *has the Lebesgue property (resp. is equinormal) if and only if* d *has the Lebesgue property (resp. is equinormal).*

The following lemma will be used in the proof of our main result.

Lemma 2.6. *([6]). Let* $(X, M, *)$ *be a fuzzy metric space. Then* $M(x, y, \cdot)$ *is nondecreasing for all* $x, y \in X$.

Theorem 2.7. *For a fuzzy metric space* $(X, M, *)$ *the following are equivalent.*

(1) For each fuzzy metric space $(Y, N, \star)$ *any continuous mapping from* (X,τ_M) *to* (Y, τ_N) *is uniformly continuous as a mapping from* $(X, M, *)$ *to* $(Y, N, \star)$.

(2) Every real valued continuous function on (X, τ_M) *is* $\mathbb{R}$*-uniformly continuous on* $(X, M, *)$.

(3) Every real valued continuous function on (X, τ_M) *is uniformly continuous on* $(X,\mathcal{U}_M)$.

(4) $(M, *)$ *is an equinormal fuzzy metric on* X.

(5) $\mathcal{U}_M$ *is an equinormal uniformity on* X.

(6) The uniformity $\mathcal{U}_M$ *has the Lebesgue property.*

(7) The fuzzy metric $(M, *)$ *has the Lebesgue property.*

Proof. (1) $\Rightarrow$ (2). Let f be a real valued continuous function on (X, τ_M) and $\varepsilon > 0$. (We may assume without loss of generality that $\varepsilon \in (0, 1)$.) Choose $n \in \mathbb{N}$ such that $1 - \varepsilon > 1/n$. By assumption f is uniformly continuous as a mapping from $(X, M, *)$ to $(\mathbb{R},M_d, \star)$, where $(M_d, \star)$ is the Euclidean fuzzy metric on $\mathbb{R}$. Hence, there exist $r \in (0, 1)$ and $s > 0$ such that $\frac{1/n}{1/n+|f(x)-f(y)|} > 1 - \varepsilon$ whenever $M(x, y, s) > 1-r$. An easy computation shows that $| f(x)-f(y) |< \varepsilon$ whenever $M(x, y, s) > 1 - r$. We conclude that f is $\mathbb{R}$-uniformly continuous on $(X, M, *)$.

(2) $\Rightarrow$ (3). Let f be a real valued continuous function on (X, τ_M) and $\varepsilon > 0$. By assumption, there exist $r \in (0, 1)$ and $s > 0$ such that $| f(x) - f(y) |< \varepsilon$ whenever $M(x, y, s) > 1 - r$. Take $n \in \mathbb{N}$ such that $1/n \leq \min\{r, s\}$. Then for all $x, y \in X$ such that $(x, y) \in U_n$, we obtain, by the above lemma, $M(x, y, s) \geq M(x, y, 1/n) > 1 - 1/n \geq$

$1-r$, so $| f(x)-f(y) | < \varepsilon$. We conclude that f is uniformly continuous on $(X, \mathcal{U}_M)$.

(3) $\Rightarrow$ (4). Let A and B be two disjoint nonempty closed subsets of (X, τ_M). Then there exists a continuous function $f : X \to [0,1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$. Since by assumption f is uniformly continuous on $(X, \mathcal{U}_M)$, for $\varepsilon = 1$ there is $n \in \mathbb{N}$ such that $| f(x) - f(y) | < 1$ whenever $M(x, y, 1/n) > 1 - 1/n$. Hence $M(a, b, 1/n) \leq 1 - 1/n$ for all $a \in A$ and $b \in B$. We conclude that $(M, *)$ is an equinormal fuzzy metric on X.

(4) $\Rightarrow$ (5). Let A and B be two disjoint nonempty closed subsets of (X, τ_M). By assumption, there exist $s > 0$ and $r \in (0,1)$ such that

$$\sup\{M(a, b, s) : a \in A,\, b \in B\} = 1 - r.$$

Put $U = \{(x, y) \in X \times X : M(x, y, s) > 1 - r\}$. Then $U \in \mathcal{U}_M$ and $U(A) \cap B = \emptyset$. Hence $\mathcal{U}_M$ is an equinormal uniformity on X.

(5) $\Rightarrow$ (6). [2] Theorem 2.3.1.

(6) $\Rightarrow$ (7). Let $\mathcal{G}$ be an open cover of X. From our assumption it follows that there is an $n \in \mathbb{N}$ such that $\{B_M(x, 1/n, 1/n) : x \in X\}$ refines $\mathcal{G}$. Hence $(M, *)$ is a Lebesgue fuzzy metric on X.

(7) $\Rightarrow$ (1). Let $(Y, N, \star)$ be a fuzzy metric space and f a continuous mapping from (X, τ_M) to (Y, τ_N). Fix $\varepsilon \in (0,1)$ and $t > 0$. There is $\delta > 0$ such that $(1-\delta) \star (1-\delta) > 1 - \varepsilon$. Since f is continuous, for each $x \in X$ there is an open neighborhood V_x of x such that $f(V_x) \subseteq B_N(f(x), \delta, \frac{t}{2})$. By assumption there exist $r = r(t, \varepsilon) \in (0,1)$ and $s > 0$ such that $\{B_M(x, r, s) : x \in X\}$ refines $\{V_x : x \in X\}$.

Now if $M(x, y, s) > 1 - r$ we have $y \in B_M(x, r, s)$, so $x, y \in V_z$ for some $z \in X$. Hence, $f(x)$ and $f(y)$ are in $B_N(f(z), \delta, \frac{t}{2})$, and thus

$$N(f(x), f(y), t) \geq N(f(x), f(z), \frac{t}{2}) \star N(f(z), f(y), \frac{t}{2}) > 1 - \varepsilon.$$

We conclude that f is uniformly continuous from $(X, M, *)$ to $(Y, N, \star)$. $\square$

It is well known (see, for instance, [2]) that a metrizable topological space admits a metric with the Lebesgue property if and only if the set of nonisolated points is compact. From this result and the preceding theorem we deduce the following corollary.

COROLLARY 2.8. *A (fuzzy) metrizable topological space admits a fuzzy metric with the Lebesgue property if and only if the set of nonisolated points is compact.*

REMARK 2.9. *Given a metrizable topological space X we denote by $\mathcal{FN}_M$ the supremum of all uniformities $\mathcal{U}_M$ induced by all the compatible fuzzy metrics for X. It is easy to see that $\mathcal{FN}_M$ is exactly the fine uniformity of X. Hence, the classical theorem that if a topological space X admits a metric d with the Lebesgue property, then the uniformity $\mathcal{U}_d$ coincides with the fine uniformity of X, can be reformulated as follows:*

If a topological space admits a fuzzy metric $(M, *)$ with the Lebesgue property, then the uniformity $\mathcal{U}_M$ coincides with the uniformity $\mathcal{FN}_M$.

We conclude the paper with an example which illustrates the obtained results.

EXAMPLE 2.10. *Let X be the set of natural numbers and let $*$ be the continuous t-norm defined by $a * b = ab$ for all $a, b \in [0, 1]$. For each $x, y \in X$ and $t > 0$ let $M(x, y, t) = 1$ if $x = y$ and $M(x, y, t) = 1/xy$ otherwise. It is routine to check that $(X, M, *)$ is a fuzzy metric space. Note that there is no metric d on X for which $(M, *)$ is the fuzzy metric induced by d (indeed, otherwise we have, for $x, y \in X$ fixed with $x \neq y$, $t = \frac{d(x,y)}{xy-1}$ for all $t > 0$, which is impossible).*

For each pair of disjoint nonempty subsets of X, A and B, we have $\sup\{M(a, b, t) : a \in A,\ b \in B\} \leq 1/2$ for all $t > 0$. From this fact it follows that $(M, *)$ is an equinormal fuzzy metric on X (so $(X, M, *)$ satisfies conditions in our theorem), and that τ_M is the discrete topology on X since for each $x \in X$, $\sup\{M(x, y, t)\colon y \neq x,\ t > 0\} \leq 1/2$ and thus $B_M(x, 1/2, 1/2) = \{x\}$.

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